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Normal blow-ups and their expected defining equations

Mark R. Johnson[a,*], Susan Morey[b]

[a] Department of Mathematical Sciences, University of Arkansas, Fayetteville, AR 72701, USA
[b] Department of Mathematics, Southwest Texas State University, San Marcos, TX 78666, USA

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Abstract

We give some criteria for the blow-up of various ideals to be normal, when one knows the defining equations have a certain "expected form". © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let R be a noetherian ring and let I be an ideal of R with grade $I > 0$. Given a presentation

$$R^m \xrightarrow{\phi} R^n \to I \to 0$$

of I, let $\boldsymbol{x} = x_1, \ldots, x_s$ be generators for the ideal $I_1(\phi)$ generated by the entries of ϕ, and let $\boldsymbol{T} = T_1, \ldots, T_n$ be variables over R. We write

$$\boldsymbol{T} \cdot \phi = \boldsymbol{x} \cdot B(\phi)$$

for some matrix $B = B(\phi)$; we call B a *Jacobian dual* of ϕ. This matrix $B(\phi)$ plays an important role in the study of the polynomial relations amongst the generators of I, or in other words, the defining equations of the *blow-up ring* $\mathscr{R}(I) = R[It] \cong \bigoplus_{i\geq 0} I^i$. Indeed, we may present the blow-up ring as

$$\mathscr{R} = R[It] \cong R[T_1, \ldots, T_n]/Q,$$

* Corresponding author.
E-mail addresses: mark@math.uark.edu (M.R. Johnson), sm26@swt.edu (S. Morey).

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such that

$$(\boldsymbol{x}B(\phi), I_s(B(\phi))) \subset Q.$$

We say that the defining ideal of $\mathcal{R}$ has *the expected form* if equality holds, i.e., if $Q = (\boldsymbol{x}B(\phi), I_s(B(\phi)))$. Of course this holds whenever the defining ideal is generated solely by the linear forms $\boldsymbol{x}B$, or equivalently that $\mathcal{R} \cong S(I)$, the symmetric algebra of I. The normality of the blow-up ring for such ideals *of linear type* has been studied by various authors [10,22].

In this work, we are interested in determining the normality of $\mathcal{R}$ when its defining ideal has the expected form. It turns out that if $R = k[x_1, \ldots, x_d]$ is a polynomial ring over a field, and ϕ has linear entries, then fairly generally the blow-up ring is regular in codimension 1. Thus, the corresponding ideal is normal (equivalently, all its powers are integrally closed) in the presence of Serre's condition (S_2). As a corollary, we obtain the normality of reduced perfect ideals of codimension 2 with linear presentation, as the defining ideal of the blow-up ring is known to be Cohen–Macaulay and have the expected form [13].

It can happen, however, that the blow-up rings of such ideals can be considerably smoother. In fact, the blow-up $\operatorname{Proj} R[It]$ itself can be smooth. For a perfect ideal of codimension 2, generated by $n = d + 1$ elements, we show that $\operatorname{Proj} \mathcal{R}$ is smooth precisely when the corresponding fiber $\mathcal{R} \otimes_R k$ is smooth. Using this, one may construct examples of four-generated homogeneous prime ideals in $\mathbf{Q}[x, y, z]$ whose blow-ups are smooth, thus negatively answering a question of Francia.

In the local case, where now (R, m) is a regular local ring of dimension d, I an ideal with presentation matrix ϕ such that $I_1(\phi) = m$, and the defining ideal of $\mathcal{R}$ has the expected form, it turns out that I need not be normal. However, what seems to be needed as a remedy now is the reducedness of the fiber $\mathcal{R} \otimes_R k$. Of course, this condition automatically holds in the graded case, whenever I is generated by forms of the same degree. With this assumption, we can show in general the normality of perfect ideals of codimension 2, generalizing our earlier result. In the case when I is minimally generated by $n = d + 1$ elements, we can show that $\mathcal{R}$ satisifes (R_1) if the fiber only satisifes (R_0).

2. Linear presentation

In this section we study the normality of I in the case of linear presentation. We begin with the following general result.

Proposition 2.1. *Let $R = k[x_1, \ldots, x_d]$ and let I be an ideal with a linear presentation matrix ϕ, with $\operatorname{ht} I_1(\phi) = d$, such that the defining ideal of $\mathcal{R}$ has the expected form. If $\mathcal{R}(I_p)$ satisfies (R_1) for every $p \in V(I), p \neq m = (x_1, \ldots, x_d)$, then $\mathcal{R}$ satisfies (R_1).*

Proof. Let $P \in \operatorname{Spec}(\mathcal{R})$ with $\dim \mathcal{R}_P \leq 1$ and set $p = P \cap R$. Since $\mathcal{R}_P$ is a localization of $\mathcal{R}(I_p)$, we may assume that $p = m$. Now since I is homogeneous, it is well-known that $\mathscr{R} \otimes_R k$ is a domain, hence $m\mathscr{R}$ is a prime ideal. As $m\mathscr{R} \subset P$, it follows that $P = m\mathscr{R}$, and that $\dim \mathscr{R} \otimes_R k = \dim \mathscr{R} - ht\, m\mathscr{R} = d$. It suffices to show that $\mathscr{R}_{m\mathscr{R}}$ is regular.

Let ϕ be a matrix with linear entries presenting I, and $I_1(\phi) = (x_1, \dots, x_d)$, and let $B(\phi)$ be its Jacobian dual. Then $B(\phi)$ defines a $k[T]$-module F by

$$k[T]^m \xrightarrow{B(\phi)} k[T]^d \to F \to 0,$$

such that $S_R(I) = S_{k[T]}(F)$.

We now modify an argument from [20]. From the presentation, we see the ideal $J = I_d(B(\phi))$ annihilates F. Thus, tensoring with $C = k[T]/J \cong \mathscr{R} \otimes_R k$, gives a presentation

$$C^m \xrightarrow{B(\phi)} C^d \to F \to 0$$

of F as a C-module. On the other hand, if we let $\mathscr{A}$ be the kernel of the natural epimorphism $S(I) \to \mathscr{R}(I)$, then by hypothesis the image of J in $S(I)$ generates $\mathscr{A}$. Hence, we have isomorphisms

$$\mathscr{R} \cong S(I)/\mathscr{A} \cong S_C(F).$$

It follows that F is a torsionfree C-module of linear type, and in particular that $\dim S_C(F) = \dim C + \mathrm{rank}_C F = d + \mathrm{rank}_C F$ [19].

But $\dim S_C(F) = \dim \mathscr{R} = d + 1$, hence F has rank one, and is isomorphic to an ideal in C, necessarily of linear type. Thus, if K denotes the field of fractions of R, then $\mathscr{R}_{m\mathscr{R}} \cong S_C(F)_{(x_1,\dots,x_d)} \cong S_K(F \otimes_R K)_{(\mathbf{x})}$ is a localization of a polynomial ring over K, hence regular. □

Corollary 2.2. *Let $R = k[x_1, \dots, x_d]$, let I be an ideal with linear presentation ϕ and $ht\, I_1(\phi) = d$, which is normal on the punctured spectrum, and suppose that $\mathscr{R}$ satisfies (S_2) and that its defining ideal has the expected form. Then I is normal.*

An important case in which one knows the defining ideal of $\mathscr{R}$ has the expected form is for perfect ideals of codimension 2. More precisely, if I is a perfect ideal of codimension 2 with a linear presentation matrix, and if $\mu(I_p) \leq \dim R_p$ for all $p \in V(I)$ with $p \neq m = (x_1, \dots, x_d)$, then $\mathscr{R}$ is Cohen–Macaulay and its defining ideal has the expected form [13, Theorem 1.3].

Corollary 2.3. *Let $R = k[x_1, \dots, x_d]$ and let I be an integrally closed perfect ideal of codimension 2, with a linear presentation matrix, satisfying $\mu(I_p) \leq \max\{2, \dim R_p - 1\}$ for every $p \in V(I)$ with $\dim R_p < d$. Then I is normal.*

Proof. Since $\mathscr{R}$ is Cohen–Macaulay, and the defining ideal has the expected form, it only remains to show that I is normal on the punctured spectrum. This holds by the condition on the local number of generators (e.g. [20, Corollary 4.2]) and the fact that I is generically normal [4]. □

We now explore the question of when the blow-up is in fact smooth. We will use the following classical fact about determinants:

Remark 2.4. If $A=(a_{ij})$ is a square matrix whose entries are functions of t, Δ_{ij} denotes the (signed) minor obtained by deleting the ith row and jth column, and $\Delta=\det(A)$, then

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t}=\sum_{i,j}\frac{\mathrm{d}a_{ij}}{\mathrm{d}t}\Delta_{ij}.$$

Proposition 2.5. *Let k be a field, $R=k[x_1,\ldots,x_d]$, and let I be an ideal generated by $n>d$ elements, with a linear presentation matrix ϕ. Let $B(\phi)$ be the Jacobian dual, let Δ_i denote its maximal minors, and let $\partial\Delta/\partial T$ denote the corresponding Jacobian matrix.*

(a) *If $\sqrt{I_{n-d}(\partial\Delta/\partial T)}$ is the irrelevant ideal of $k[T_1,\ldots,T_n]$, then* Proj $\mathscr{R}$ *is smooth.*
(b) *Suppose that the defining ideal of $\mathscr{R}$ has the expected form, $n=d+1$, that k is perfect and char $k\nmid d$.*

Then Proj $\mathscr{R}$ *is smooth if and only if $\sqrt{I_1(\partial\Delta/\partial T)}$ is the irrelevant ideal of $k[T_1,\ldots,T_n]$.*

Proof. We compute the Jacobian matrix of the elements $\boldsymbol{x}B(\phi),\Delta_i$ of the defining ideal of $\mathscr{R}$:

$$\mathscr{J}=\left(\begin{array}{c|c} B^t & \phi^t \\ \hline 0 & \partial\Delta/\partial T\end{array}\right).$$

Since the defining ideal has codimension $n-1$, the singular locus of $\mathscr{R}$ is defined by an ideal containing in particular the $(n-1)$-sized minors of $\mathscr{J}$. But the ideal of $(n-1)$-sized minors of $\mathscr{J}$ clearly contains the ideal $I_{d-1}(B(\phi))\cdot I_{n-d}(\partial\Delta/\partial T)$. Thus, $V(I_{n-1}(\mathscr{J}))\subset V(I_{n-d}(\partial\Delta/\partial T))\cup V(I_{d-1}(B(\phi)))$.

On the other hand, by Remark 2.4, $I_1(\partial\Delta/\partial T)\subset I_{d-1}(B(\phi))$. Hence, in fact we have that $V(I_{n-1}(\mathscr{J}))\subset V(I_{n-d}(\partial\Delta/\partial T))$.

Now in case (a), it follows that if $p\in$ Proj $\mathscr{R}$, then the local ring $\mathscr{R}_p$ is regular. Thus, all the local rings of the blow-up are regular, hence Proj $\mathscr{R}$ is smooth.

For the converse (b), suppose now that the defining ideal has the expected form, and that that Proj $\mathscr{R}$ is smooth. This implies that $\sqrt{I_{n-1}(\mathscr{J})}$ contains the irrelevant ideal of $R[T_1,\ldots,T_n]$, hence for $1\le i\le n$ monomials T_i^k belong to the ideal $I_{n-1}(\mathscr{J})$ for some k. Moreover, we see at least, modulo $(x_1,\ldots,x_d)$, that the Jacobian ideal is contained in the ideal generated by the maximal minors Δ_i of $B(\phi)$, and their partial derivatives $I_1(\partial\Delta/\partial T)$. But by Euler's formula, $d\Delta_i\in(\partial\Delta_i/\partial T_1,\ldots,\partial\Delta_i/\partial T_n)$. Thus, as d is invertible in k, all the Δ_i belong to $\partial\Delta/\partial T$. Thus modulo $(x_1,\ldots,x_d)$, the Jacobian ideal is contained in $I_1(\partial\Delta/\partial T)$, and hence $T_i^k\in(x_1,\ldots,x_d,I_1(\partial\Delta/\partial T))$. But by the homogeneity, $T_i^k\in I_1(\partial\Delta/\partial T)$. Hence, $\sqrt{I_1(\partial\Delta/\partial T)}$ is the irrelevant ideal. □

Corollary 2.6. *Let k be a field, $R=k[x_1,\dots,x_d]$, let I be an ideal generated by $n>d$ elements, with a linear presentation, and suppose that the defining ideal of $\mathscr{R}$ has the expected form, and that* $\dim \mathscr{R}\otimes_R k=d$.

(a) *If* $\mathrm{Proj}(\mathscr{R}\otimes_R k)$ *is smooth then so is* $\mathrm{Proj}\,\mathscr{R}$.
(b) *If in addition $n=d+1$, k is perfect and* $\mathrm{char}\, k\nmid d$, *then* $\mathrm{Proj}\,\mathscr{R}$ *is smooth if and only if* $\mathrm{Proj}\,(\mathscr{R}\otimes_R k)$ *is smooth.*

Corollary 2.7. *Let k be a perfect field with* $\mathrm{char}\, k\nmid d$, *let $R=k[x_1,\dots,x_d]$, let ϕ be a $d+1\times d$ matrix with linear entries satisfying $\mathrm{ht}\, I_t(\phi)\geq d-t+2$ for every $2\leq t\leq d$ and let $I=I_d(\phi)$. Then* $\mathrm{Proj}\,R[It]$ *is smooth if and only if the hypersurface* $Z(\det B(\phi))\subset \mathbf{P}^d_k$ *is smooth.*

Francia has asked (cf. [15]) whether a one-dimensional prime ideal in a regular local ring with a smooth blow-up is necessarily a complete intersection. We are now able to give a negative answer to this question. To obtain an example in $k[x,y,z]$ with a smooth blow-up we may choose the entries of ϕ to be sufficiently general. But of course, if k is algebraically closed, this ideal is not prime. We construct a prime ideal over the rationals, using the method employed in [7, Theorem 2.13].

Example 2.8. Let $R=\mathbf{Q}[x,y,z]$, and let I be the ideal generated by the maximal minors of the matrix

$$\begin{pmatrix} 0 & x-y & y-z \\ z-y & 0 & x-y+z \\ z & 0 & x-2y \\ x & y & z \end{pmatrix}.$$

Then I is a prime ideal of codimension 2 and $\mathrm{Proj}\,R[It]$ is smooth.

Proof. One may check that the partials of the determinant of the Jacobian dual of this matrix is in fact an irrelevant ideal. Thus, Proposition 2.5 (or Corollary 2.7) implies that the blow-up is smooth.

To prove that I is prime, one verifies that $\mathbf{Q}[z]\subset\mathbf{Q}[x,y,z]/I$ is a Noether normalization, has degree 6 over $\mathbf{Q}[z]$, and that the polynomial $f(x,z)=x^6+4x^5z-15x^4z^2+3z^3+x^2z^4-xz^5$ belongs to $I\cap\mathbf{Q}[x,z]$. Thus, [23, Proposition 10.4.19] implies that I is prime if and only if f is irreducible. But f is irreducible if and only if its dehomogenization $1+4z-15z^2+3z^3+z^4-z^5$ is. This latter polynomial is obviously irreducible, e.g. via reduction modulo 2. □

A conjecture of the second author [12] asserts that, for any perfect Gorenstein ideal of codimension 3, of linear type on the punctured spectrum, in an odd dimensional local (Gorenstein) ring, the defining ideal of $\mathscr{R}$ has the expected form. The following gives an example of such an ideal whose blow-up is smooth.

Example 2.9. Let k be a field, let $R=k[x,y,z]$, and let $I=(x^2-y^2,y^2-z^2,xy,yz,xz)$. Then I is a Gorenstein ideal with linear presentation, $\mathrm{Proj}\,R[It]$ is smooth, but $R[It]$ does not satisfy (S_2).

Proof. Note that I is generated by the 4×4 Pfaffians of the alternating matrix

$$\phi=\begin{pmatrix} 0 & 0 & z & 0 & y\\ 0 & 0 & 0 & x & y\\ -z & 0 & 0 & y & x\\ 0 & -x & -y & 0 & -z\\ -y & -y & -x & z & 0 \end{pmatrix}.$$

Hence by the Buchsbaum–Eisenbud structure theorem, I is Gorenstein and ϕ is a presentation matrix of I. The fact that $R[It]$ does not satisfy (S_2) is a consequence of [14].

To prove $\operatorname{Proj} R[It]$ is smooth, we may use Proposition 2.5. Writing $(T_1,\dots,T_5)\phi=(x,y,z)B(\phi)$, we see that the Jacobian dual is the 3×5 matrix

$$B(\phi)=\begin{pmatrix} 0 & -T_4 & -T_5 & T_2 & T_3\\ -T_5 & -T_5 & -T_4 & T_3 & T_1+T_2\\ -T_3 & 0 & T_1 & T_5 & -T_4 \end{pmatrix}.$$

One easily sees that 3 of the 10 maximal minors of $B(\phi)$ are redundant, and the ideal of maximal minors is generated by the 7 cubics

$$T_1T_3T_4+T_2T_3T_4-T_3^2T_5+T_4^2T_5,$$

$$T_3^2T_4-T_2T_3T_5-T_4T_5^2,$$

$$T_1T_2T_3+T_2^2T_3-T_3^3+T_2T_4T_5+T_3T_5^2,$$

$$T_1T_3T_4-T_1T_2T_5+T_4^2T_5-T_5^3,$$

$$T_1^2T_4+T_1T_2T_4-T_4^3-T_1T_3T_5+T_4T_5^2,$$

$$T_3T_4^2+T_1T_4T_5-T_3T_5^2,$$

$$T_1^2T_2+T_1T_2^2-T_1T_3^2-T_2T_4^2+T_1T_5^2+T_2T_5^2.$$

A routine verification shows that the two-sized minors of the matrix of all the partial derivatives generate an irrelevant ideal. Hence the blow-up is smooth. □

In fact, it holds for this example that the defining ideal of $\mathscr{R}$ has the expected form. This may be verified using the computer algebra system *Macaulay* [2].

The Grauert–Riemenschneider vanishing theorem (as reformulated by Sancho de Salas [17]) states for an ideal I in a local Cohen–Macaulay ring, essentially of finite type over the complex numbers, with $\operatorname{Proj} R[It]$ smooth, the associated graded ring $gr_{I^n}R$ is Cohen–Macaulay for all $n\gg 0$. For the previous example, the Grauert–Riemenscheider theorem now implies (at least for $k=\mathbf{C}$) that the associated graded ring of all sufficiently large powers of I is Cohen–Macaulay. In our case, this holds already for the second power. Indeed, $I^n=m^{2n}$ for all $n\geq 2$, where $m=(x,y,z)$, so

$R[I^n t]$ is Cohen–Macaulay for all $n \geq 2$ [21], hence $gr_{I^n}(R)$ is Cohen–Macaulay for all $n \geq 2$.

The fact that I is not integrally closed would also follow from [3, Corollary 3.2], as an integrally closed perfect Gorenstein ideal of codimension 3 in a local Gorenstein ring is a complete intersection.

3. Codimension 2

In this section we consider the case of perfect ideals of codimension 2. Let (R, m) be a d-dimensional regular local ring, let $x_1, \dots, x_d$ be a regular system of parameters and let I be a perfect ideal of codimension 2. Then by the Hilbert–Burch theorem, I is generated by the maximal minors of its $n \times n-1$ presentation matrix ϕ. We assume also that $I_1(\phi) = (x_1, \dots, x_d)$. We would like to know when $\mathscr{R}$ is normal (or at least satisfies (R_1)) assuming its defining ideal has the expected form.

But unlike the graded case, this need not occur without further assumptions. We may already see this in the case $d=2$: if (R, m) is a two-dimensional regular local ring with regular system of parameters x, y, and ϕ has the form $\phi = \binom{\psi}{x\ y}$, then $(T_1, T_2, T_3)\phi = (x, y)B$, for some Jacobian dual B, and it is well-known (and easy to see) that the defining ideal of $\mathscr{R}$ has the expected form, i.e., $\mathscr{R} \cong R[T_1, T_2, T_3]/((x, y)B, \det B)$. Thus if the entries of ψ lie in m^2 then I is not normal; indeed, $m\mathscr{R}$ has a unique minimal prime that is a singular point of $\mathscr{R}$.

From this example, one might expect that I might be normal if ψ contains "sufficiently many" regular parameters. But this appears awkward to make precise. On the other hand, this example is lacking another attribute of the graded case, namely that the fiber $\mathscr{R} \otimes_R k$ is a domain. In fact, $\mathscr{R} \otimes_R k \cong k[T_1, T_2, T_3]/(T_3^2)$, so the fiber is not even reduced. It turns out that this is necessarily the case.

Theorem 3.1. *Let (R, m, k) be a regular local ring of dimension d, let I be a perfect ideal of codimension 2 with $n = \mu(I) > d$, $\mu(I_p) \leq \dim R_p$ and with $\mathscr{R}(I_p)$ satisfying (R_k) for every $p \in V(I) - \{m\}$, suppose that $I_1(\phi) = m$, and that any of the following conditions hold*:

(a) *The defining ideal of $\mathscr{R}$ has the expected form*;
(b) *$\mathscr{R}$ is Cohen–Macaulay and $I_{n-d}(\phi) = m^{n-d}$*;
(c) *After elementary row operations, if ϕ' denotes the last $n-d$ rows of ϕ, $I_{n-d}(\phi') = m^{n-d}$.*

If $\mathscr{R} \otimes_R k$ satisfies (R_{k-1}) then $\mathscr{R}$ satisfies (R_k).

Proof. These conditions are equivalent if k is infinite [13, Theorem 1.2], and in any case imply (a). Write $I_1(\phi) = (x_1, \dots, x_d)$, where $x_1, \dots, x_d$ is a regular system of parameters. Then using (a), we may write the defining ideal Q of $\mathscr{R}$ as $Q = (\boldsymbol{x})B + I_d(B)$, for a $d \times n-1$ Jacobian dual B of ϕ. Since $ht\, Q = n-1$, it follows that $Q = (\boldsymbol{x}B) : (\boldsymbol{x})$ ([9, 1.8 and 1.5]), so that Q is an $(n-1)$-residual intersection of $(x_1, \dots, x_d)$.

Furthermore, since Q is prime, $ht(\boldsymbol{x})+Q > n-1$, hence Q is a geometric $(n-1)$-residual intersection of $(x_1,\ldots,x_d)$. Since by hypothesis $\mathscr{R}$ is regular locally in codimension k at primes not containing $m\mathscr{R}$, we are done once we show the following result. □

Lemma 3.2. *Let R be a local Cohen–Macaulay ring, let I be a strongly Cohen–Macaulay ideal of grade g, and let J be a geometric s-residual intersection of I. Suppose that R satisfies (R_{s+k}), that R/I satisfies (R_{s-g+k}), and that $R/I+J$ satisfies (R_{k-1}). Then $(R/J)_p$ is regular for every $p \in V(I+J)$ with $\dim(R/J)_p \le k$.*

Proof. One has that $ht\,J = s$ and $ht\,I + J = s + 1$, and both are Cohen–Macaulay ideals [5]. Let $p \in V(I+J)$ with $\dim(R/J)_p \le k$. Then I_p and $(I+J)_p$ are regular. Furthermore, J_p is still a geometric s-residual intersection of I_p. Hence, replacing R by R_p we may assume that R is regular, R/I is regular and that $R/I+J$ is regular. We must now show that R/J is regular.

Write $I = (x_1,\ldots,x_g)$, where $x_1,\ldots,x_g$ is part of a regular system of parameters, $J = (a_1,\ldots,a_s) : I$ for some $a_i \in I$, and $\boldsymbol{a} = \boldsymbol{x}B$, for some $g \times s$ matrix B. Then $J = (a_1,\ldots,a_s) + I_g(B)$ [8, Example 3.4]. Of course $ht\,I_g(B) \le s-g+1$. But on the other hand, as R is regular, $s+1 = ht\,I+J = ht(x_1,\ldots,x_g)+J = ht(x_1,\ldots,x_g)+I_g(B) \le ht(x_1,\ldots,x_g)+ht\,I_g(B) \le g+s-g+1 = s+1$. Hence it follows that $ht\,I_g(B) = s-g+1$, and that $I_g(B)$ is a determinantal ideal of the generic height. As is well-known, $R/I_g(B)$ is Cohen–Macaulay, and the singular locus of $R/I_g(B)$ contains $V(I_{g-1}(B))$. But since $I+J = (\boldsymbol{x})+I_g(B)$, and $\boldsymbol{x}$ is regular on $R/I_g(B)$, we may conclude that $R/I_g(B)$ is regular. Hence some $g-1$ sized minor of B is invertible, and we may assume, after elementary operations, that $B = \begin{pmatrix} Id & 0 \\ 0 & * \end{pmatrix}$, where Id denotes the $g-1 \times g-1$ identity matrix. Hence, $I_g(B)$ is generated by the entries of the unspecified block. Therefore, $J = I_1(\boldsymbol{x}B) + I_g(B) = (x_1,\ldots,x_{g-1}, I_g(B)x_g) + I_g(B) = (x_1,\ldots,x_{g-1}) + I_g(B)$. But then $I+J = (J, x_g)$, and x_g is regular on R/J. It thus follows that R/J is regular. □

Remark 3.3. With the assumptions of Theorem 3.1 one has $k \le 2$, and $\mathscr{R} \otimes_R k$ is Cohen–Macaulay. In particular, the hypothesis on the fiber cone is equivalent to either its reducedness $(k=1)$ or its normality $(k=2)$.

Corollary 3.4. *Let (R,m,k) be a regular local ring of dimension d and let I be an integrally closed perfect ideal of codimension 2 satisfying $n = \mu(I) > d$, $\mu(I_p) \le \max\{2, \dim R_p - 1\}$ for every $p \in V(I) - \{m\}$, $I_1(\phi) = m$, $I_{n-d}(\phi) = m^{n-d}$, $\mathscr{R}$ is Cohen–Macaulay and $\mathscr{R} \otimes_R k$ is reduced. Then I is a normal ideal.*

Notice that in the case of linear presentation Corollary 3.4 generalizes our earlier result Corollary 2.3.

Corollary 3.5. *Let (R,m,k) be a regular local ring of dimension d, and let I be a reduced perfect ideal of codimension 2 generated by $n = d+1$ elements satisfying $\mu(I_p) \le \max\{2, \dim R_p - 1\}$ for every $p \in V(I) - \{m\}$, and assume that one row of* *a minimal presentation matrix ϕ of I generates m and $\mathscr{R} \otimes_R k$ is reduced. Then $\mathscr{R}$ is a normal Cohen–Macaulay ring.*

Note that in the case $n=d+1$, the fiber $\mathscr{R} \otimes_R k$ is now a hypersurface defined by the image of the determinant of a Jacobian dual $B(\phi)$ (as we had also seen in Corollary 2.7). In this situation, it is especially easy to verify its reducedness: it suffices to verify that *det $B(\phi)$ is squarefree modulo m*. In fact, this condition, together with $I_1(\phi)=m$, implies the row condition [1] and the Cohen–Macaulayness of $\mathscr{R}$ [11,16] (at least if R contains the rationals, cf. [7, Proposition 2.5]).

Recently, Huckaba and Huneke [7] have characterized the four-generated perfect ideals of codimension 2, in a three-dimensional regular local ring, whose blow-up ring is normal and not Cohen–Macaulay. Corollary 3.5 gives on the other hand a criterion for such a blow-up ring to be both normal and Cohen–Macaulay.

Finally we would like to point out that the condition when the fiber $\mathscr{R} \otimes_R k$ is reduced is not a necessary condition for normality. This can already be surmised from the proof of Lemma 3.2, which shows that the fiber is a specialization of $\mathscr{R}$; hence one might only expect that it would be a complete intersection in codimension 1. As an explicit example, let $R=k[[x,y]]$, and let $I=(x^2+y^3, x^2y, x^3)$. Then $I_1(\phi)=(x,y)$ and I is integrally closed [6, Example 2.3]. Hence by Zariski's theory of integrally closed ideals in two-dimensional regular local rings, I is normal, and also $R[It]$ is Cohen–Macaulay [6]. On the other hand, it is easy to see that the fiber $\mathscr{R} \otimes_R k$ is not reduced.

4. The case $n = d + 1$

In terms of the study of the defining equations of the Rees algebra $\mathscr{R}$, the case $n=d+1$ is in some sense the first non-trivial situation to consider (cf. e.g. [7,11,18,20]). In this case, we can prove the most general normality result so far, by resorting to a direct computation of Jacobians. As a consequence, the assumptions on the ground field are more restrictive than previously, so when the cases overlap, the earlier results still give the sharpest statements.

Proposition 4.1. *Let k be a perfect field, let $R=k[[x_1,\dots,x_d]]$, with* char $k \nmid d$, *and let I be an ideal generated by $n=d+1$ elements, with $I_1(\phi)=m$, such that $\mathscr{R}(I_p)$ satisfies (R_1) for every $p \in V(I)-\{m\}$ and that the defining ideal of $\mathscr{R}$ has the expected form. If $\mathscr{R} \otimes_R k$ satisfies (R_0) then $\mathscr{R}$ satisfies (R_1).*

Proof. As in the proof of Proposition 2.1, it suffices to show that $\mathscr{R}$ is regular locally at every minimal prime of $m\mathscr{R}$. Set $S=R[T_1,\dots,T_{d+1}]$. Let $B(\phi)$ be the Jacobian dual of ϕ such that the defining ideal of $\mathscr{R}$ is generated by the entries of $(x_1,\dots,x_d)B(\phi)$ and the maximal minors Δ_i of $B(\phi)$. Let "–" denote images in $S \otimes_R k = k[T_1,\dots,T_{d+1}]$. By assumption, we may write $\mathscr{R} \otimes_R k \cong k[T_1,\dots,T_{d+1}]/(\bar{\Delta}_1,\dots,\bar{\Delta}_r)$, where $\bar{\Delta}_i$ are squarefree polynomials.

We may use the Jacobian criterion. Let $\mathscr{J}$ be the Jacobian matrix of the defining ideal of $\mathscr{R}$ with respect to $x_1,\ldots,x_d$ and $T_1,\ldots,T_{d+1}$. It suffices to show that $I_d(\mathscr{J})$ is not contained in any minimal prime of $m\mathscr{R}$, or equivalently, since $n = d+1$, that $ht\, I_d(\bar{\mathscr{J}}) \geq 2$.

But over $k[T_1,\ldots,T_{d+1}]$, the Jacobian has the form

$$\bar{\mathscr{J}} = \left(\begin{array}{c|c} \bar{B}^t & 0 \\ \hline * & \partial\bar{\Delta}/\partial T \end{array}\right).$$

In particular, the ideal of d-sized minors contains the product $I_{d-1}(\bar{B})I_1(\partial\bar{\Delta}/\partial T)$. Since the $\bar{\Delta}_i$ are maximal minors of $\bar{B}$, by Remark 2.4, $I_1(\partial\bar{\Delta}/\partial T) \subset I_{d-1}(\bar{B})$. Thus it suffices to show that the ideal of the partials of the $\bar{\Delta}_i$ has height at least 2. In fact we show the partials of any $F = \bar{\Delta}_i$ generate an ideal of height at least 2. By Euler's formula

$$dF = \sum_{i=1}^{n} T_i \partial F/\partial T_i.$$

Since d is a unit, it follows that $(\partial F/\partial T) = (F, \partial F/\partial T)$ is the Jacobian ideal of F. But on the other hand, as F is squarefree, the ring $k[T_1,\ldots,T_{d+1}]/(F)$ is reduced. It follows that the Jacobian ideal has height at least 2. □

Corollary 4.2. *Let (R, m, k) be an equicharacteristic regular local ring of dimension d, with k perfect and $\operatorname{char} k \nmid d$, and let I be an ideal which is normal on the punctured spectrum, with $\mu(I) = d+1$, $I_1(\phi) = m$, and suppose further $\mathscr{R}$ satisfies (S_2), that its defining ideal has the expected form, and that $\mathscr{R} \otimes_R k$ is reduced. Then I is normal.*

Proof. Since all the assumptions are preserved under completion, while the property of I being normal descends, we may complete and use Proposition 4.1. □

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