Topological groups in which all countable subgroups are closed

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1. Introduction

It is well known that an Abelian group $G$ endowed with the maximal precompact topological group topology $\tau_b$ (called the Bohr topology of $G$) does not contain infinite compact subsets and all subgroups of $G^\# = (G, \tau_b)$ are closed (see [13] and [9, Lemma 2.1], respectively). Further, the group $G^\#$ is pseudocompact iff $G$ is finite (see [9, Theorem 2.2] or [3, Theorem 9.9.42] or [18] for more general facts).

A similar functorial approach to topologizations of Abelian groups is considered in [10], where the maximal $\omega$-narrow topological group topology on Abelian groups is studied. We recall that a topological group is $\omega$-narrow if it can be covered by countably many translates of any neighborhood of the neutral element. For an Abelian group $G$, let $G^{\square}$ be the underlying group $G$ endowed with its maximal $\omega$-narrow group topology. The topological groups $G^{\square}$ and $G^\#$ share many properties. In particular, since the identity isomorphism of $G^{\square}$ onto $G^\#$ is continuous, all subgroups of $G^{\square}$ are closed. It is easy to verify that all countable subgroups of $G^{\square}$ are even discrete [10, Corollary 2.6]. Therefore, if $G$ is uncountable, then $G^{\square}$ is a non-discrete $\omega$-narrow topological group in which all countable subgroups are closed and discrete.

We consider here topological Abelian groups that have a property weaker than the groups of the form $G^\#$ or $G^{\square}$. More precisely, our main objective is to study the class $\mathcal{C}C$ of topological Abelian groups $G$ such that all countable subgroups of $G$ are closed. It is shown that all countably compact subsets of a bounded torsion group in $\mathcal{C}C$ are finite, while in general countably compact subsets of any group in $\mathcal{C}C$ are countable and compact.

It was proved by the author in 1992 that there exist arbitrarily big pseudocompact groups in $\mathcal{C}C$; however all these groups did not contain non-trivial convergent sequences. For every infinite cardinal $\kappa$ satisfying $\kappa^\omega = \kappa$, we construct here a pseudocompact Abelian group $G \in \mathcal{C}C$ of cardinality $\kappa$ which contains non-trivial convergent sequences.

We show, however, that all countably pseudocompact groups as well as all countably pracompact groups in the class $\mathcal{C}C$ are finite.

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Further, we show in Theorem 2.6 that all countably compact subsets of any group in \(\mathbb{C}\mathbb{C}\) are countable and compact. In other words, an arbitrary group in \(\mathbb{C}\mathbb{C}\) reminds the group \(\mathbb{Q}_{\mathbb{P}\mathbb{R}}\) in this respect.

The case of pseudocompact Abelian groups is even more interesting. It is shown in [17] that for every infinite cardinal \(\kappa\), there exists a pseudocompact Boolean group \(G\) of cardinality \(2^\kappa\) such that every subgroup \(H\) of \(G\) with \(|H| \leq \kappa\) is closed in \(G\). Since \(G\) is Boolean, Theorem 2.1 implies that such a group \(G\) does not contain infinite compact subsets. It is natural to ask, therefore, whether a pseudocompact Abelian group \(G\) can contain non-trivial convergent sequences provided all countable subgroups of \(G\) are closed. We answer this question in Theorem 2.8 in the affirmative and show that there exist arbitrarily big pseudocompact Abelian groups \(G \in \mathbb{C}\mathbb{C}\) that contain non-trivial convergent sequences. By Theorem 2.1, none of these groups can be torsion.

We also show in Proposition 2.9 and Theorem 2.10 that if a group \(G \in \mathbb{C}\mathbb{C}\) is either countably pseudocompact or countably pracompact (the corresponding definitions are given below), then it is finite.

It is clear that every Boolean if Abelian. For Abelian groups, we will use additive notation, except for the circle group \((\mathbb{R}, +)\). It is natural to ask whether the pseudocompact groups\(\mathbb{C}\mathbb{C}\) are closed. In the case of a bounded torsion group \(G\), Theorem 2.6 implies that such a group \(G\) is Abelian (equivalently, \(nx = 0\_G\) if \(G\) is Abelian) for each \(x \in G\). The minimum integer \(n > 0\) with this property is called the period of the bounded torsion group \(G\).

The subgroup of \(G\) generated by a set \(A \subseteq G\) is \((A)\) and the cyclic subgroup of \(G\) generated by an element \(x \in G\) is \((x)\). Given an Abelian group \(G\) and a positive integer \(n\), we put \(G[n] = \{x \in G; nx = 0\_G\}\). It is clear that \(G[n]\) is a subgroup of \(G\).

A subset \(Y\) of an Abelian group \(G\) is independent if for any pairwise distinct elements \(y_1, \ldots, y_k \in Y\) and any integers \(n_1, \ldots, n_k\), the equality \(n_1y_1 + \cdots + n_ky_k = 0\_G\) implies that \(n_1y_1 = \cdots = n_ky_k = 0\_G\).

For a topological Abelian group \(G\), \(G^\wedge\) denotes the dual group of \(G\), i.e., the group of all continuous homomorphisms of \(G\) to the circle group \(\mathbb{T}\). The group \(G^\wedge\) carries the topology of uniform convergence on compact subsets of \(G\). The canonical evaluation homomorphism \(\alpha_G\) of \(G\) to the second dual group \(G^{\wedge\wedge} = (G^\wedge)^\wedge\) is defined by the rule

\[\alpha_G(\chi)(\xi) = \chi(\xi),\]

for all \(\xi \in G\) and \(\chi \in G^\wedge\). If \(\alpha_G\) is a topological isomorphism of \(G\) onto \(G^{\wedge\wedge}\), then the group \(G\) is called reflexive. Pontryagin-van Kampen's duality theorem states that every locally compact Abelian group is reflexive.

2. Countably compact sets in groups from the class \(\mathbb{C}\mathbb{C}\)

In the case of a bounded torsion group \(G \in \mathbb{C}\mathbb{C}\), the topological structure of compact sets in \(G\) is trivial—they are finite:

**Theorem 2.1.** Let \(G\) be a topological bounded torsion Abelian group such that all countable subgroups of \(G\) are closed. Then every countably compact subset of \(G\) is finite.

**Proof.** We divide the argument into three parts. Let us show first that \(G\) does not contain non-trivial convergent sequences. If not, we claim that there exists a non-trivial convergent sequence \(\{x_n; n \in \omega\} \subseteq G\) satisfying the following condition:

\[(*)\text{ for every finite subgroup } F \subseteq G, \ (x_n) \cap F = [e] \text{ for almost all } n \in \omega.\]

Indeed, denote by \(H\) a subgroup of \(G\) of the minimum possible period \(N\) such that \(H\) contains a non-trivial convergent sequence \(\xi = \{y_n; n \in \omega\}\). We can also assume that \(y_n \neq y_m\) if \(n \neq m\). Let us verify that \(\xi\) satisfies condition \((*)\).

If not, one can find an infinite set \(A \subseteq \omega\) such that for every \(n \in A\), there exist an integer \(k_n\) with \(0 < k_n < N\) and an element \(f_n \in F\) satisfying \(k_ny_n = f_n \neq e\), where \(e\) is the neutral element of \(G\). Since \(F\) is finite, there exist an infinite set \(B \subseteq A\), an integer \(k < N\), and \(f \in F\) such that \(k_n = k\) and \(kyn = f \neq e\) for each \(n \in B\). Let \(y^*\) be the limit point of the sequence \(\{y_n; n \in \omega\}\). Again, we can assume that \(y^* \neq y_n\) for each \(n \in \omega\). It follows from \(y_n \rightarrow y^*\) that \(kyn = f\). Pick any \(m \in B\) and put \(z_n = y_n - y_m\) for every \(n \in B\). Then \(kzn = e\) for each \(n \in B\), \(zn \rightarrow z^* = y^* - y_m \neq e\), and \(kz^* = k(y^* - y_m) = f - f = e\). It follows that \(\{z_n; n \in B\}\) is a non-trivial sequence lying in the subgroup \(G[k]\) of \(G\) and the period of \(G[k]\) is less than or equal to \(k\), where \(k < N\). This contradicts our choice of \(H\) and proves the claim.
Let \( \{x_n : n \in \omega \} \) be a non-trivial sequence in \( G \) converging to an element \( x^* \in G \) and satisfying \((*)\). Choosing a subsequence, if necessary, we can assume that \( e \neq x_0 \neq x_m \neq x^* \) if \( n \neq m \) and that all elements \( x_n \)'s have the same order, say \( p \).

We can also assume that \( x^* \neq e \). Indeed, if \( x^* = e \), pick any \( y^* \in G \setminus \{e\} \) with \( py^* = e \). We put \( y_n = x_n + y^* \) for each \( n \in \omega \). Then \( y_n \to y^* \). Let \( F \) be a finite subgroup of \( G \) and \( F' = F + \{y^*\} \). Since the group \( F' \) is finite, there exists a finite set \( C \subseteq \omega \) such that \( (y_n) \cap F' = \{e\} \), for each \( n \in \omega \setminus C \). If, however, \( (y_n) \cap F \neq \{e\} \) for some \( n \in \omega \setminus C \), then there are \( k \in \mathbb{Z} \) and \( f \in F \) such that \( kx_n + y^* = f \neq e \). It follows that \( kx_n = f - ky^* \in F' \). Since \( n \notin C \), we have that \( kx_n = e \) and, hence, \( f = ky^* \). Further, since \( p \) is the order of both \( x_n \) and \( y^* \), we conclude that \( k = ky^* = f \), which is a contradiction. We have thus shown that \( (y_n) \cap F = \{e\} \) for almost all \( n \in \omega \), i.e., the sequence \( y \) does not satisfy condition \((*)\).

Our second step is to show that the existence of the sequence \( y \) implies that there exists a countable subgroup \( H \) of \( G \) which contains infinitely many \( y_n \)'s, but \( y^* \notin H \). Since \( H \) is closed in \( G \), this will contradict the convergence \( y_n \to y^* \).

Indeed, let \( C = \{y^*\} \) and choose \( n_0 \in \omega \) such that \( (y_{n_0}) \cap C = \{e\} \). Put \( y_0 = y_{n_0} \). Suppose that for some \( n \in \omega \), we have defined pairwise distinct elements \( z_0, \ldots, z_n \in \eta \) such that \( (z_0, \ldots, z_n) \cap C = \{e\} \). Let \( F_n = C + (z_0, \ldots, z_n) \). Then \( F_n \) is a finite subgroup of \( G \), so there exists \( m \in \omega \) such that \( (y_m) \cap F_n = \{e\} \). We put \( z_{n+1} = y_m \) and claim that \( (z_0, \ldots, z_n, z_{n+1}) \cap C = \{e\} \). Indeed, otherwise one can find \( z \in (z_0, \ldots, z_n) \) and \( k, l \in \mathbb{Z} \) such that \( z + kz_{n+1} = ly^* - z \in F_n \). By the choice of \( y_m \), we have that \( ky_m = e \) and, hence, \( z = ly^* \neq e \). Therefore, \( (z_0, \ldots, z_n) \cap C = \{e\} \), which contradicts our inductive assumption. Note that \( z_{n+1} \neq z_i \), for each \( i \leq n \).

Once the sequence \( S = \{z_n : n \in \omega\} \) is constructed, we put \( H = \langle S \rangle \). It is clear from the construction that \( H \) is countable and contains infinitely many \( y_n \)'s. Since \( (z_0, \ldots, z_n) \cap C = \{e\} \) for each \( n \in \omega \), we conclude that \( H \cap C = \{e\} \). In particular, \( y^* \notin H \). It follows that \( H \) is not closed in \( G \), which is again a contradiction. This proves that \( G \) does not contain non-trivial convergent sequences.

Finally, we are in the position to show that every countably compact subset \( K \) of \( G \) is finite. Suppose to the contrary that \( K \) is infinite and take a countable infinite subset \( T \) of \( K \). Then \( H = \langle T \rangle \) is a countable (hence closed) subgroup of \( G \) and \( T \subseteq H \). Since \( K \) is countably compact and \( H \) is countable and closed, \( T^* = H \cap K \) is an infinite compact subset of \( H \). Clearly, the countable infinite compact space \( T^* \) contains non-trivial convergent sequences. This contradicts the fact that all convergent sequences in \( H \) are trivial and finishes the proof. \( \square \)

The following fact complements Theorem 2.1:

**Proposition 2.2.** If \( G \) is a pseudocompact Abelian group and all subgroups of \( G \) are closed, then \( G \) is finite.

**Proof.** Suppose to the contrary that \( G \) is infinite. It is well known that every infinite Abelian group contains a proper subgroup of countable infinite index [12]. Let \( N \) be such a subgroup of \( G \). Since \( N \) is closed in \( G \), the quotient group \( G/N \) is Hausdorff and pseudocompact. However, every countable Tychonoff pseudocompact space is compact. Thus, \( G/N \) is a countable infinite compact topological group, which is impossible since every infinite compact group has cardinality greater than or equal to \( 2^{\omega} \) (see [3, Theorem 9.11.2]). \( \square \)

We will show in Theorem 2.6 below that all (countably) compact subsets of a topological Abelian group in the class \( \mathcal{CC} \) are countable. The following lemma about independent subsets of the groups in the class \( \mathcal{CC} \) is a step towards the proof of this fact. It is worth mentioning that, according to [14, Lemma 1.4], every independent subset of an Abelian group \( H \) endowed with the Bohr topology is closed and discrete in \( H \). We deduce the same conclusion for independent subsets of an Abelian group \( H \in \mathcal{CC} \) under an additional assumption on \( p \)-primary components of the torsion part of \( H \).

First we recall that every torsion Abelian group \( H \) is a direct sum \( H = \bigoplus \mathbb{H}_p \), where \( \mathbb{H}_p \) is the set of prime numbers and for every \( p \in \mathbb{P} \), each element of the group \( H_p \) distinct from \( 0_H \) has order \( p^n \) for some integer \( n \geq 1 \) [16, Theorem 4.11]. The subgroups \( H_p \) of \( H \) are called \( p \)-primary components of \( H \).

**Lemma 2.3.** Let \( H \) be a topological Abelian group such that its torsion part \( t(H) \) has at most finitely many nontrivial \( p \)-primary components. If \( H \in \mathcal{CC} \), then every countably independent subset of \( H \) is closed and discrete.

**Proof.** We start with a simple but useful observation that will be applied several times in the argument that follows. If \( K \) and \( G \) are countable subgroups of \( H, K \subseteq G, \) and \( K \) is of a finite index in \( G \), then \( G \) can be covered by finitely many pairwise disjoint cosets of \( K \) in \( G \). Since \( |K| \leq |G| \leq \omega, K \) is closed in \( H \) and in \( G \). Therefore, each of these cosets is open in \( G \). We conclude that \( K \) is open in \( G \).

Let \( X = \{x_i : i \in \omega \} \) be a countable independent subset of \( H \), where \( x_i \neq x_j \) if \( i \neq j \). Since the subgroup of \( H \) generated by \( X \) is closed in \( H \), we can assume without loss of generality that \( H = \langle X \rangle \). Therefore, every element \( h \in H \setminus \{0_H \} \) can be written in the canonical form \( h = \sum_{i=0}^{m} n_i x_i \), where \( m \in \omega, n_i \in \mathbb{Z} \) for each \( i \leq m \), and \( n_m x_m \neq 0_H \) (it may happen that \( n_i = 0 \) for some \( i < m \)). We put

\[
L(h) = \sum_{i=0}^{m} n_i.
\]
For every $p \in \mathbb{P}$, let $X_p$ be the set of all elements of $X$ that have $p$-power orders. Let also $X_{\infty}$ be the part of $X$ consisting of elements of infinite order.

**Claim 1.** The set $X_{\infty}$ is closed in $H$.

Let $H_{\infty}$ be the subgroup of $H$ generated by $X_{\infty}$. Then $H_{\infty}$ is closed in $H$, so it suffices to verify that $X_{\infty}$ is closed in $H_{\infty}$. If $n$ is an integer with $n \geq 2$, then

$$H(n) = \{ g \in H_{\infty} : L(g) \text{ is a multiple of } n \}$$

is a subgroup of $H_{\infty}$ and $|H_{\infty}/H(n)| = n$. In fact, the restriction of $L$ to $H_{\infty}$ is a homomorphism of $H_{\infty}$ to $\mathbb{Z}$ which sends each $x \in X_{\infty}$ to 1, and $H(n) = L^{-1}(n\mathbb{Z})$. Hence, by the above observation, $H(n)$ is open in $H_{\infty}$. It is clear that $H(n) \cap X_{\infty} = \emptyset$, so the group $H(n)$ does not contain accumulation points of $X_{\infty}$. We have thus proved that if $h \in H_{\infty}$ satisfies $|L(h)| \neq 1$, then $h$ is not an accumulation point of $X_{\infty}$. In particular, $0_H$ is not in the closure of $X_{\infty}$.

Suppose therefore that $h \in H_{\infty}$ satisfies $|L(h)| = 1$. Let $h = \sum_{i=0}^m n_i x_i$ be a canonical form of $h$. Then $n_i$ is odd for some $s \leq m$, and we put

$$H^* = \left\{ \sum_{i=0}^n k_i x_i \in H_{\infty} : n \geq m, \; k_i \text{ is even} \right\}.$$

It is clear that $H^*$ is a subgroup of $H_{\infty}$ and $H_{\infty} = H^* \cup (H^* + x_i)$. Again, the cosets $H^*$ and $H^* + x_i$ are disjoint, so $H^*$ is an open and closed subgroup of $H$. It follows easily from the definition of $H^*$ that $h^* \in H^* + x_i$ and $(H^* + x_i) \cap X_{\infty} = \{x_i\}$. Thus $h^*$ cannot be an accumulation point of $Y$ either and we infer that $X_{\infty}$ is closed in $H_{\infty}$ and in $H$.

**Claim 2.** The set $X_p$ is closed in $H$ for each $p \in \mathbb{P}$.

Let $H_p$ be the subgroup of $H$ generated by $X_p$. Then $H_p$ is closed in $H$. As in the proof of Claim 1, it suffices to verify that $X_p$ is closed in $H_p$. Let $\varphi$ be a homomorphism of $H_p$ to $\mathbb{Z}/p\mathbb{Z}$ such that $\varphi(y) = 1$ for each $y \in X_p$. Then the kernel $K$ of $\varphi$ is an open subgroup of $H_p$ and $K \cap X_p = \emptyset$. Hence $K$ does not contain accumulation points of $X_p$. In particular, $0_H \notin X_p$.

Take an arbitrary element $h \in H_p \setminus X_p$, $h \neq 0$. Let $h = \sum_{i=0}^m n_i x_i$ be the canonical form of $h$. Then $n_m x_m \neq 0_H$ and the order of $x_m$ equals $p^k$ for some $k \in \mathbb{N}^+$. Clearly $p^k$ does not divide $n_m$. Consider the homomorphism $\varphi$ of $H_p$ to $\mathbb{Z}/p\mathbb{Z}$ that sends each $x_i$ with $i \neq m$ to 0 and $x_m$ to 1. Then $l = \varphi(h) \neq 0$ and $N = \varphi^{-1}(0)$ is a subgroup of index $p$ in $H_p$. Hence $N$ is open in $H_p$ and $U = N + h$ is an open neighborhood of $h$ in $H_p$. Since $X_p \setminus \{x_n\} \subseteq N$ and $l \neq 0$, we see that $U \cap X_p \subseteq \{x_m\}$. Hence $V = U \setminus \{x_m\}$ is an open neighborhood of $h$ in $H_p$ disjoint from $X_p$. Thus $X_p$ is closed in $H_p$ and in $H$. This proves Claim 2.

Finally, since $H$ has at most finitely many nontrivial $p$-primary components, $X_p = \emptyset$ for almost all $p \in \mathbb{P}$. It now follows from Claim 1 and Claim 2 that $X = X_{\infty} \cup \bigcup_{p \in \mathbb{P}} X_p$ is closed in $H$. In addition, if $x \in X$, then the subgroup $K_x$ of $H$ generated by the set $X \setminus \{x\}$ is closed in $H$ and does not contain $x$. Since $X \setminus \{x\} \subseteq K_x$, we see that $x \neq X \setminus \{x\}$. Therefore, the set $X$ is discrete. \[\square\]

The example below shows that the assumption on the torsion part of $H$ in Lemma 2.3 cannot be omitted:

**Example 2.4.** Let $H = \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_p$, where $\mathbb{P}$ is the set of prime numbers. Then the torsion group $H$ admits a Hausdorff topological group topology $\tau$ such that $(H, \tau) \in \mathcal{C}$ and $(H, \tau)$ contains a (countable) independent non-closed subset.

Indeed, let $\mathbb{P} = \{p_n : n \in \omega\}$ be a faithful enumeration of the prime numbers. Suppose that $\xi$ is a free ultrafilter on $\omega$. For every $A \in \xi$, let

$$K_A = \bigoplus_{n \in A} \mathbb{Z}_{p_n}.$$ 

It is easy to see that the family

$$\{K_A : A \in \xi\}$$

is a local base at $0_H$ for a Hausdorff topological group topology $\tau$ on $H$. Notice that each $K_A$ is a subgroup of $H$. We claim that $(H, \tau) \in \mathcal{C}$.

A direct verification shows that every subgroup $G$ of $H$ has the form $G = K_B = \bigoplus_{n \in B} \mathbb{Z}_{p_n}$, for a certain subset $B$ of $\omega$. Therefore, it suffices to verify that $K_B$ is closed in $(H, \tau)$ for every $B \subseteq \omega$. If $B \in \xi$, then $K_B$ is an open subgroup of $(H, \tau)$ and, hence, $K_B$ is closed. Suppose that $B \notin \xi$. Then $A = \omega \setminus B \in \xi$. Take an arbitrary element $h \in H \setminus K_B$. There exists a finite set $F \subseteq \omega$ such that $h \in K_F$. Put $A^* = A \setminus F$. Then $A^* \in \xi$ and $K_{A^*}$ is an open subgroup of $(H, \tau)$. It is clear that
Clearly, $X$ in infinite order. Since the sets $B \cup F$ and $A^*$ are disjoint, we see that $K_{B \cup F} \cap K_{A^*} = \{0_H\}$. Taking into account that $h \notin K_B$, we conclude that $(K_B - h) \cap K_{A^*} = \emptyset$ or, equivalently, $(h + K_{A^*}) \cap K_B = \emptyset$. Thus, $K_B$ is closed in $(H, \tau)$ and, hence, $(H, \tau) \in \mathcal{C}$. For every $p \in \mathbb{P}$, let $a_p$ be a generator of the subgroup $\mathbb{Z}_p$ of $H$. Then the set $Y = \{a_p; \ p \in \mathbb{P}\}$ is independent. However, every neighborhood $K_A$ (with $A \in \xi$) of $0_H$ in $(H, \tau)$ contains infinitely many elements $a_p$, i.e., the set $Y$ accumulates at $0_H$ in $(H, \tau)$. Therefore, $Y$ is not closed. \hfill \Box

It is easy to see that the group $(H, \tau)$ in Example 2.4 is not precompact. This suggests the following:

**Problem 2.5.** Let $G$ be a precompact (or even pseudocompact) group in $\mathcal{C}$. Is every countable independent subset of $G$ closed?

Let us turn back to the study of countably compact subsets of the groups in $\mathcal{C}$.

**Theorem 2.6.** Every countably compact subset of a group $G \in \mathcal{C}$ is countable and compact.

**Proof.** Suppose for a contradiction that $K$ is a countably compact uncountable subset of $G$. For every integer $n \geq 1$, let $G[n] = \{x \in G; \ nx = e\}$, where $e$ is the neutral element of $G$. It is clear that the subgroups $G[n]$ are closed in $G$. Hence Theorem 2.1 implies that the intersections $K \cap G[n]$ are finite. It follows that there exists a countable set $C \subseteq K$ such that every element of $K \setminus C$ has infinite order.

We claim that $K \setminus C$ contains an independent set $X$ of size $\aleph_1$. Indeed, let $X$ be a maximal independent subset of $K \setminus C$. Clearly, $X$ is non-empty since the set $\{x\}$ is independent for each $x \in K \setminus C$. Suppose that $X$ is countable. Then, for every $x \in K \setminus (X \cup C)$, there exist $n \in \mathbb{Z}$ and $y \in X$ such that $nx = y$. Clearly, $n \neq 0$. Since the set $K^* = K \setminus (X \cup C)$ is uncountable, we can find a non-zero integer $n^*$, an element $g^* \in X$, and an uncountable set $Y \subseteq K^*$ such that $n^*x = g^*$ for each $x \in Y$. Pick any $x^* \in K^*$. Then $n^*(x - x^*) = e$ for each $x \in K^*$, which means that the countably compact set $K - x^*$ has uncountable intersection with the closed subgroup $G[n^*]$ of $G$. The latter is impossible in view of Theorem 2.1, whence our claim follows.

We apply the above claim to choose an infinite independent set $X = \{x_n; \ n \in \omega\} \subseteq K \setminus C$. Let $H$ be the subgroup of $G$ generated by $X$. Then the torsion part of $H$ is trivial and, by Lemma 2.3, $X$ is closed and discrete in $H$. Since $H$ is a countable closed subgroup of $G$ and $X \subseteq H \cap K$, this contradicts the countable compactness of $K$.

We have thus proved that $K$ is countable. Finally, every countably compact countable space is compact. \hfill \Box

**Corollary 2.7.** Every countably compact group $G \in \mathcal{C}$ is finite.

**Proof.** By Theorem 2.6, the group $G$ is compact and countable. However, a compact group is either finite or uncountable, whence the required conclusion follows. \hfill \Box

Notice that Proposition 2.9 and Theorem 2.10 below extend the conclusion of Corollary 2.7 to wider classes of topological (not necessarily Abelian) groups.

In the following theorem we show that some groups in the class $\mathcal{C}$ can contain infinite compact subsets, even if they are pseudocompact.

**Theorem 2.8.** For every infinite cardinal $\kappa$ satisfying $\kappa^{\omega} = \kappa$, there exists a pseudocompact Abelian group $G$ of cardinality $\kappa$ such that all countable subgroups of $G$ are closed, but $G$ contains non-trivial convergent sequences. Furthermore, $G$ contains an infinite cyclic metrizable subgroup.

**Proof.** Let $\kappa$ be an infinite cardinal satisfying $\kappa^{\omega} = \kappa$. Let also $H = \mathbb{Z}(\kappa)$ be the direct sum of $\kappa$ copies of the group $\mathbb{Z}$, i.e., $H$ is the free Abelian group of cardinality $\kappa$. Denote by $X = \{x_\alpha; \ \alpha < \kappa\}$ a free algebraic basis for $H$.

Our strategy is to construct a family $\mathcal{H} = \{h_\alpha; \ \alpha < \kappa\}$ of homomorphisms of $H$ to the circle group $\mathbb{T}$ which separates elements of $H$ and, hence, introduces a precompact Hausdorff topological group topology $\tau$ in $H$. We have to guarantee, additionally, that the topological group $G = (H, \tau)$ will be pseudocompact, all countable subgroups of $G$ will be closed, and that $G$ will contain an infinite cyclic metrizable subgroup, say $C$. Since pseudocompact groups are precompact [8, Theorem 1.1], no infinite subgroup of $G$ will be discrete. Therefore, $C$ will contain non-trivial convergent sequences.

Since $|H| = \kappa = \kappa^{\omega}$, there exists an enumeration $\{(S_\alpha, y_\alpha); \ \alpha < \kappa\}$ of all pairs $(S, y)$, where $S$ is a countable subgroup of $H$ and $y \in H \setminus S$. At the step $\alpha < \kappa$ of our construction of the family $\mathcal{H}$, we will choose $h_\alpha$ to satisfy $h_\alpha(S_\alpha) = \{1\}$ and $h_\alpha(y) \neq 1$. Hence every countable subgroup of $G = (H, \tau)$ will be closed.
To guarantee the existence of an infinite cyclic metrizable subgroup of $G$, it suffices to construct $\mathcal{H}$ in such a way that the family $\{h_\alpha : C_0: \alpha < \kappa\}$ will be countable, where $C_0 = \{x_0\}$. In fact, each $h_\alpha(x_0)$ will be of finite order in $T$ and, hence, the sequence $[n!x_0 : n \in \mathbb{N}]$ will converge to $e$.

Finally, the pseudocompactness of $G$ will follows from the fact that the group $G$, when identified with a subgroup of $T^\omega$, fills all countable subproducts of $T^\omega$. To take care of this, we enumerate all elements of countable subproducts, say,

$$\bigcup \{T^A : A \subseteq \kappa, |A| \leq \omega\} = \{z_\alpha : 0 < \alpha < \kappa\}.$$  

This is possible since $\kappa^\omega = \kappa$. We can also assume that each $z$ in the left side set appears $\kappa$ times as $z_\alpha$. For every $z \in T^\omega$, we put $\supp(z) = A$. It is easy to see that the above enumeration can be chosen to satisfy $\supp(z_\alpha) \subseteq \alpha$, for each $\alpha \in \kappa \setminus \{0\}$.

To start, we put $D_{0,0} = S_0 + (x_0) + (y_0)$ and define a homomorphism $h_{0,0}$ of $D_{0,0}$ to $T$. First, we put $h_{0,0}(S_0) = \{1\}$. If $(y_0) \cap S_0 = \{e\}$, then we let $h_{0,0}(y_0) = 1$. Otherwise, let $k = \min(n \in \mathbb{N}^+: ny_0 \in S_0)$, choose $t \in T$ with $t \neq 1$ and $t^k = 1$, and put $h_{0,0}(y_0) = t$. This is possible since $y_0 \notin S_0$ and, hence, $k \geq 2$. By [3, Lemma 1.1.5], $h_{0,0}$ extends to a homomorphism of $S_0 + (y_0)$ to $T$ which is denoted by the $h_{0,0}$ as well. Notice that $h_{0,0}(S_0')$ is a finite subgroup of $T$. Similarly, $h_{0,0}$ extends to a homomorphism of $D_{0,0} = S_0' + (x_0)$ to $T$ such that the element $h_{0,0}(x_0)$ has finite order.

Suppose that for some $\alpha \in \kappa \setminus \{0\}$, we have defined a family $[h_{\beta,v}: \beta \leq v < \alpha]$, where every $h_{\beta,v}$ is a homomorphism of a subgroup $D_{\beta,v}$ of $H$ to $T$, such that the following conditions hold for all $\beta, \gamma, \mu, v < \alpha$:

1. $|D_{\beta+1}| \leq |\alpha| + \omega$;
2. $y_0 \in D_{\beta+1}$ if $\beta \leq \mu < v$;
3. $S_0 \cap (y_0) \subseteq D_{\beta+1}$;
4. $D_{\gamma,\mu} \subseteq D_{\beta,v}$ if $\gamma < \beta$;
5. $h_{\beta,v}(y_0) = h_{\beta,\mu}$ if $\beta \leq \mu < v$;
6. $h_{\beta,v}(S_0) = \{1\}$ and $h_{\beta,v}(y_0) \neq 1$;
7. $(h_{\beta,v}(x_0))$ is a finite subgroup of $T$;
8. if $\beta > 0$, there exists $g_{\beta} \in D_{\beta,v}$ such that $h_{\gamma,v}(g_{\beta}) = z_{\beta}(\gamma')$ for each $\gamma' \in \supp(y_\beta)$.

Here is a short explanation of conditions (i)-(viii). Clearly, (i) controls the size of the subgroups $D_{\beta,v}$, while (ii) and (v) imply that there will be a homomorphism $h_{\beta,v}: H \to T$ extending each of the homomorphisms $h_{\beta,v}$ with $\beta \leq v < \kappa$. Condition (vi) says that $h_{\beta}$ separates $y_\beta$ and $S_\beta$, while (vii) guarantees that the family of restrictions $h_{\beta,v}: H_\beta$ with $\beta < \kappa$ will be countable, where $H_\beta = \{C_\beta\}$. Finally, (viii) implies that the subgroup $G = h(H)$ will fill all countable faces of the product group $T^\omega$, where $h$ is the diagonal product of the family $[h_{\beta,v}: \beta < \kappa]$.

It is clear that conditions (i)-(viii) hold true at the step 0. We have to define subgroups $D_{0,\alpha}$ of $H$, for each $\beta < \alpha$, extend homomorphisms $h_{\beta,v}$ over $D_{0,\alpha}$ whenever $\beta < v < \alpha$, and define a homomorphism $h_{\alpha,a}$ of $D_{0,a}$ to $T$. First, we put $E_{\beta,a} = \bigcup_{\beta < \alpha} D_{\beta,v}$ for every $\beta < \alpha$. It follows from (ii) that $E_{\beta,a}$ is a subgroup of $H$, while (i) implies that $|E_{\beta,a}| \leq |\alpha| + \omega$. Denote by $F_{\alpha}$ the subgroup of $H$ generated by the set $\bigcup_{\beta < \alpha} E_{\beta,a}$. Again, we have that $|F_{\alpha}| \leq |\alpha| + \omega$. Since $H$ is a free Abelian group of cardinality $\kappa > |\alpha|$, there exists an element $g_\alpha 

\text{in } H \text{ distinct from } e \text{ such that } (g_\alpha) \cap F = [e]$. We now put $D_{\alpha,a} = F_{\alpha} + S_{\alpha} + (\langle g_\alpha, y_\alpha \rangle)$ and $P = C_\alpha + S_{\alpha}$. Notice that $P$ is a subgroup of $D_{\alpha,a}$ since, by (ii), $x_0 \in F_{\alpha} \subseteq D_{\alpha,a} \geq S_{\alpha}$. To define a homomorphism $h_{\alpha,a}: D_{\alpha,a} \to T$, we consider two cases.

Case 1. $y_\alpha \in P$. Let $\pi : P \to H$ be the quotient homomorphism. Then $\pi(C_\alpha) = P/S_\alpha$ and, since $y_\alpha \notin S_\alpha$, we see that $\pi(y_\alpha)$ is a non-trivial subgroup of the cyclic group $P/S_\alpha$. Hence $\pi(y_\alpha) = m\pi(x_0)$ for some integer $m \neq 0$. Further, either $P/S_\alpha$ is infinite or $\pi(y_\alpha)$ has a finite order $k > 1$. Notice that in the latter case, $k$ does not divide $m$—otherwise $\pi(y_\alpha)$ would be the identity of $P/S_\alpha$. If $P/S_\alpha$ is infinite, choose $t \in T$ such that $t^m = 1$. If $P/S_\alpha$ is finite, i.e., $|P/S_\alpha| = k$, choose $t \in T$ such that $t^k = 1$ and $t^m \neq 1$. This is possible since $k$ does not divide $m$.

Let $p : P/S_\alpha \to T$ be a homomorphism such that $p(\pi(x_0)) = t$. Let $h = p \circ \pi$. Then $h$ is a homomorphism of $P$ to $T$, $h(S_\alpha) = \{1\}$, and $h(y_\alpha) = t^m \neq 1$. We denote by $h_{\alpha,a}$ an arbitrary extension of $h$ to a homomorphism of $D_{\alpha,a}$ to $T$. It is clear that either $h_{\alpha,a}(x_0)2m = h(x_0)^{2m} = t^{2m} = 1$ or $h_{\alpha,a}(x_0)^k = h(x_0)^k = t^k = 1$, so $h_{\alpha,a}$ satisfies (vii). It is also clear that $h_{\alpha,a}$ satisfies (vi) since $h_{\alpha,a}(y_\alpha)^{2m} \neq 1$.

Case 2. $y_\alpha \notin P$. Let $\pi : D_{\alpha,a} \to D_{\alpha,a}/P$ be the quotient homomorphism. Then $z = \pi(y_\alpha)$ is distinct from the neutral element of $D_{\alpha,a}/P$. Hence we can find an integer $m \geq 1$, an element $t \in T$ with $t \neq 1$ and $t^m = 1$, and a homomorphism $p : D_{\alpha,a}/P \to T$ such that $p(z) = t$. The homomorphism $h_{\alpha,a} = p \circ \pi$ satisfies (vi) and (vii) since $h_{\alpha,a}(y_\alpha)^{2m} \neq 1$ and $h_{\alpha,a}(x_0) = 1$.

For every $\beta < \kappa$, we put $h_{\beta,a} = h_{\beta,v} + (g_\alpha)$. It is clear that $|D_{\beta,a}| \leq |\alpha| + \omega$ for each $\beta < \alpha$. Let us define a homomorphism $h_{\beta,a} : D_{\beta,a} \to T$ extending the homomorphisms $h_{\beta,v}$ with $\beta < \kappa$. Since $(g_\alpha) \cap E_{\beta,a} = \{e\}$, it suffices to put $h_{\beta,a}(D_{\beta,v}) = h_{\beta,v}$ if $\beta < \kappa$. Then $h_{\beta,a}(g_\alpha) = z_{\beta}(\beta)$ provided that $\beta \in \supp(y_\beta)$—otherwise we put $h_{\beta,a}(g_\alpha) = 1$. A direct verification shows that the family of homomorphisms $[h_{\beta,a}: \beta < \kappa]$ satisfies conditions (v)-(viii). This completes our construction at the stage $\alpha$.

Given $\beta < \kappa$, we put $D_{\beta,a} = \bigcup_{\beta < \alpha} D_{\beta,v}$. It follows from (ii) that $D_{\beta,a}$ is a subgroup of $H$, so (v) implies that there exists a homomorphism $h_{\beta,a} : D_{\beta,a} \to T$ which coincides with $h_{\beta,v}$ for each $\nu$ satisfying $\beta < \nu < \kappa$. Since the group $T$ is divisible, each $h_{\beta,a}$ admits a homomorphic extension over $H$ which is again denoted by $h_{\beta,a}$.
Let \( h \) be the diagonal product of the family \([h_\beta: \beta < \kappa]\). Clearly, \( h \) is a homomorphism of \( H \) to \( \mathbb{T}^\kappa \). We claim that \( h \) is a monomorphism. Indeed, if \( y \in H \) and \( y \neq e \), there exists a countable subgroup \( S \) of \( H \) which does not contain \( y \). Hence \( (S, y) = (S_\beta, y_\beta) \) for some \( \beta < \kappa \). Since \( h_\beta \) and \( h_{\beta, \beta} \) coincide on \( S_\beta, y_\beta \), it follows from (vi) that \( h_\beta(y_\beta) \neq 1 \) and, therefore, \( h(y_\beta) \) is distinct from the neutral element of \( G = h(H) \). We conclude that \( h \) is an isomorphism of \( H \) onto \( G \).

We claim that the group \( G \), considered with the topology inherited from \( \mathbb{T}^\kappa \), is pseudocompact. By [3, Theorem 2.4.15], it suffices to verify that \( \pi_\kappa(G) = \mathbb{T}^\kappa \) for each countable set \( \kappa \subseteq \kappa \), where \( \pi_\kappa \) is the projection of \( \mathbb{T}^\kappa \) to \( \mathbb{T}^\kappa \). In what follows we will write \( \pi_\gamma \) in place of \( \pi_{\gamma \wedge \kappa} \) for \( \gamma < \kappa \). So, let \( A \) be a non-empty countable subset of \( \kappa \) and take any point \( \xi \in \mathbb{T}^\kappa \). Then \( \xi = z_\gamma \) for some \( \beta \in \kappa \setminus \{0\} \). Put \( v = h(\xi_\beta) \). Since \( \pi_\gamma(v) = h_\gamma(y) \) for each \( \gamma < \kappa \), it follows from (vii) that \( \pi_\gamma(v) = h_\gamma(\xi_\beta (\gamma)) = z(\gamma) \) for each \( \gamma \in A = \text{sup}(z_\beta) \subseteq \beta \). In other words, we have that \( \pi_\kappa(v) = \xi \). This proves that \( \pi_\kappa(G) = \mathbb{T}^\kappa \), so \( G \) is pseudocompact.

Similarly, it is easy to see that all countable subgroups of \( G \) are closed. Indeed, let \( T \) be a countable subgroup of \( G \) and \( z \in G \setminus T \). Take a countable subgroup \( S \) of \( H \) and an element \( y \in H \) such that \( h(S) = T \) and \( h(y) = z \). Then \( y \notin S \) since \( h(y) \) is an isomorphism. Hence there exists \( \beta < \kappa \) such that \( (S, y) = (S_\beta, y_\beta) \). It follows from (vi) and our definition of \( h \) that \( \pi_\beta(T) = h_\beta(S) = h_{\beta, \beta}(S_\beta) = \{1\} \) and \( \pi_\beta(z) = h_\beta(y_\beta) = h_{\beta, \beta}(y_\beta) \neq 1 \). Thus the projection \( \pi_\beta \) of \( G \) to \( T \) separates \( z \) from \( T \), so \( z \) is not in the closure of \( T \). This proves that \( T \) is closed in \( G \).

Let \( z^* = h(x_\beta) \in G \). Since \( h : H \to G \) is an isomorphism, \( C = \langle z^* \rangle \) is an infinite cyclic subgroup of \( G \). Clearly, \( C \) is closed in \( G \) since it is countable. It follows from (vii) that \( \pi_\kappa(z^*) = h_\kappa(x_0) = h_{\kappa, \kappa}(x_0) \) is an element of finite order in \( T \), for each \( \beta < \kappa \), and the set \( \{\pi_\beta(z^*): \beta < \kappa \} \) is countable. Hence the family of restrictions \( \{\pi_\beta | C: \beta < \kappa \} \) is also countable and the subgroup \( C \) of \( G \) is metrizable. This finishes the proof of the theorem. \( \square \)

It was mentioned in the introduction that for every infinite cardinal \( \kappa \), there exists a pseudocompact Boolean group \( G \) of cardinality \( 2^\kappa \) such that every subgroup \( H \) of \( G \) with \( |H| \leq \kappa \) is closed in \( G \). It turns out that several natural conditions stronger than pseudocompactness imposed on a group \( G \in \mathbb{C}_\mathbb{C} \) force \( G \) to be finite.

Following [15], we say that a space \( X \) is countably pseudocompact if for every countable set \( A \subset X \), there exists a countable set \( B \subset X \) such that \( A \subset B \) and the set \( B \) is pseudocompact. It is clear that countably pseudocompact spaces are pseudocompact, but not vice versa. Let us show that every countably pseudocompact group in \( \mathbb{C}_\mathbb{C} \) is finite:

**Proposition 2.9.** Let \( G \) be a countably pseudocompact group, not necessarily Abelian, such that every countable subgroup of \( G \) is closed. Then \( G \) is finite.

**Proof.** It is easy to see that the closure of every countable subset of \( G \) is countable and compact. Indeed, given a countable set \( A \subset G \), take a countable set \( B \subset G \) such that \( A \subset B \) and \( B \) is pseudocompact. Denote by \( H \) the subgroup of \( G \) generated by \( B \). Then \( H \) is a countable closed subgroup of \( G \), so \( A \subset B \subset \mathbb{C}_\mathbb{C} \subset H \). It follows that \( B \) is a countable pseudocompact space, so it must be compact. Since \( A \subset B \), we conclude that the closure of \( A \) is countable and compact as well.

If \( G \) is infinite, take a countable infinite subgroup \( C \) of \( G \). Then \( C \) is closed in \( G \). We already know that the closure of \( C \) in \( G \), i.e., the same group \( C \) is compact. This is, however, impossible since every infinite compact group has cardinality at least \( 2^\omega \). Therefore \( G \) must be finite. \( \square \)

We recall that a space \( X \) is said to be countably pracom pact (see [2, Ch. III, Sec. 4]) if \( X \) contains a dense set \( Y \) such that every infinite subset of \( Y \) has an accumulation point in \( X \). It is clear that countably compact spaces are countably pracom pact, while every countably pracom pact space is pseudocompact. Let us show that, similarly to Proposition 2.9, countably pracom pact groups in \( \mathbb{C}_\mathbb{C} \) are finite. Our argument requires several modifications in the proof of Proposition 2.9.

**Theorem 2.10.** Let \( G \) be a countably pracom pact topological group such that all countable subgroups of \( G \) are closed. Then \( G \) is finite.

**Proof.** Let \( Y \) be a dense subset of \( G \) witnessing that \( G \) is countably pracom pact. We claim that the closure in \( G \) of every countable subset of \( Y \) is compact and countable. Indeed, take a countable set \( C \subset Y \). Then the subgroup \( H \) of \( G \) generated by \( C \) is countable and hence closed in \( G \). Since \( C \subset H \), the closure of \( C \) in \( G \), say, \( F \) is contained in \( H \). In particular, \( F \) is countable. Evidently, \( C \) is dense in \( F \) and every infinite subset of \( C \) has an accumulation point in \( F \), so \( F \) is countably pracom pact. We see that \( F \) is a countable pseudocompact space and therefore \( F \) is compact.

Suppose to the contrary that \( G \) is infinite. The group \( G \), being countably pracom pact, must be pseudocompact and hence pracom pact. In particular, the set \( Y \) is uncountable—otherwise \( G \) would be compact and countable as the closure of the countable set \( Y \), while infinite compact groups have cardinality at least \( 2^\omega \).

Let us show that \( Y \) contains an infinite subset \( X = \{x_n: n \in \omega\} \) without accumulation points in \( G \). Take an arbitrary element \( x_0 \in Y \) and put \( H_0 = \langle x_0 \rangle \). Then \( H_0 \) is a countable closed subgroup of \( G \). Since \( Y \) is uncountable and \( H_0 \) is closed in \( G \), we can find a point \( y \in Y \) and a symmetric open neighborhood \( U_0 \) of the neutral element \( e \) in \( G \) such that \( yU_0 \cap H_0 = \emptyset \). Then \( yU_0 \cap H_0U_0 = \emptyset \), so \( H_0U_0 \) is not dense in \( G \).

Suppose that for some \( n \in \omega \), we have defined points \( x_0, \ldots, x_n \in Y \), countable subgroups \( H_0, \ldots, H_n \) of \( G \), and open neighborhoods \( U_0, \ldots, U_n \) of \( e \) in \( G \) satisfying the following conditions for all \( i, j \in n \):

- \( x_i \notin U_j \) for all \( i < j \).
- The group \( H_{ij} \) generated by \( x_i \) and \( x_j \) is compact.
- The subgroup \( H_i \) generated by \( x_0, \ldots, x_i \) is countable.
- The subgroup \( H_{ij} \) generated by \( x_i \) and \( x_j \) is countable.
(i) \( x_i \in H_i \cap Y \);
(ii) \( x_j \notin H_i U_i \) whenever \( i < j \);
(iii) \( H_j \subseteq H_i \) if \( i \leq j \);
(iv) \( \bigcup_{i \leq n} H_i U_i \) is not dense in \( G \).

Let \( O = G \setminus \bigcup_{i \leq n} H_i U_i \) (the closure is taken in \( G \)). By (iv), the open set \( O \) is not empty. We claim that \( O \) is uncountable.

Indeed, since the group \( G \) is precompact, it can be covered by finitely many translates of the set \( O \). This implies that \(|O| = |G| \geq |Y| > \omega \).

Take a point \( x_{n+1} \in O \cap Y \) and denote by \( H_{n+1} \) the subgroup of \( G \) generated by the set \( H_n \cup \{ x_{n+1} \} \). Then \( H_{n+1} \) is a countable closed subgroup of \( G \), so there exists a point \( z \in O \setminus H_{n+1} \). Let \( U_{n+1} \) be a symmetric open neighborhood of \( e \) in \( G \) such that \( zU_{n+1} \cap H_{n+1} = \emptyset \). Then \( zU_{n+1} \cap H_{n+1} U_{n+1} = \emptyset \), so the definition of \( O \) implies that the open neighborhood \( O \cap zU_{n+1} \) of \( z \) is disjoint from \( \bigcup_{i=0}^{n+1} H_i U_i \). It is easy to see that the sets \( \{ x_i : i \leq n+1 \}, \{ H_i : i \leq n+1 \}, \) and \( \{ U_i : i \leq n+1 \} \) satisfy (i)–(iv) at the step \( n+1 \). This finishes our construction.

Suppose that we have defined the families \( X = \{ x_i : i \in \omega \}, \{ H_i : i \in \omega \}, \) and \( \{ U_i : i \in \omega \} \) satisfying (i)–(iv). Let us show that the infinite set \( X \subseteq Y \) is closed and discrete in \( G \). If not, \( X \) has an accumulation point \( x \in G \). Since the subgroup \( H \) of \( G \) generated by \( X \) is countable and closed in \( G \), we see that \( x \in H \). It follows from (i) that \( X \subseteq \bigcup_{i \in \omega} H_i \), which in its turn implies that \( x \neq H \subseteq \bigcup_{i \in \omega} H_i \). Take \( i \in \omega \) such that \( x \in H_i \). Then \( H_i U_i \) is an open neighborhood of \( x \) and (ii) implies that \( x_0 \notin H_i U_i \) for each \( n > i \). Hence the intersection \( X \cap H_i U_i \) is finite, i.e., \( X \) cannot accumulate at \( x \). This proves that the subset \( X \subseteq Y \) is closed and discrete in \( G \). However, the latter fact contradicts our choice of the set \( Y \). Thus the group \( G \) must be finite. □

Remark 2.11. The referee raised the natural question of whether there exist relations between the classes \( \mathcal{EC} \) and \( \mathcal{ECF} \), where the latter one consists of the topological Abelian groups whose countably compact subsets are finite. We know, after Theorem 2.1, that all bounded torsion groups from \( \mathcal{EC} \) are in \( \mathcal{ECF} \), but Theorem 2.8 shows that there exist pseudocompact groups in \( \mathcal{EC} \setminus \mathcal{ECF} \). Let us show that the inclusion \( \mathcal{ECF} \subseteq \mathcal{EC} \) is not valid either.

Take an ultrafilter \( \xi \in \beta \omega \setminus \omega \) and let \( X = \omega \cup \{ \xi \} \) be a subspace of \( \beta \omega \). Denote by \( G \) the free Abelian topological group \( A(X) \) on the space \( X \). Then \( G \) is countable, all precompact subsets of \( G \) (i.e., the subsets that can be covered by finitely many translates of any neighborhood of the neutral element in \( G \)) are finite, but \( G \) contains nonclosed subgroups. Since countably compact subsets of a topological group are precompact, this will imply that \( G \in \mathcal{ECF} \setminus \mathcal{EC} \). The fact that \( G \) contains nonclosed subgroups is easy—it suffices to take the subgroup \( D \) of \( G \) generated by the dense subset \( \omega \) of \( X \). Clearly \( D \) is a proper dense subgroup of \( G \).

Suppose that \( C \) is an infinite precompact subset of \( G = A(X) \). Then, by [3, Lemma 7.5.2], there exists a bounded subset \( Y \) of \( X \) and an integer \( n \geq 1 \) such that \( C \subseteq (Y)_n \), where

\[
(Y)_n = \{ \pm y_1 \pm y_2 \pm \cdots \pm y_k : y_1, y_2, \ldots, y_k \in Y, k \leq n \}.
\]

As usual, boundedness of \( Y \) in \( X \) means that every continuous real-valued function on \( X \) is bounded on \( Y \). Clearly, all bounded subsets of \( X \) are finite, so the sets \( (Y)_n \) and \( C \) are finite as well. Therefore, \( G \in \mathcal{ECF} \setminus \mathcal{EC} \).

We do not know, however, if there exist pseudocompact groups in \( \mathcal{ECF} \setminus \mathcal{EC} \).

3. Reflexive groups in the class \( \mathcal{EC} \)

According to Theorem 2.8 in [1], every pseudocompact Abelian group \( G \) is reflexive provided that all compact subsets of \( G \) are finite. Our Theorem 2.6 implies that all compact subsets of a group in \( \mathcal{EC} \) are at most countable. The latter fact suggests the question of whether pseudocompact groups in \( \mathcal{EC} \) are reflexive. First we answer this question in the positive for pseudocompact torsion groups:

Proposition 3.1. Let \( G \) be a pseudocompact Abelian torsion group such that all countable subgroups of \( G \) are closed. Then \( G \) is reflexive.

Proof. It follows from [3, Theorem 9.11.5] (see also [7,11]) that a pseudocompact Abelian torsion group is bounded torsion. So Theorem 2.1 implies that all compact subsets of \( G \) are finite. The reflexivity of \( G \) is now immediate from [1, Theorem 2.8]. □

It turns out that one cannot drop ‘torsion’ in the above proposition. We will show that the groups constructed in Theorem 2.8 are not reflexive, thus implying that the answer to our question is “no”. Our argument makes use of the following lemma from [5]. We reproduce its proof here for the sake of completeness. Notice that we use 1 to denote the neutral element of the circle group \( \mathbb{T} \).

Lemma 3.2. Let \( G \) be a precompact reflexive group. Then every closed metrizable subgroup of \( G \) is compact.
Proof. Let $M$ be a closed metrizable subgroup of $G$. Denote by $\varrho M$ the completion of $M$, i.e., $\varrho M$ is a compact group that contains $M$ as a dense topological subgroup. As a consequence of Außenhofer–Chasco’s theorem (see [4,6]) we have that $M^{\wedge\wedge} = \varrho M$.

If $M$ is not compact, there is $\Psi \in M^{\wedge\wedge}$ with $\Psi \notin \alpha_M(M)$. Define a homomorphism $\psi : G^{\wedge} \to T$ by $\psi(\chi) = \Psi(\chi |_M)$ for each $\chi \in G^{\wedge}$. Then $\psi$ is continuous and hence $\psi \in G^{\wedge\wedge}$. Since $\alpha_G$ is surjective by hypothesis, there must be $g \in G$ with $\psi = \alpha_G(g)$. As $M$ is closed in $G$, it follows that $g \in M$, for otherwise there is $\chi \in G^{\wedge}$ with $\chi |_M = 1$ and $\chi(g) \neq 1$ and, hence, $\chi(g) = \alpha_G(g)(\chi) = \psi(\chi) = \Psi(\chi |_M) = 1$. But $g \in M$ implies $\Psi = \alpha_M(g)$, against our choice of $\Psi$.

Theorem 3.3. Pseudocompact Abelian groups in the class $\mathcal{C}$ need not be reflexive.

Proof. By Theorem 2.8, there exists a pseudocompact Abelian group $G$ in $\mathcal{C}$ that contains a closed infinite cyclic metrizable subgroup, say $C$. If $G$ were reflexive, its closed metrizable subgroup $C$ would be compact by Lemma 3.2. But every infinite compact group has cardinality greater than or equal to $2^\omega$, while $C$ is countable. This contradiction shows that $G$ is not reflexive.

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