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On the Connectivity Properties of the Solution Set of Parametrized Families of Compact Vector Fields

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INTRODUCTION

A useful tool in the study of global continua of solutions of nonlinear partial differential equations is a connectivity result on the fixed-points set of a 1-parameter family of maps that goes back to Leray and Schauder [17] and it was proved in its full generality by Browder [8]. In describing it, we shall restrict ourselves to the case of parametrized families of compact vector fields defined in open subsets of Banach spaces.

Let X be a Banach space, I be the unit interval $[-1, 1] \subset \mathbb{R}$ and U be an open subset of $I \times X$. Let $\bar{f}\colon \bar{U} \to X$ be a compact map such that

$$\begin{aligned} x \neq \bar{f}(\lambda, x) \qquad & \textit{for} \quad (\lambda, x) \in \partial U \\ \deg(\mathrm{Id} - \bar{f}(\lambda, \cdot), U_\lambda, 0) \neq 0 \qquad & \textit{for some } \lambda \in [-1, 1]. \end{aligned} \tag{*}$$

Then

(i) *there exists a connected subset $\mathscr{C}$ of the solution set $\mathscr{S} = \{(\lambda, x) \in U\colon x - \bar{f}(\lambda, x) = 0\}$ joining $\mathscr{S}_{-1} = \mathscr{S} \cap \{-1\} \times X$ with $\mathscr{S}_1 = \mathscr{S} \cap \{1\} \times X$.*

Actually, by a topological argument, statement (i) is equivalent to either one of the following assertions:

(ii) *$\mathscr{S}_{-1}$ cannot be separated from $\mathscr{S}_1$ in $\mathscr{S}$.*

(iii) *If $g\colon \mathscr{S} \to \mathbb{R}$ is a real valued continuous map such that $g(\mathscr{S}_{-1}) \subset \mathbb{R}_-$ and $g(\mathscr{S}_1) \subset \mathbb{R}_+$ then $g(\lambda, x) = 0$ for some $(\lambda, x) \in \mathscr{S}$.*

The equivalence between (ii) and (iii) is straightforward, whereas the equivalence between (i) and (ii) is known as Whyburn's lemma.

Let us remark that, starting from the celebrated Rabinowitz result, the above statements in the (i) and (ii) form have been used by several people in the study of global continua of solutions for nonlinear P.D.E. (cf. [5, 24]).

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Instead, the (iii) form was used in [4, 22] to obtain existence results for nonlinear P.D.E. arising as compact perturbations of Fredholm operators with one dimensional kernel.

In this note, under assumption (*), we shall give an improvement of the statements (i)–(iii) for n-parameter families of compact vector fields. Actually, (ii) and (iii) have a word-by-word reformulation to the n-parameter case and they are related with some classical results on fixed points and dimension theory (cf. Remarks to Theorem (1.1)). In order to give a nontrivial improvement of (i) we shall describe it in terms of Čech cohomology. Indeed, for a family of compact vector fields with parameters in the n-interval $I = [-1, 1]^n \subset \mathbb{R}^n$, under assumption (*), we have that *there exists a closed connected set* $\mathscr{C}$ *of solutions such that the homomorphism induced in cohomology by the projection* $p\colon \mathscr{C} \to I$

$$p^* : \check{H}^n(I, \partial I) \to \check{H}^n(\mathscr{C}, \mathscr{C} \cap p^{-1}(\partial I))$$

is nontrivial.

In particular, we obtain that $\mathscr{C}$ covers all of I and $\dot{\mathscr{C}} = \mathscr{C} \cap p^{-1}(\partial I)$ covers all of ∂I. Moreover, the Lebesgue covering dimension of $\mathscr{C}$ and $\dot{\mathscr{C}}$ is at least n and $n-1$, respectively (cf. Theorem 1.1). Notice that the above assertion reduces to [17, 8] result for $n = 1$.

In Theorem 1.2 we extend the above result to families of compact vector fields parametrized by whole of $\mathbb{R}^n$. In this case, if there are bounds for the solutions lying over compact subsets of $\mathbb{R}^n$ then (*) implies that there exists a closed connected set of solutions of dimension $\geqslant n$, covering all of $\mathbb{R}^n$.

The paper is organized as follows: In Section 1, Theorems 1.1 and 1.2 are stated and some consequences are pointed out. Section 2 is devoted completely to the proofs of the above theorems. Propositions 2.2 and 2.3 are of independent interest.

In Section 3, in order to illustrate the significance of Theorem 1.2, we shall give two examples arising from semilinear boundary value problems at resonance. We would like to add in passing that the conclusions in Theorems 1.1 and 1.2 applied to the first example answer a question raised in [22].

We would like to point out that although our methods are inspired by the Alexander and Yorke paper [3], our results are of a different nature. More precisely, in the main theorem of [3], from only local assumptions on the map f defined on an open set homeomorphic to $\mathbb{R} \times X$, they show the existence of a homologically nontrivial continuum of solutions which does not necessarily cover all of $\mathbb{R}$. On the contrary our assumptions are not of local character, but no restrictions on the topological type of the domain of f are needed. By imposing bounds on the solutions lying over compact subsets of $\mathbb{R}$ we obtain a connected set of solutions covering all of the parameter space.

We would also remark that our conclusions and more general results can be obtained either by using the transfer homomorphism of Dold [10] or by techniques developed in [18]. However the methods in this paper are (in our opinion) more elementary. In fact, the only tools needed are some basic properties of cohomology products and continuity of Čech theory.

Finally, we shall express our gratitude to Alexander for sending his preliminary reports [1, 2].

1. Preliminaries and Statement of the Main Results

Given any pair (X, A) of normal spaces, we shall denote by $\check{H}^n(X, A)$ its nth Čech-cohomology group with integer coefficients based on Alexander–Spanier cochains. This cohomology theory is useful for our purpose with regard to its tautness and its continuity property (for these topics, see [23, Chap. 6]). Given any cohomology class $\xi \in \check{H}^n(X, A)$ and any inclusion of pairs i: $(Y, B) \hookrightarrow (X, A)$, the cohomology class $i^*(\xi) \in \check{H}^n(Y, B)$ is called the *restriction* of ξ *to* (Y, B) and it is denoted by $\xi|_{(Y,B)}$.

We recall that a closed subset C of a topological space X *separates* two subsets A, B of X if $X \backslash C$ is a disjoint union of two open sets U, V containing A and B, respectively. We say that A, B are *separated* in X if the empty set separates them.

Given any n-tuple $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$ we denote by $|\lambda| = \max_{1 \leqslant i \leqslant n} |\lambda_i|$.

We consider $\mathbb{R}^n$ endowed with this norm. The unit ball of $\mathbb{R}^n$ is denoted by I and its boundary by $\dot{I}$. Given any subset A of $I \times X$, $\dot{A}$ stands for the part of A lying over the boundary of I, that is $\dot{A} = \{(\lambda, x) \in A : \lambda \in \dot{I}\}$ (this should not be confused with the boundary ∂A of A). Finally, we denote by A_λ the section of A at $\lambda \in I$, that is, $A_\lambda = \{x \in X$: $(\lambda, x) \in A\}$.

Let X be a Banach space and let U be an open (in the relative topology) bounded subset of $I \times X$. By a parametrized *family of compact vector fields* we mean a map $f: U \to X$ of the form $f(\lambda, x) = x - \bar{f}(\lambda, x)$, where $\bar{f}: U \to X$ is a compact map.

We assume, once and for all, that f is *admissible*, that is, f does not have zeroes on the boundary ∂U of U (in $I \times X$). In this case, the Leray–Schauder degree $\deg(f_\lambda, U_\lambda, 0)$ of the map

$$f_\lambda(\cdot) = f(\lambda, \cdot): \bar{U}_\lambda \to X$$

is defined and it is independent of $\lambda \in I$.

Let $\mathscr{S}$ be the *solution set* of the equation $f(\lambda, x) = 0$ in U. Then $\mathscr{S}$ is a compact subset of U, being the set of zeroes of a family of compact vector fields.

Let $p: I \times X \to I$ be the projection map. Then p induces various maps of pairs such as $p: (\bar{U}, \dot{\bar{U}}) \to (I, \dot{I})$, $p: (\mathscr{S}, \dot{\mathscr{S}}) \to (I, \dot{I})$ which we still denote with the same symbol p. For each $1 \leqslant i \leqslant n$, let $\mathscr{S}_i^{+-} = \{(\lambda, x) \in \mathscr{S} : \lambda_i = \pm 1\}$. Clearly,

$$\dot{\mathscr{S}} = \bigcup_{+-} \bigcup_{i} \mathscr{S}_i^{+-}.$$

Finally, let us consider the class $\xi = p^*(\tilde{e}_n) \in \check{H}^n(\bar{U}, \dot{\bar{U}})$, where $\tilde{e}_n \in \check{H}^n(I, \dot{I})$ in the canonical generator of $\check{H}^n(I, \dot{I}) \simeq \mathbb{Z}$.

Now we are able to formulate our main result.

THEOREM 1.1. *Let $f : \bar{U} \to X$ be a family of compact vector fields. Assume that f is admissible and that*

$$\deg(f_\lambda, U_\lambda, 0) \neq 0 \qquad \textit{for some} \quad \lambda \in I. \tag{1.1}$$

Then the following holds:

(a) *If C_i is any n-tuple of closed subsets of $\mathscr{S}$ such that C_i separates $\mathscr{S}_i^-$ from $\mathscr{S}_i^+$, then $\bigcap_{i=1}^n C_i \neq \varnothing$.*

(b) *If $g_i : \mathscr{S} \to \mathbb{R}$, $1 \leqslant i \leqslant n$, are continuous functions such that $g_i(\mathscr{S}_i^-) \subset \mathbb{R}^-$ and $g_i(\mathscr{S}_i^+) \subset \mathbb{R}^+$, then $\bigcap_{i=1}^n g_i^{-1}(0) \neq \varnothing$, i.e., the map $g = (g_1, \ldots, g_n): \mathscr{S} \to \mathbb{R}^n$ has a zero in $\mathscr{S}$.*

(c) *There exists a connected component $\mathscr{C}$* of $\mathscr{S}$ *such that the restriction of ξ to $(\mathscr{C}, \dot{\mathscr{C}})$ is nontrivial in $\check{H}^n(\mathscr{C}, \dot{\mathscr{C}})$.*

Now we would like to point out some consequences of Theorem 1.1 and make some comments. First of all, notice that the homomorphism $p^*: \check{H}^n(I, \dot{I}) \to \check{H}^n(\mathscr{C}, \dot{\mathscr{C}})$ in statement (c) is not the only nontrivial one induced by the projection. Indeed, from the naturality of the exact sequence of a pair it follows that also $p^*: \check{H}^{n-1}(\dot{I}) \to \check{H}^{n-1}(\dot{\mathscr{C}})$ is nontrivial. Hence,

$p: (\mathscr{C}, \dot{\mathscr{C}}) \to (I, \dot{I})$ *is essential* (i.e., it cannot be deformed in this class of maps to a map into $\dot{I}$). *In particular it is onto.* (1.2)

$p: \dot{\mathscr{C}} \to \dot{I}$ *is essential* (i.e., it cannot be deformed to a constant map). *In particular it is onto.* (1.3)

Moreover, since from the homological characterization of Lebesgue covering dimension of compact metric spaces [16, Theorem VIII.4] we have that if $\dim X < n$ then for any closed subset C of X, $\check{H}^n(X, C) = 0$. It follows from (c) in Theorem 1.1 that

$$\dim \mathscr{C} \geqslant n \qquad \text{and} \qquad \dim \dot{\mathscr{C}} \geqslant n - 1. \tag{1.4}$$

Let us make some remarks to the statements (a) and (b). Since for $n = 1$, (a) reduces to (ii) of the Introduction it can be regarded as a higher order connectivity property of the solution set. Such types of properties and this relation with dimension theory have been considered long ago by several topologists (a good reference for this topic is Chap. 5, Vol. II of Kuratowski). For example, a theorem due to Eilenberg and Otto [12] states that:

dim $X < n$ if for any n-tuple of closed disjoint pairs (A_i, B_i), $i = 1,..., n$, there exists closed sets $C_i, i = 1,..., n$, such that C_i separates A_i from B_i and $\bigcap_{i=1}^{n} C_i = \varnothing$.

Hence, (a) in Theorem 1.1 implies that dim $\mathscr{S} \geqslant n$. On the other hand, (b) can be regarded as an improvement to the solution set over n-intervals of a principle for the existence of zeroes of maps $g: I^n \to \mathbb{R}^n$ that goes back to Poincaré [20] and it was observed by Miranda [19] to be equivalent to Brower's theorem.

Finally, let us remark that if f is defined in the whole of $\mathbb{R}^n \times X$ then Theorem 1.1 can be applied over intervals I that do not intersect the set of bifurcation points from infinity (see [18]). Furthermore, by "patching" it can be extended to subsets more general than intervals.

In Theorem 1.2 we have applied the above results in order to study connectivity properties of the solution set of a family of compact vector fields parametrized by $\mathbb{R}^n$. Namely, let $U \subset \mathbb{R}^n \times X$ be an open set that is locally bounded over $\mathbb{R}^n$ (each $\lambda \in \mathbb{R}^n$ has a neighborhood V such that $\{(\lambda, x) \in U: \lambda \in V\}$ is bounded) and let $f: \bar{U} \to X$ be a parameterized family of compact vector fields without zeroes on the boundary of U. Since U is locally bounded, the solution set $\mathscr{S}$ of the equation $f(\lambda, x) = 0$ is locally compact and the restriction of $p: \mathbb{R}^n \times X \to \mathbb{R}^n$ to $\mathscr{S}$ is a proper map. If C is a closed subset of $\mathscr{S}$, we shall denote by $\check{H}_c^n(C)$ the nth Čech-cohomology group of C with compact supports. The map $p|_C$ being proper induces a cohomology homomorphism $p^*: \check{H}_c^n(\mathbb{R}^n) \to \check{H}_c^n(C)$. Let $e \in \check{H}_c^n(\mathbb{R}^n) \simeq \mathbb{Z}$ be a generator of this group.

THEOREM 1.2. *Let U be a locally bounded open subset of $\mathbb{R}^n \times X$, $f: \bar{U} \to X$ be a parametrized family of compact vector fields such that $o \notin f(\partial U)$. Assume that*

$$\deg(f_\lambda, U_\lambda, 0) \neq 0 \qquad \textit{for some } \lambda \in \mathbb{R}^n.$$

Then there exists a connected subset $\mathscr{C}$ of $\mathscr{S}$ such that

$$p^*(e) \in \check{H}_c^n(\mathscr{C}) \qquad \textit{is nontrivial.}$$

In particular, $p|_{\mathscr{C}}$ is essential as a proper map from $\mathscr{C}$ into $\mathbb{R}^n$, the projection of $\mathscr{C}$ covers all of $\mathbb{R}^n$ and the topological dimension of $\mathscr{C}$ is at least n.

Remark 1.3. According to Theorem 1.1, the topological dimension of $\mathscr{C}$ at each point is at least n.

Remark 1.4. When f is defined on the whole of $\mathbb{R}^n \times X$, then the above result can be applied provided that there are "a priori bounds" for the solution set $\mathscr{S}$ over compact subsets of $\mathbb{R}^n$ (i.e., if $K \subset \mathbb{R}^n$ is compact, then the set $\{(\lambda, x) \in \mathscr{S} : \lambda \in K\}$ is bounded).

Remark 1.5. Let $f : \bar{U} \to X$ be a parametrized family of compact vector fields as in Theorem 1.2 and let $g : \mathscr{S} \to \mathbb{R}^n$ be any continuous function such that for each $t \in [0, 1]$ the map $h_t(\lambda, x) = t\lambda + (1-t)\, g(\lambda, x)$ has the property

$$(\lambda_n, x_n) \in \mathscr{S}, |\lambda_n| \to +\infty, n \to +\infty$$

$$\text{implies } |h_t(\lambda_n, x_n)| \to +\infty \quad \text{as} \quad n \to +\infty \tag{1.5}$$

(or equivalently h_t is a proper map). Then the system

$$f(\lambda, x) = 0$$

$$g(\lambda, x) = 0$$

has at least one solution.

Note that (1.5) is verified if $g : \mathscr{S} \to R^n$ is such that

$$\langle g(\lambda, x), \lambda \rangle > 0 \qquad \text{for sufficiently large } \lambda.$$

Finally, we would like to add in passing that conditions similar to the above ones appear frequently in literature on solving equations defined as nonlinear perturbations of Fredholm operators (cf. [7, 9, 14]).

2. Theorem Proofs

Proof of Theorem 1.1. First of all, we show the equivalence between (a) and (b). Then we prove that, under assumption (1.1), (b) holds. Finally we prove (c) in several steps.

Proof of (a) $\Leftrightarrow$ (b). Clearly (a) implies (b) since $C_i = g_i^{-1}(0)$ separates $\mathscr{S}_i^-$ from $\mathscr{S}_i^+$. On the other hand, if C_i separates $\mathscr{S}_i^-$ from $\mathscr{S}_i^+$ then, using Urysohn's lemma, we can construct continuous functions $g_i : \mathscr{S} \to \mathbb{R}$ such that

$$g_{i|\mathscr{S}_i^-} = -1, \qquad g_{i|\mathscr{S}_i^+} = +1, \qquad \text{and} \qquad g_i^{-1}(0) = C_i.$$

Proof of (b). First of all, we extend the map $g = (g_1, ..., g_n) : \mathscr{S} \to \mathbb{R}^n$ to all of $\bar{U}$. This is possible since $\mathbb{R}^n$ is an absolute retract. We still denote by g such an extension. Now, let $V = p^{-1}(\mathring{I}) \cap U$, i.e., the part of U lying over the interior of I. Then V is an open subset of $\mathbb{R}^n \times X$, and its boundary

$$\partial V \text{ is contained in } \partial U \text{ (relative to } I \times X) \cup \dot{U}. \tag{2.1}$$

Let $G: \bar{V} \to \mathbb{R}^n \times X$ be the compact vector field defined by $G(\lambda, x) = (g(\lambda, x), f(\lambda, x))$. By the assumption in (b) and (2.1), we have that $G(\lambda, x) \neq 0$ for $(\lambda, x) \in \partial V$, hence $\deg(G, V, O)$ is defined.

Consider now the homotopy $H: \bar{V} \times [0, 1] \to \mathbb{R}^n \times X$ defined by

$$H((\lambda, x), t) = (t\lambda + (1-t)\, g(\lambda, x), f(\lambda, x)).$$

We shall prove that H is an admissible homotopy. In fact, $H((\lambda, x), t) \neq 0$ whenever $(\lambda, x) \in \partial U$ (relative to $I \times X$) and $t \in [0, 1]$ since in this case $f(\lambda, x) \neq 0$. Moreover, if $(\lambda, x) \in \dot{U}$ and $f(\lambda, x) = 0$ then (λ, x) belongs to some $\mathscr{S}_i^+$ or $\mathscr{S}_i^-$, but in this case $g_i(\lambda, x)$ has the same sign of λ_i and hence $t\lambda_i + (1-t)\, g_i(\lambda, x) \neq 0$ for all $t \in [0, 1]$. Therefore $H((\lambda, x), t) \neq 0$ on $\partial V x[0, 1]$. Hence

$$\deg(G, V, 0) = \deg(H_1, V, 0) \qquad \text{(by homotopy invariance)},$$

where $H_1(\lambda, x) = (\lambda, f(\lambda, x)) = (\lambda, x) - (0, \bar{f}(\lambda, x))$.

Since $\bar{h}(\lambda, x) = (0, \bar{f}(\lambda, x))$ sends $\bar{V}$ into $\{0\} \times X \subset \mathbb{R}^n \times X$ by the reducing property of degree we get that

$$\deg(H_1, V, 0) = \deg(H_1|_{\bar{V}'}, V', 0),$$

where

$$V' = V \cap \{0\} \times X.$$

But $H_1|_{\bar{V}'} = f_o$. Hence $\deg(G, V, 0) \neq 0$ by assumption (1.1). This implies that for some $(\lambda, x) \in V$ we have that $g(\lambda, x) = 0$ and $f(\lambda, x) = 0$. The latter means that $(\lambda, x) \in \mathscr{S}$. But the extension of g agrees with g on $\mathscr{S}$. Hence we have the desired result.

The proof of (c) will be derived in several steps. Roughly speaking, we first consider the case of finite dimensional X and we prove that $\xi_{|(\mathscr{S}, \dot{\mathscr{S}})} \neq 0$ by showing that its cup-product with some element induced by f is nontrivial. Then by an approximation argument, we will be able to establish $\xi_{|(\mathscr{S}, \dot{\mathscr{S}})} \neq 0$ in the case that X is any Banach space. The existence of a connected component $\mathscr{C}$ of $\mathscr{S}$ such that $\xi_{|(\mathscr{C}, \dot{\mathscr{C}})}$ will follow by a general argument. The proofs of the former results, although conceptually very simple, involve some technical devices.

In particular, we partially use singular homology and cohomology theories (denoted as usually by $H_k(-)$ and $H^k(-)$) due to the lack of references in textbooks on products in Čech theory. On the other hand, we avoid the use of Alexander duality which is the basic tool in [3].

Once and for all, we shall assume that $\mathbb{R}^k$, $k \geqslant 0$, is oriented over $\mathbb{Z}$ by the continuous choice of orientation classes $0_p \in H_k(\mathbb{R}^k, \mathbb{R}^k \backslash \{p\})$, $p \in \mathbb{R}^k$ and we shall denote by $e_k \in H^k$ $(\mathbb{R}^k, \mathbb{R}^k \backslash \{0\})$ the dual class of the orientation class 0_p at $p=0$ (for orientation and products the standard references are in [10, 15]).

Assume now that $X=\mathbb{R}^m$ and let (V, W) be an open neighborhood pair of $(\mathcal{S}, \dot{\mathcal{S}})$ in U such that

$$W \cap \{0\} \times \mathbb{R}^m = \varnothing. \tag{2.2}$$

Under this assumption the projection p can be considered as a map of pairs $p\colon (V, W) \to (\mathbb{R}^n, \mathbb{R}^n \backslash \{0\})$. First of all, we shall prove

Claim 1. *$p^*(e_n)$ is nontrivial in $H^n(V, W)$.* Let $Z = V \backslash \mathcal{S}$. The map f induces a map of pairs $f\colon (V, Z) \to (\mathbb{R}^m, \mathbb{R}^m \backslash \{0\})$. To prove our claim, it is enough to show that the cup-product

$$p^*(e_n) \cup f^*(e_m) \neq 0 \qquad \text{in } H^{n+m}(V, W \cup Z).$$

By the well-known properties of cup-products, we have that

$$\eta = p^*(e_n) \cup f^*(e_m) = \Delta^*(p^*(e_n) \times f^*(e_m)) = [(p \times f) \circ \Delta]^*\,(e_{n+m}), \tag{2.3}$$

where the right-hand side is the cohomology homomorphism induced by the map of pairs given by

$$\begin{aligned}(V, W \cup Z) &\xrightarrow{\Delta} (V \times V, W \times V \cup V \times Z)\\ &\xrightarrow{p \times f} (\mathbb{R}^n \times \mathbb{R}^m, (\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}^m \cup (\mathbb{R}^n \times (\mathbb{R}^m \backslash \{0\}))\\ &= (\mathbb{R}^{n+m}, \mathbb{R}^{n+m} \backslash \{0\}).\end{aligned}$$

Set $V' = V \cap p^{-1}(\mathring{I})$. Clearly V' is open not only in $I \times \mathbb{R}^m$ but also in $\mathbb{R}^{n+m}$. Moreover, $K = V - W \cup Z$ is a compact subset of V'. Thus, we have that $(V', V' - K) \subset (V, W \cup Z)$. We shall show that the restriction of η to $(V', V' - K)$ is nontrivial in $H^{n+m}(V', V' - K)$. This implies the nontriviality of η and hence also that of $p^*(e_n)$.

For this, let us denote by G the restriction of $(p \times f) \circ \Delta$ to V'. G can be viewed as a map of pairs $G\colon (V', V' - K) \to (\mathbb{R}^{n+m}, \mathbb{R}^{n+m} - \{0\})$ since by (2.2), $G^{-1}(0) \subset K$. Let $O_K \in H_{n+m}(V', V' - K)$ be the fundamental class of V' around K. By [10, Proposition 5.5, p. 67] we have that $G_*(O_K) = \deg(G, V', O) \cdot O_{n+m} = H_{n+m}(\mathbb{R}^{n+m}, \mathbb{R}^{n+m} - \{0\})$. Notice that (2.3) implies that $\eta_{|(V'V'-K)} = G^*(e_{n+m})$ in $H^{n+m}(V', V' - K)$. By taking the Kroenecker pairing with O_K we have that

$$\langle \eta_{|(V',\, V'-K)}; O_K \rangle = \langle G^*(e_{n+m}); O_K \rangle = \langle e_{n+m}; G_*(O_K) \rangle = \deg(G, V', 0)$$

being $\langle e_{n+m}, O_{n+m} \rangle = 1$.

But we have already proved in (b) that if G and V' are as above then $\deg(G, V', 0) \neq 0$. Thus Claim 1 is proved.

Claim 2. $\xi|_{(\mathscr{S}, \dot{\mathscr{S}})} \neq 0$. Being $I \times \mathbb{R}^m$ an euclidean neighborhood retract it is well known that

$$\check{H}^n(\mathscr{S}, \dot{\mathscr{S}}) = \varinjlim H^n(V, W), \tag{2.4}$$

where (V, W) ranges over all open neighborhood pairs of $(\mathscr{S}, \dot{\mathscr{S}})$. This limit can be also taken over all open neighborhood pairs (V, W) verifying (2.2), since this family is cofinal in the family of all neighborhood pairs. But if (V, W) verifies (2.2.) by Claim 1 and the commutativity of the diagram

$$\begin{tikzcd} H^n(V, W) & \\ H^n(I, \dot{I}) \arrow[u, "p^*"] & H^n(R^n, R^n - \{0\}) \arrow[l, leftrightarrow, "\simeq"'] \arrow[ul, "p^*"'] \end{tikzcd}$$

we have that if $\tilde{e}_n$ is a generator of $\check{H}^n(I, \dot{I}) \simeq H^n(I, \dot{I})$ then $p^*(\tilde{e}_n) \neq 0$ in $H^n(V, W)$. From this and (2.4) it follows that also $\xi|_{(\mathscr{S}, \dot{\mathscr{S}})} = p^*(\tilde{e}_n)$ is nontrivial in $\check{H}^n(\mathscr{S}, \dot{\mathscr{S}})$ and Claim 2 is proven.

Now let X be any Banach space. We prove

Claim 3. *$\xi_{|(\mathscr{S}, \dot{\mathscr{S}})}$ is nontrivial in $\check{H}^n(\mathscr{S}, \dot{\mathscr{S}})$.* Let (V, W) be an open neighborhood pair of $(\mathscr{S}, \dot{\mathscr{S}})$ in U such that

$$W \cap \{0\} \times X = \varnothing. \tag{2.5}$$

Set $V_\delta = \{u : \operatorname{dist}(u, \mathscr{S}) < \delta\}$ then it can be easily proved that given an open neighborhood pair (V, W) of $(\mathscr{S}, \dot{\mathscr{S}})$, there exists $\delta > 0$ such that $(V_\delta, \dot{V}_\delta) \subset (V, W)$. Using [18, Lemma 2.3.], there exists $\bar{f}' : \bar{U} \to X$ with finite dimensional range X' such that the map $\bar{f}' : \bar{U} \to X$ defined by $f'(\lambda, x) = x - \bar{f}'(\lambda, x)$, verifies the following two properties:

(i) $\deg(f'_o, U_o, 0) = \deg(f_o, U_o, 0)$;

(ii) $\mathscr{S}' = \{(\lambda, x) : f'(\lambda, x) = 0\} \subset V_\delta \subset V$ and $\dot{\mathscr{S}}' = \{(\lambda, x) \in \mathscr{S}' : |\lambda| = 1\} \subset \dot{V}_\delta \subset W$.

By the reduction property of the degree, the restriction of f' to $\bar{U} \cap X'$ has the same degree as f'. Namely

$$\deg(f'|_{\bar{U}_o \cap X'}, U_o \cap X', 0) = \deg(f'_o, U_o, 0) \neq 0.$$

Since $\mathscr{S}' = \mathscr{S}' \cap (\mathbb{R}^n \times X')$ and $\dot{\mathscr{S}}' = \dot{\mathscr{S}}' \cap (\mathbb{R}^n \times X')$, Claim 2 applies to the restriction of f' to $\bar{U} \cap \mathbb{R}^n \times X'$ and hence

$$\xi|_{(\mathscr{S}', \dot{\mathscr{S}}')} \neq 0 \qquad \text{in } \check{H}^n(\mathscr{S}', \dot{\mathscr{S}}').$$

Furthermore, from the commutativity of the diagram

$$\begin{tikzcd} \check{H}^n(V, W) \arrow[rr, "i^*"] & & \check{H}^n(\mathscr{S}', \dot{\mathscr{S}}') \\ & H^n(I, \dot{I}) \arrow[ul, "p^*"] \arrow[ur, "p^*"'] & \end{tikzcd}$$

we have that $\xi_{|(V,W)} \neq 0$ in $\check{H}^n(V, W)$ for any pair of open neighborhoods (V, W) of $(\mathscr{S}, \dot{\mathscr{S}})$ verifying (2.3). The tautness property of Čech cohomology for normal spaces tells us that

$$\check{H}^n(\mathscr{S}, \dot{\mathscr{S}}) = \varinjlim \check{H}^n(V, W),$$

where (U, V) ranges over all open neighborhoods of $(\mathscr{S}, \dot{\mathscr{S}})$ verifying (2.5). Hence Claim 3 is established.

Claim 4. *Existence of a component $\mathscr{C}$ of $\mathscr{S}$ such that*

$$\xi_{|(\mathscr{C}, \dot{\mathscr{C}}) \neq 0} \qquad \textit{in} \quad \check{H}^n(\mathscr{C}, \dot{\mathscr{C}}).$$

This follows from Claim 3 and Propositions 2.2–2.3 (i).

Let $(X, \dot{X})$ be a pair of normal spaces. For any closed subset C of X, let $\dot{C} = C \cap \dot{X}$. Given any nontrivial cohomology class $\xi \in \check{H}^n(X, \dot{X})$, let $\mathscr{F}$ be the family of all closed subsets C of X such that $\xi|_{(C, \dot{C})} \neq 0$ in $\check{H}^n(C, \dot{C})$.

DEFINITION 2.1. A closed subset C of X is said to be *ξ-irreducible* if C is a minimal element in the family $\mathscr{F}$ partially ordered by the inclusion (cf. [6]).

PROPOSITION 2.2. *Let $(X, \dot{X})$ be a pair of compact spaces, and $\xi \in \check{H}^n(X, \dot{X})$ be nontrivial. Then a ξ-irreducible set exists.*

Proof. If $\{C_\alpha\}$ is a chain in $\mathscr{F}$ then, by the continuity property of Čech cohomology, $\bigcap_\alpha C_\alpha$ is a lower bound of the chain. Hence the existence of a ξ-irreducible set follows from Zorn's lemma.

PROPOSITION 2.3. *Under the same assumptions as in* 2.1, *if C is a ξ-irreducible set then*

(i) *C is connected*

(ii) *$C \backslash \dot{C}$ is connected.*

Proof. (i) Let U, V be a separation of C. By [13, Theorem 3.12, p. 33], we have that $\check{H}^n(C, \dot{C}) \simeq \check{H}^n(U, \dot{U}) \times \check{H}^n(V, \dot{V})$. By this isomorphism $\xi_{|(C,\dot{C})}$ is sent into $(\xi_{|(U,\dot{U})}, \xi_{|(V,\dot{V})})$. Therefore, if $\xi_{|(U,\dot{U})} \neq 0$ by the minimality of C, being U-closed also in X, we have that V must be empty.

(ii) Let U, V be a separation of $C \backslash \dot{C}$. Since the closures, with respect to C, of U and V are contained in $U \cup \dot{C}$ and $V \cup \dot{C}$, we have that $U' = U \cup \dot{C}$ and $V' = V \cup \dot{C}$ are closed subsets of C. Further, $U' \cup V' = C$ and $U' \cap V' = \dot{C}$. By [13, Theorem 5.4, p. 266] (C, U', V') is a proper triad. Hence, by [13, Theorem 14.2(c), p. 37], we have that

$$\check{H}^n(C, \dot{C}) \simeq \check{H}^n(U', \dot{C}) \times \check{H}^n(V', \dot{C}').$$

But $\dot{U}' = \dot{C}$ and $\dot{V}' = \dot{C}$. The same argument, as in (i), shows that either U or V must be the empty set.

Proof of Theorem 1.2. For $k \in \mathbb{N}$, let $I_k = \{\lambda \in \mathbb{R}^n : |\lambda| \leqslant k\}$, $\dot{I}_k = \partial I_k$, $\mathcal{S}_k = \mathcal{S} \cap p^{-1}(I_k)$, and $\dot{\mathcal{S}}_k = \mathcal{S} \cap p^{-1}(\dot{I}_k)$. For each $k \in \mathbb{N}$, the restriction of f to $\bar{U} \cap p^{-1}(I_k)$ verifies the assumptions of Theorem 1.1 and so, if e^k is a generator of $\check{H}^n(I_k, \dot{I}_k)$ then

$$p^*(e^k) \neq 0 \qquad \text{in } \check{H}^n(\mathcal{S}_k, \dot{\mathcal{S}}_k) \text{ (by Claim 3).} \tag{2.6}$$

Set $D_k = \{\lambda \in \mathbb{R}^n \mid |\lambda| \geqslant k\}$ and $\mathcal{S}'_k = \mathcal{S} \cap p^{-1}(D_k)$. We have the following commutative diagram

$$\begin{array}{ccc}
\check{H}^n(\mathcal{S}, \mathcal{S}'_k) & \xrightarrow[\simeq]{\text{exc.}} & \check{H}^n(\mathcal{S}_k, \dot{\mathcal{S}}_k) \\
\uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle p^*} \\
\check{H}^n(\mathbb{R}^n, D_k) & \xrightarrow[\simeq]{\text{exc.}} & \check{H}^n(I_k, \dot{I}_k)
\end{array}$$

where the horizontal arrows are isomorphisms being excisions. From this and (2.6) it follows that $p^*: \check{H}^n(\mathbb{R}^n, D_k) \to \check{H}^n(\mathcal{S}, \mathcal{S}'_k)$ is nontrivial for each k. On the other hand $\mathcal{S}'_k$ and D_k are cofinal in the family of all cobounded subsets of $\mathcal{S}$ and $\mathbb{R}^n$, respectively. Hence by [23, Theorem 15, p. 322] we have that $\check{H}^n_c(\mathcal{S}) = \varinjlim \check{H}^n(\mathcal{S}, \mathcal{S}'_k)$ and $\check{H}^n_c(\mathbb{R}^n) = \varinjlim \check{H}^n(\mathbb{R}^n, D_k)$. This shows that $p^*: \check{H}^n_c(\mathbb{R}^n) \to \check{H}^n_c(\mathcal{S})$ is nontrivial and therefore $p^*(e) \neq 0$ in $\check{H}^n_c(\mathcal{S})$.

Let $\mathcal{S}^+$ denote the Alexander compactification of $\mathcal{S}$. The map $p: \mathcal{S} \to \mathbb{R}^n$, being a proper map, induces a map $p_1: \mathcal{S}^+ \to \mathbb{S}^n$ (= the euclidean n-dimensional sphere). Consider now the following commutative diagram

$$\begin{array}{ccc} \check{H}^n_c(\mathbb{R}^n) & \xrightarrow{\theta} & \check{H}^n(\mathbb{S}^n, \infty) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} \\ \check{H}^n_c(\mathscr{S}) & \xrightarrow{\theta} & \check{H}^n(\mathscr{S}^+, \infty), \end{array}$$

where θ denotes the natural isomorphism (cf. [23, Theorem 11, p. 321]. Set $\xi = p_1^* \circ \theta(e)$. Then $\xi \neq 0$ in $\check{H}^n(\mathscr{S}^+, \infty)$. From Proposition 2.2 we derive that there exists a ξ-irreducible subset $\mathscr{C}_1$ of $\mathscr{S}^+$. Moreover by Proposition 2.2(ii), $\mathscr{C} = \mathscr{C}_1 \backslash \{\infty\}$ is a connected subset of $\mathscr{S}$. Finally, the commutativity of the following diagram

$$\begin{array}{ccc} \check{H}^n_c(\mathbb{R}^n) & \xrightarrow{\theta} & \check{H}^n(\mathbb{S}^n, \infty) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p_1^*} \\ \check{H}^n_c(\mathscr{C}) & \xrightarrow{\theta} & \check{H}^n(\mathscr{C}_1, \infty) \end{array}$$

gives $p^*(e) \neq 0$ in $\check{H}^n_c(\mathscr{C})$.

3. Two Examples

(1) Consider the following nonlinear boundary value problem (cf. [21]). Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Let $\mathscr{L}$ be a linear uniformly elliptic partial differential operator with real valued coefficients which are smooth on $\bar{\Omega}$. Let $\{\mathscr{B}_i\}$ be a normal family of m smooth boundary operators of order $\leqslant 2m-1$, which cover $\mathscr{L}$ on $\partial\Omega$. Let L be the operator $\mathscr{L}$ acting on real functions satisfying $\mathscr{B}_i u = 0$ on $\partial\Omega$, $1 \leqslant i \leqslant m$. Then L can be considered as a closed operator on $L^2(\Omega)$. Moreover, L is a Fredholm operator, i.e., the range of L, $R(L)$, is closed in $L^2(\Omega)$ and $R(L)^\perp$ and the kernel of L, $N(L)$, are finite dimensional. Assume that index of $L > 0$.

Let $g: \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a uniformly bounded continuous function such that the limits

$$g_\pm(x) = \lim_{u \to \pm\infty} g(x, u) \qquad \text{exist for every } x \in \Omega.$$

Let T be a linear map from $R(L)^\perp$ into $N(L)$. Define

$$M_T(z) = \int_{Tz>0} g_+(x)\, z(x)\, dx + \int_{Tz<0} g_-(x)\, z(x)\, dx.$$

Finally, assume that $(\mathscr{L}, \mathscr{B})$ has the "unique continuation property," that is if $v \in N(L)$ vanishes on a set of positive measure, then $v = 0$.

We are interested in studying the nonlinear functional equation

$$Lu = G(u), \tag{3.1}$$

where G is the Nemytskii operator generated by g.

LEMMA 3.1. *Assume that*

$$M_T(z) > 0 \qquad \text{for all } z \in R(L)\backslash\{0\}. \tag{3.2}$$

If V denotes the L_2-orthogonal complement of $TR(L)^\perp$ in $N(L)$ then there exists a connected subset $\mathscr{C}$ of the solution set of (3.1) *such that the projection of $\mathscr{C}$ on V covers all of V and the topological dimension of $\mathscr{C}$ is at least $n = \dim V$.*

Proof. Via the Ljapunov–Schmidt method, Eq. (3.1) is equivalent to the system

$$\begin{aligned} w &= KQG(v + Tz + w) \\ o &= (I - Q)\, G(v + Tz + w) \qquad \text{for} \quad v + Tz + w \in V \oplus TR(L)^\perp \oplus N(L)^\perp, \end{aligned}$$

where Q is the projection onto $R(L)$ and K is the compact inverse of the restriction of L to $N(L)^\perp$. Consider the map

$$\bar{f}\colon V \times Z \times N(L)^\perp \to Z \times N(L)^\perp$$

defined by

$$\bar{f}(v, z, w) = (z - (I - Q)\, G(v + Tz + w), KQG(v + Tz + w)).$$

Then $\bar{f}$ is compact. Set $f(v, z, w,) = (z, w) - \bar{f}(v, z, w)$.

Now, by the uniform boundedness of G and (3.2) (cf. [21]), where the existence of a solution for such a problem was treated) it follows that there exists $k > 0$ and

$$\langle f(v, z, w,), (z, w)\rangle \geqslant k\, \|(z, w)\|^2, \tag{3.3.}$$

where v ranges on bounded intervals and $\|(z, w)\|$ is sufficiently large. Then (3.3) implies "a priori bounds" on the solution set $\{(z, v, w)\colon f(z, v, w) = o\}$. Hence, cf. Remark 1.3, there exists an open subset U of $V \times Z \times W$, locally bounded over V such that f does not have zeroes on ∂U. Fix the parameter $v = 0$, by Krasnosel'kii's theorem, we have that f_0 has degree 1. Hence Theorem 1.2 applies and we have obtained the desired results.

(2) Consider now the following nonlinear eigenvalue problem studied by Berestyckii and Brezis in [5] (we follow the terminology and notations used in [5]) in a bounded domain Ω with smooth boundary Γ, find $u \in H^2(\Omega)$ satisfying

$$\begin{aligned} &\mathscr{L}u(x) = \lambda g(x, u(x)) \qquad \text{in } \Omega \\ &u_{|\Gamma} \text{is constant} \qquad -\int_\Gamma \frac{\partial u}{\partial \nu}\, d\Gamma = I, \end{aligned} \tag{3.4}$$

where $\mathscr{L}$ is a second order uniformly elliptic operator in divergence form, $\partial/\partial\nu$ is the outward conormal derivative on Γ associated with $\mathscr{L}$, I is a given positive constant, and $g\colon \bar{\Omega} \times \mathbb{R} \to [0, +\infty)$ is a nonzero continuous function verifying (1.4)–(1.6) in [5]. Define λ^* by

$$I/\lambda^* = \lim_{z \to +\infty} \int_\Omega g(x, z)\, dz \qquad \text{so that } 0 \leqslant \lambda^* < +\infty.$$

In [5], the existence of a "sufficiently large" connected component of solutions (λ, u) of (3.4) in $(\lambda^*, +\infty) \times H^1(\Omega)$ was established. We shall briefly indicate how, via Theorem 1.2, a global conclusion can be given.

LEMMA 3.2. *Under the previous assumptions, there exists a connected set $\mathscr{C}$ of solutions (λ, u) of* B.V.P. (2.2) *in $(\lambda^*, +\infty) \times H^1(\Omega)$ such that the projection of $\mathscr{C}$ on $(\lambda^*, +\infty)$ covers all of $(\lambda^*, +\infty)$.*

The proof of Lemma 3.2 is a consequence of Theorem 1.2 via a reparametrization of $(\lambda^*, +\infty)$, the a priori estimates of [5, Lemma 3.4] and the computation of the topological degree in [5, Lemma 4.2]. Indeed, by a reparametrization of $(\lambda^*, +\infty)$ and following [5], we have that B.V.P. (3.4) is equivalent to the functional equation

$$u = (\lambda^* + e^\mu)\, Ru + Lu + w_0$$

in $E = \{u \in H^1(\Omega) \colon u_{|\Gamma} \text{ is constant}\}$, with $\mu \in \mathbb{R}$. $L\colon L^2(\Omega) \to H^1(\Omega)$ is the solution (bounded, linear) operator of the problem

$$\begin{aligned} &\mathscr{L}u + u = f \qquad \text{in} \quad \Omega \\ &\int \frac{\partial u}{\partial \nu}\, d\Gamma = 0 \end{aligned}$$

$R\colon E \to E$ is the compact operator

$$Ru(x) = L(g(x, u(x)))$$

and $w_o \in E$ is the unique solution of the problem

$$\mathscr{L}w_o + w_o = 0$$

$$\int_\Gamma \frac{\partial w_0}{\partial \nu}\, d\Gamma = I.$$

Let f: $\mathbb{R} \times E \to E$ defined by $f(\mu, u) = u - (\lambda^* + e^\mu) Ru + Lu + w_o$. Then f is a one-parameter family of compact vector fields. Moreover, by the a priori estimates given in [5, Lemma 3.4 and Remark 2] we have that for each compact subset Λ of $\mathbb{R}$ there exists a $C > 0$ (depending on Λ) such that $f(\mu, u) = 0$ and $\mu \in \Lambda$ implies $\|u\|_E < C$. Hence f is defined on the closure of an open subset U of $\mathbb{R} \times E$, locally bounded over $\mathbb{R}$ and $f(\mu, u) \neq 0$ on ∂U. Then, the Leray–Schauder degree $\deg(f_\mu, U_\mu, o)$ is defined. Finally, from [5, Lemma 4.2]one has $\deg(f_\mu, U_\mu, 0) = -1$. Now Lemma 3.2 follows from Theorem 1.2.

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