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The infinite families of optimal double loop networks*

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Let N, s be integers, $1 < s < N$. The *double loop network* (DLN) $G(N;s)$ is a digraph with N nodes $0, 1, \ldots, N-1$ and $2N$ arcs $\{i \to i+1, i+s: i = 0, 1, \ldots, N-1\}$, where nodes are always represented by residues modulo N. DLN's have been widely studied lately as practical models in the design of local area networks and parallel processing architectures [1, 3, 5]. Let $d(N;s)$ be the diameter of $G(N;s)$, and let $d(N) = \min\{d(N;s): 1 < s < N\}$. The network $G(N;s)$ is said to be *optimal* if $d(N;s) = d(N)$. Wong and Coppersmith initiated the studies of finding optimal $G(N;s)$ for every $N \geq 4$ and they established a good lower bound $\mathrm{lb}(N) = \lfloor \sqrt{3N} \rfloor - 2$ for $d(N)$ [5]. A network $G(N;s)$ is said to be *tight optimal* if $d(N;s) = \mathrm{lb}(N)$. Obviously, a tight optimal DLN is certainly optimal but the converse is not true. For example, if $N = 3(t+1)^2$ then $\mathrm{lb}(N) = 3t + 1$ but $d(N) = d(N; 3t+5) = 3t + 2$ for every $t \geq 1$. In accordance with this situation, a network $G(N;s)$ is said to be *nearly tight optimal* if $d(N;s) = d(N) = \mathrm{lb}(N) + 1$. A lot of infinite families of tight optimal and only one infinite family of nearly tight optimal DLN's, namely $\{G(3(t+1)^2; 3t+2: t \geq 1\}$, have been found by several authors since 1987. However the union of all the known infinite families of optimal DLN's can not even include a DLN with N nodes for every $N \leq 50$ [1, 2, 4]. In this note we list explicitly 69 infinite families of tight optimal and 33 infinite families of nearly tight optimal DLN's such that for every N, $4 \leq N \leq 300$, at least one of our families contain an optimal $G(N;s)$ (see Tables 1, 2, 3).

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Table 1
The case $\mathrm{lb}(N) = 3t - 1$, $N = N(t)$

$N(t)$	$d(N)$	t	s	$N(t)$								
				$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$
$3t^2+1$	$3t-1$	$t=2e, e \geq 1$	$6e^2-3e+1$	–	13	–	49	–	109	–	193	–
$3t^2+t-8$	$3t-1$	$t \geq 9$	$3t+6$	–	–	–	–	–	–	–	–	244
$3t^2+t-7$	$3t-1$	$t=25e^2-10e-6, e \geq 1$	$(45e^2-15e-10)t-5e^2-4$	–	–	–	–	–	–	–	–	245
$3t^2+t-6$	$3t-1$	$t \geq 7$	$3t-3$	–	–	–	–	–	–	148	194	246
$3t+t-5$	$3t-1$	$t \geq 6$	$3t-3$	–	–	–	–	–	109	149	195	247
$3t^2+t-4$	$3t-1$	$t=3e+3, e \geq 1$	$18e^2+35e+15$	–	–	–	–	–	110	–	–	248
		$t=3e+2, e \geq 1$	$9e^2+10e+2$	–	–	–	–	76	–	–	196	–
		$t=2e+3, e \geq 1$	$6e^2+16e+10$	–	–	–	–	76	–	150	–	248
$3t^2+t-3$	$3t$	$t=4e+4, e \geq 1$	$3t+3$	–	–	–	–	–	–	–	197	–
	$3t-1$	$t=e^2+4e+1, e \geq 1$	$(3e+9)t-2e-7$	–	–	–	–	–	111	–	–	–
		$t=e^2+4e+2, e \geq 1$	$3t^2-3et+e^2-2$	–	–	–	–	–	–	151	–	–
		$t=9e^2+3e-3, e \geq 1$	$(18e^2+9e-5)t-3e^2-2$	–	–	–	–	–	–	–	–	249
$3t^2+t-2$	$3t-1$	$t \geq 3$	$3t+3$	–	–	28	50	78	112	152	198	250
$3t^2+t-1$	$3t-1$	$t \geq 2$	$3t+3$	–	13	29	51	79	113	153	199	251
$3t^2+t$	$3t-1$	$t \geq 1$	$3t$	4	14	30	52	80	114	154	200	252
$3t^2+2t-8$	$3t-1$	$t=5e, e \geq 1$	$30e^2+7e-2$	–	–	–	–	77	–	–	–	–
		$t=5e+1, e \geq 1$	$45e^2+27e$	–	–	–	–	–	112	–	–	–
		$t=5e+2, e \geq 1$	$15e^2+17e+4$	–	–	–	–	–	–	153	–	–
		$t=5e+4, e \geq 1$	$60e^2+107e+42$	–	–	–	–	–	–	–	–	253
$3t^2+2t-7$	$3t$	$t=2e+6, e \geq 1$	$60e^2+41e+64$	–	–	–	–	–	–	–	201	–
		$t=3e+6, e \geq 1$	$9e^2+35e+31$	–	–	–	–	–	–	–	–	254
$3t^2+2t-6$	$3t-1$	$t=2e+3, e \geq 1$	$3e^2+18e+21$	–	–	–	–	79	–	155	–	255
		$t=5e+3, e \geq 1$	$15e^2+17e+4$	–	–	–	–	–	–	–	202	–
$3t^2+2t-5$	$3t$	$t=2e+5, e \geq 1$	$3t-2$	–	–	–	–	–	–	156	–	256
	$3t-1$	$t=4e, +e \geq 1$	$36e^2+3e-3$	–	–	–	51	–	–	–	203	–
		$t=4e+2, e \geq 1$	$12e^2+11e+2$	–	–	–	–	–	115	–	–	–
$3t^2+2t-4$	$3t-1$	$t \geq 3$	$3t-2$	–	–	29	52	81	116	157	204	257
$3t^2+2t-3$	$3t$	$t \geq 4$	$3t-2$	–	–	–	53	82	117	158	205	258
$3t^2+2t-2$	$3t-1$	$t \geq 2$	$3t+4$	–	14	31	54	83	118	159	206	259
$3t^2+2t-1$	$3t$	$t=2e+1, e \geq 1$	$6e+4$	–	–	32	–	84	–	160	–	260
	$3t-1$	$t=2e, e \geq 1$	$6e^2-6e+4$	–	15	–	55	–	119	–	207	–
$3t^2+2t$	$3t-1$	$t \geq 1$	$3t$	5	16	33	56	85	120	161	208	261

Table 2
The case $\mathrm{lb}(N) = 3t$, $N = N(t)$

$N(t)$	$d(N)$	t	s	$N(t)$								
				$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$
$3t^2+2t+1$	$3t$	$t \ge 1$	$3t+1$	6	17	34	57	86	121	162	209	262
$3t^2+2t+2$	$3t$	$t = 8e^2-8e+3, e \ge 1$	$24e^3-36e^2+23e-5$	–	–	35	–	–	–	–	–	–
		$t = 8e^2+1, e \ge 1$	$96e^4-24e^3+56e^2-9e+10$	–	–	–	–	–	–	–	–	263
$3t^2+3t-6$	$3t$	$t = 2e+5, e \ge 1$	$6e^2+36e+53$	–	–	–	–	–	–	162	–	264
		$t = 3e+5, e \ge 1$	$18e^2+63e+52$	–	–	–	–	–	–	–	210	–
$3t^2+3t-5$	$3t+1$	$t = 25e^2-10e-6, e \ge 1$	$(45e^2-15e-9)t-4$	–	–	–	–	–	–	–	–	265
$3t^2+3t-4$	$3t+1$	$t = 2e+4, e \ge 1$	$12e^2+48e+40$	–	–	–	–	–	122	–	212	–
$3t^2+3t-3$	$3t$	$t \ge 4$	$3t+5$	–	–	–	57	87	123	165	213	267
$3t^2+3t-2$	$3t$	$t \ge 3$	$3t$	–	–	34	58	88	124	166	214	268
$3t^2+3t$	$3t$	$t \ge 1$	$3t+2$	6	18	36	60	90	126	168	216	270
$3t^2+3t+1$	$3t$	$t \ge 1$	$3t+2,$	7	19	37	61	91	127	169	217	271
$3t^2+3t+2$	$3t+1$	$t = 2e, e \ge 1$	$3t+2$	–	20	–	62	–	128	–	218	–
$3t^2+4t-14$	$3t+1$	$t = 9, 10, 13, \pm \ge 16$	$3t+9$	–	–	–	–	–	–	–	–	265
$3t^2+4t-13$	$3t$	$t \ge 7$	$3t+9$	–	–	–	–	–	–	162	211	266
$3t^2+4t-12$	$3t+1$	$t = 7, 8, 11, \pm \ge 14$	$3t^2+t-20$	–	–	–	–	–	–	163	212	–
$3t^2+4t-11$	$3t$	$t = 3e+4, e \ge 1$	$9e^2+31e+21$	–	–	–	–	–	–	164	–	–
$3t^2+4t-10$	$3t+1$	$t = 25e^2-10e-6, e \ge 1$	$(15e^2-9e-4)t+5e-1$	–	–	–	–	–	–	–	–	269
$3t^2+4t-9$	$3t$	$t \ge 5$	$3t-3$	–	–	–	–	86	123	166	215	270
$3t^2+4t-8$	$3t$	$t = 2e+3, e \ge 1$	$6e^2+25e+24$	–	–	–	–	87	–	167	–	271
$3t^2+4t-7$	$3t$	$t = 5e+4, e \ge 1$	$45e^2+87e+37$	–	–	–	–	–	–	–	–	272
	$3t+1$	$t = 20e-14, e \ge 1$	$300e^2+385e+123$	–	–	–	–	–	125	–	–	–
$3t^2+4t-6$	$3t+1$	$t = 2e+3, e \ge 1, \ge 3$	$6e^2+19e+9$	–	–	–	–	89	–	–	–	273
$3t^2+4t-5$	$3t$	$t = 2e+2, e \ge 1$	$6e^2+13e+6$	–	–	–	59	–	127	–	219	–
		$t = 3e+1, e \ge 1$	$18e^2+17e-1$	–	–	–	59	–	–	170	–	–
		$t = 3e+3, e \ge 1$	$9e^2+19e+7$	–	–	–	–	–	127	–	–	274
$3t^2+4t-4$	$3t+1$	$t = 2e+4, e \ge 1$	$6e^2+18$	–	–	–	–	–	128	–	220	–
	$3t$	$t = 4e+1, e \ge 1$	$12e^2+13e+3$	–	–	–	–	91	–	–	–	275
		$t = 4e+3, e \ge 1$	$36e^2+69e+30$	–	–	–	–	–	–	171	–	–
$3t^2+4t-3$	$3t$	$t \ge 2$	$3t+6$	–	17	36	61	92	129	172	221	276
$3t^2+4t-2$	$3t+1$	$t \ge 4$	$3t$	–	–	–	62	93	130	173	222	277
$3t^2+4t-1$	$3t$	$t \ge 1$	$3t$	6	19	38	63	94	131	174	223	278
$3t^2+4t$	$3t$	$t = 2e+1, e \ge 1$	$6e^2+7e+4$	–	–	39	–	95	–	175	–	279
	$3t+1$	$t = 2e, e \ge 1$	$6e$	–	20	–	64	–	132	–	224	–
$3t^2+4t+1$	$3t$	$t \ge 1$	$3t+3$	8	21	40	65	96	133	176	225	280

Table 3
The case $\mathrm{lb}(N) = 3t + 1$, $N = N(t)$

$N(t)$	$d(N)$	t	s	$N(t)$								
				$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$	$t = 7$	$t = 8$	$t = 9$
$3t^2 + 4t + 2$	$3t + 1$	$t \geq 1$	$3t + 3$	9	22	41	66	97	134	177	226	281
$3t^2 + 5t - 6$	$3t + 1$	$t \geq 8$	$3t - 2$	–	–	–	–	–	–	–	226	282
$3t^2 + 5t - 4$	$3t + 1$	$t \geq 6$	$3t + 7$	–	–	–	–	–	134	178	228	284
$3t^2 + 5t - 3$	$3t + 1$	$t \geq 5$	$3t + 7$	–	–	–	–	97	135	179	229	285
$3t^2 + 5t - 2$	$3t + 1$	$t = 3e + 3, e \geq 1$	$18e^2 + 43e + 23$	–	–	–	–	–	136	–	–	286
		$t = 3e + 2, e \geq 1$	$9e^2 + 14e + 4$	–	–	–	–	98	–	–	230	–
		$t = 4e + 3, e \geq 1$	$36e^2 + 72e - 34$	–	–	–	–	–	–	180	–	–
$3t^2 + 5t$	$3t + 1$	$t \geq 2$	$3t + 1$	–	22	42	68	100	138	182	232	288
$3t^2 + 5t + 1$	$3t + 1$	$t \geq 1$	$3t + 4$	9	23	43	69	101	139	183	233	289
$3t^2 + 5t + 2$	$3t + 1$	$t \geq 1$	$3t + 4$	10	24	44	70	102	140	184	234	290
$3t^2 + 6t - 14$	$3t + 2$	$t = 24e^2 - 8e - 7, e \geq 1$	$(18e^2 - 6e - 4)t - 12e^2 + e$	–	–	–	–	–	–	–	–	283
$3t^2 + 6t - 13$	$3t + 1$	$t = 4e, e \geq 1$	$12e^2 + 3e - 2,$	–	–	–	–	–	–	–	277	–
$3t^2 + 6t - 10$	$3t + 1$	$t = 2e + 5, t \geq 1$	$6e^2 + 39e + 60$	–	–	–	–	–	–	179	–	287
$3t^2 + 6t - 9$	$3t + 1$	$t = 6e + 2, e \geq 1$	$90e^2 + 93e + 14$	–	–	–	–	–	–	–	231	–
$3t^2 + 6t - 8$	$3t + 2$	$t = 2e + 5, e = 1, e \geq 3$	$6e^2 + 39e + 55$	–	–	–	–	–	–	181	–	–
$3t^2 + 6t - 7$	$3t + 2$	$t = 6e^2 + 5e - 5, e \geq 1$	$3t^2 + 3t - 6e - 13$	–	–	–	–	–	137	–	–	–
$3t^2 + 6t - 6$	$3t + 1$	$t = 3e + 3, e \geq 1$	$18e^2 + 51e + 32$	–	–	–	–	–	138	–	–	291
		$t = 3e + 1, e \geq 1$	$9e^2 + 15e + 5$	–	–	–	66	–	–	183	–	–
	$3t + 2$	$t = 18e^2 - 12e - 1, e \geq 1$	$(36e^3 - 13e)t + 6e^2 - 10e - 2$	–	–	–	–	99	–	–	–	–
$3t^2 + 6t - 5$	$3t + 2$	$t = 4, t \geq 8$	$3t + 8$	–	–	–	67	–	–	–	235	292
$3t^2 + 6t - 4$	$3t + 1$	$t \geq 3$	$3t + 8$	–	–	41	68	101	140	185	236	293
$3t^2 + 6t - 3$	$3t + 2$	$t \geq 6$	$3t^2 + 3t - 1$	–	–	–	–	–	141	186	237	294
$3t^2 + 6t - 2$	$3t + 2$	$t = 6e^2 + 2e - 3, e \geq 1$	$(9e^2 + 3e - 3)t - 3e - 3$	–	–	–	–	103	–	–	–	–
		$t = 12e^2 - 4e - 2, e \geq 1$	$2t^2 + 5t + 2e$	–	–	–	–	–	142	–	–	–
		$t = 8e^2 - 1, e \geq 1$	$(12e^2 + 3e + 1)t + 4e^2 - 6e - 1$	–	–	–	–	–	–	187	–	–
		$t = 4e^2 + 6e - 2, e \geq 1$	$(6e^2 + 12e)t + 6e - 2$	–	–	–	–	–	–	–	238	–
		$t = 24e^2 - 16e + 1, e \geq 1$	$(54e^2 + 36e + 6)t + 3e - 3$	–	–	–	–	–	–	–	–	295
$3t^2 + 6t - 1$	$3t + 2$	$t = 6e^2 + e - 2, e \geq 1$	$2t^2 + 5t - 2e + 1$	–	–	–	–	–	104	–	–	–
		$t = 6e^2 + 2e - 1, e \geq 1$	$(18e^2 + 4)t - 12e + 2$	–	–	–	–	–	–	188	–	–
		$t = 6e^2 + 5e - 2, e \geq 1$	$3t + 6e + 7$	–	–	–	–	–	–	–	–	296
	$3t + 1$	$t = 2e + 2, e \geq 1$	$6e^2 + 21e + 15$	–	–	–	71	–	143	–	239	–
$3t^2 + 6t$	$3t + 1$	$t = 3e + 2, e \geq 1$	$9e^2 + 9e + 2$	–	–	45	72	–	144	189	–	297
		$t = 3e + 2, e \geq 1$	$9e^2 + 19e + 5$	–	–	–	–	105	–	–	240	–
$3t^2 + 6t + 1$	$3t + 2$	$t \geq 2$	$3t + 5$	–	25	46	73	106	145	190	241	298
$3t^2 + 6t + 2$	$3t + 1$	$t \geq 1$	$3t + 5$	11	26	47	74	107	146	191	242	299
$3t^2 + 6t + 3$	$3t + 2$	$t \geq 1$	$3t + 5$	12	27	48	75	108	147	192	243	300

Note that there are 72 N's with $d(N) = \text{lb}(N) + 1$ and only 9 of them are covered by the above-mentioned infinite family of nearly tight optimal DLN's.

The following are the basic ideas of our approach:

(1) Use a geometrical consideration on a class of plane figures called L-shape tiles, and represent an L-shape tile by some indeterminate parameters.

(2) Give a simple characterization of an L-shape tile which can be implemented by a DLN. In fact, we show that the sufficient condition given in [5, Theorem 4] is also necessary.

(3) For a given $N = N_1 \geq 4$, if $d(N_1) = \text{lb}(N_1) + 1$ (respectively $d(N_1) = \text{lb}(N_1)$) and there is a nearly tight (respectively tight) optimal $G(N_1; s_1)$, then we try to generate an infinite family of nearly tight (respectively tight) optimal DLN's by careful consideration, and this family contains $G(N_1; s_1)$. Our strategy works pretty well, at least for $N_1 \leq 300$.

The details of our approach will appear elsewhere.

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