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On the existence of cycle frames and almost resolvable cycle systems

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ABSTRACT

Suppose H is a complete m-partite graph $K_m(n_1, n_2, \ldots, n_m)$ with vertex set V and m independent sets $G_1, G_2, \ldots, G_m$ of $n_1, n_2, \ldots, n_m$ vertices respectively. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_m\}$. If the edges of λH can be partitioned into a set $\mathcal{C}$ of k-cycles, then $(V, \mathcal{G}, \mathcal{C})$ is called a k-cycle group divisible design with index λ, denoted by (k, λ)-CGDD. A (k, λ)-cycle frame is a (k, λ)-CGDD $(V, \mathcal{G}, \mathcal{C})$ in which $\mathcal{C}$ can be partitioned into holey 2-factors, each holey 2-factor being a partition of $V \setminus G_i$ for some $G_i \in \mathcal{G}$. Stinson et al. have resolved the existence of $(3, \lambda)$-cycle frames of type g^u. In this paper, we show that there exists a (k, λ)-cycle frame of type g^u for $k \in \{4, 5, 6\}$ if and only if $g(u-1) \equiv 0 \pmod{k}$, $\lambda g \equiv 0 \pmod{2}$, $u \geq 3$ when $k \in \{4, 6\}$, $u \geq 4$ when $k = 5$, and $(k, \lambda, g, u) \neq (6, 1, 6, 3)$. A k-cycle system of order n whose cycle set can be partitioned into $(n-1)/2$ almost parallel classes and a half-parallel class is called an almost resolvable k-cycle system, denoted by k-ARCS(n). Lindner et al. have considered the general existence problem of k-ARCS(n) from the commutative quasigroup for $k \equiv 0 \pmod{2}$. In this paper, we give a recursive construction by using cycle frames which can also be applied to construct k-ARCS(n)s when $k \equiv 1 \pmod{2}$. We also update the known results and prove that for $k \in \{3, 4, 5, 6, 7, 8, 9, 10, 14\}$ there exists a k-ARCS$(2kt+1)$ for each positive integer t with three known exceptions and four additional possible exceptions.

1. Introduction

Throughout this paper, we use C_k or k-cycle for a cycle of length k and K_n for the complete graph on n vertices. For a graph G, we use λG to represent the multi-graph obtained from G by replacing each edge of G with λ copies of it.

A graph H is called a *complete m-partite graph* if its vertex set V can be partitioned into m subsets $G_1, G_2, \ldots, G_m$ such that every pair of vertices, not both from the same partition, is an edge of H. Each subset G_i $(i = 1, 2, \ldots, m)$ is called its independent set. A complete m-partite graph is denoted by $K_m(n_1, n_2, \ldots, n_m)$, where n_i is the cardinality $|G_i|$ of G_i $(i = 1, 2, \ldots, m)$.

Let J be a set of positive integers. Suppose H is a complete m-partite graph $K_m(n_1, n_2, \ldots, n_m)$ with vertex set V and m independent sets $G_1, G_2, \ldots, G_m$ of $n_1, n_2, \ldots, n_m$ vertices respectively. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_m\}$ (called the group set). If the edges of λH can be partitioned into a set of cycles $\mathcal{C}$ with cycle lengths from J, then $(V, \mathcal{G}, \mathcal{C})$ is called a *cycle group divisible design* with index λ, denoted by (J, λ)-CGDD.

When $J = \{k\}$, we write (J, λ)-CGDD as (k, λ)-CGDD. The *type* of the CGDD $(V, \mathcal{G}, \mathcal{C})$ is the multiset of sizes $|G|$ of the $G \in \mathcal{G}$. We use the "exponential" notation for its description: type $1^i 2^j 3^k \ldots$ denotes i occurrences of groups of size 1, j occurrences of groups of size 2, and so on.

A factor of a graph G is a subgraph F for which $V(F) = V(G)$. An r-factor of G is a factor that is regular of degree r. Clearly, a 2-factor is disjoint union of cycles. A (J, λ)-CGDD $(V, \mathcal{G}, \mathcal{C})$ is *resolvable* (denoted by RCGDD) if the collection $\mathcal{C}$ of cycles can be partitioned into 2-factors.

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Theorem 1.1 ([2,3,13–15,17,18]). *For* $k \geq 3$ *and* $u \geq 2$, *there exists a* $(k, 1)$-RCGDD *of type* g^u *if and only if* $g(u-1) \equiv 0 \pmod 2$, $gu \equiv 0 \pmod k$, $k \equiv 0 \pmod 2$ *if* $u = 2$, *and* $(g, u, k) \notin \{(2, 3, 3), (6, 3, 3), (2, 6, 3), (6, 2, 6)\}$.

Definition 1.1. A (J, λ)-*cycle frame* of type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ is a (J, λ)-CGDD of type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}(V, \mathcal{G}, \mathcal{C})$ in which the collection $\mathcal{C}$ of cycles can be partitioned into *holey* 2-*factors*, each holey 2-factor being a 2-regular graph on the vertex set $V \setminus G_j$ for some $G_j \in \mathcal{G}$.

Actually, a (J, λ)-cycle frame is a (J, λ)-CGDD whose cycle set can be partitioned into holey 2-factors, each of which omits one group in the group set. It is not difficult to show that if there exists a (k, λ)-cycle frame of type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$, then $g_i (1 \leq i \leq s)$ must be even and for each group $G_j \in \mathcal{G}$ there are exactly $\frac{\lambda |G_j|}{2}$ holey 2-factors with G_j as their holes and hence there are $\frac{1}{2} \sum_{i=1}^{s} g_i u_i$ holey 2-factors in total. Thus, we have the following theorem when $s = 1$.

Theorem 1.2. *If there exists a* (k, λ)-*cycle frame of type* g^u, *then* (1) $\lambda g \equiv 0 \pmod 2$, (2) $g(u-1) \equiv 0 \pmod k$, (3) $u \geq 3$ *when* $k \equiv 0 \pmod 2$ *and* (4) $u \geq 4$ *when* $k \equiv 1 \pmod 2$.

Cycle frames have been proved to be useful for the construction of resolvable cycle systems [6,10,14,15]. Below are some known results on (k, λ)-cycle frames.

Theorem 1.3 ([7,20]). *There exists a* $(3, \lambda)$-*cycle frame of type* h^u *if and only if* $u \geq 4$, $\lambda h \equiv 0 \pmod 2$ *and* $h(u-1) \equiv 0 \pmod 3$.

Theorem 1.4 ([5,12]). *There is a* $(m, 2)$-*cycle frame of type* 1^n *if and only if* $n \equiv 1 \pmod m$.

In this paper, we shall mainly consider (k, λ)-cycle frames of type g^u with $k = 4, 5, 6$. We shall prove the following main result.

Theorem 1.5. *There exists a* (k, λ)-*cycle frame of type* g^u *for* $k \in \{4, 5, 6\}$ *if and only if* $g(u-1) \equiv 0 \pmod k$, $\lambda g \equiv 0 \pmod 2$, $u \geq 3$ *when* $k \in \{4, 6\}$, $u \geq 4$ *when* $k = 5$, *and* $(k, \lambda, g, u) \neq (6, 1, 6, 3)$.

Cycle frames can also be applied to construct almost resolvable cycle systems which have been considered by Lindner et al. recently. So, we shall investigate the existence of almost resolvable cycle systems in this paper.

A k-cycle system of order n is a pair $(V, \mathcal{C})$, where $\mathcal{C}$ is a collection of k-cycles which partition the edges of K_n with vertex set V. Clearly, a k-cycle system of order n is a $(k, 1)$-CGDD of type 1^n. A k-cycle system of order n exists if and only if $3 \leq k \leq n$, $n \equiv 1 \pmod 2$ and $n(n-1) \equiv 0 \pmod{2k}$ [1,19]. A resolvable k-cycle system of order n exists if and only if $3 \leq k \leq n$, n and k are odd, and $n \equiv 0 \pmod k$ [2].

If $(V, \mathcal{C})$ is a k-cycle system of order n and $n \equiv 1 \pmod{2k}$, then the k-cycle system is not resolvable. In this case, Hanani, Vanstone, Lindner et al. started the research of the existence of an almost resolvable k-cycle system. A collection of $(n-1)/k$ disjoint k-cycles is called an *almost parallel class*. In a k-cycle system of order $n \equiv 1 \pmod{2k}$, the maximum possible number of almost parallel classes is $(n-1)/2$ in which case a *half-parallel class* containing $(n-1)/2k$ disjoint k-cycles is left over. A k-cycle system of order n whose cycle set can be partitioned into $(n-1)/2$ almost parallel classes and a half-parallel class is called an almost resolvable k-cycle system, denoted by k-ARCS(n). Lindner et al. have considered the general existence problem of almost resolvable k-cycle system from the commutative quasigroup for $k \equiv 0 \pmod 2$. We summarize the known results for k-ARCS(n)s as follow.

Theorem 1.6 ([8,9,16,21]). *There exists a* k-ARCS$(2kt+1)$ *for* $k \in \{3, 4, 6, 10, 14\}$ *and* $t \geq 1$, *except for* $(k, t) \in \{(3, 1), (3, 2), (4, 1)\}$ *and except possibly for* $(k, t) \in \{(4, 5), (4, 7), (14, 2)\}$.

Theorem 1.7 ([16]). *Let* $k \equiv 0 \pmod 2$ *and* $k \geq 8$. *If there exists a* k-ARCS$(2k+1)$, *then there exists a* k-ARCS$(2kt+1)$ *except possibly for* $t = 2$.

In this paper, we give a recursive construction by using cycle frames which can also be applied when $k \equiv 1 \pmod 2$. Thus, we can prove the existence of an almost resolvable cycle system k-ARCS$(2kt+1)$ with $k \in \{5, 7, 8, 9\}$ as the application of cycle frames. We shall update the known results and prove the following theorem.

Theorem 1.8. *Let* $k \geq 3$, $t \geq 1$ *be integers and* $n = 2kt + 1$. *There exists a* k-ARCS(n) *for* $k \in \{3, 4, 5, 6, 7, 8, 9, 10, 14\}$, *except for* $(k, n) \in \{(3, 7), (3, 13), (4, 9)\}$ *and except possibly for* $(k, n) \in \{(4, 41), (4, 57), (8, 33), (14, 57)\}$.

2. Recursive constructions

For the recursive constructions of cycle frames, we start with the definition of a group divisible design.

Let K be a set of positive integers. Suppose H is a complete m-partite graph $K_m(n_1, n_2, \ldots, n_m)$ with vertex set V and m independent sets $G_1, G_2, \ldots, G_m$ of $n_1, n_2, \ldots, n_m$ vertices respectively. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_m\}$. If the edges of λH can be partitioned into a set $\mathcal{B}$ of complete graphs with the numbers of their vertices from K, then $(V, \mathcal{G}, \mathcal{B})$ is called a *group* *divisible design* with index λ, denoted by (K, λ)-GDD. When $K = \{k\}$, we write (K, λ)-GDD as (k, λ)-GDD. The *type* of the GDD $(V, \mathcal{G}, \mathcal{B})$ is the multiset of sizes $|G|$ of the $G \in \mathcal{G}$ and we usually use the "exponential" notation for its description as that in the definition of CGDD.

A (K, λ)-GDD with group type 1^v is called a *pairwise balanced design*, denoted by (K, λ, v)-PBD. A (K, λ)-*frame* is a (K, λ)-GDD $(V, \mathcal{G}, \mathcal{B})$ in which $\mathcal{B}$ can be partitioned into holey parallel classes, each holey parallel class being a partition of $V \setminus G_j$ for some $G_j \in \mathcal{G}$. Obviously, a $(3, \lambda)$-frame is also a $(3, \lambda)$-cycle frame. For later use, we need the following known results on pairwise balanced designs and $(2, \lambda)$-frames.

Theorem 2.1 ([7]). *There exists a* $(K, 1, v)$-PBD *for the following parameters.*

1. $K = \{3, 4, 5\}$, $v \neq 6, 8$.
2. $K = \{3, 5\}$, $v \equiv 1 \pmod 2$.
3. $K = \{4, 5, 6\}$, $v \neq 7 - 12, 14, 15, 18, 19, 23$.
4. $K = \{4, 7\}$, $v \equiv 1 \pmod 3$ *and* $v \neq 10, 19$.

Theorem 2.2 ([7]). *There exists a* $(2, \lambda)$-*frame of type* h^u *if and only if* $u \geq 3$ *and* $h(u-1) \equiv 0 \pmod 2$.

Before giving our recursive constructions, we still need some other definitions in graph theory. If G and H are graphs, the *wreath product* $G \wr H$ of G and H is the graph obtained by replacing each vertex u of G with a copy $H(u)$ of H, joining each vertex of $H(u)$ to each vertex of $H(v)$ if u and v are adjacent in G, and having no edges joining vertices of $H(u)$ to vertices of $H(v)$ if u and v are not adjacent in G.

Here, we need the wreath product of C_k and $\overline{K_p}$ for our constructions, where $\overline{K_p}$ denotes the complement of K_p. The graph $C_k \wr \overline{K_p}$ is called *m-resolvable* if its edge set can be partitioned into p 2-factors each of which consists of pk/m cycles of length m. Next, we shall show that $C_k \wr \overline{K_t}$ is k-resolvable for all positive t and all $k \geq 3$ with some definite exceptions.

Lemma 2.3 ([2]). *Suppose k is an odd integer and p is a prime,* $3 \leq k \leq p$. *Then* $C_k \wr \overline{K_p}$ *is p-resolvable.*

Construction 2.4. *Let* $k \geq 3$. *Suppose* $C_k \wr \overline{K_p}$ *is k-resolvable. Then* $C_{k+2} \wr \overline{K_p}$ *is* $(k+2)$-*resolvable.*

Proof. Suppose $C_k = (1, 2, \ldots, k-1, k)$ and the vertex set of $\overline{K_p}$ is $I_p = \{1, 2, \ldots, p\}$. Let the vertex set of $C_k \wr \overline{K_p}$ be $I_p \times I_k$. Suppose the jth cycle in the ith 2-factor of the known $C_k \wr \overline{K_p}$ is $C_j^i = ((b_{j,1}^i, 1), (b_{j,2}^i, 2), \ldots, (b_{j,k}^i, k))$, $b_{j,s}^i \in I_p$, $1 \leq i, j \leq p$, $1 \leq s \leq k$. Let

$$M_j^i = ((b_{j,1}^i, 1), (b_{j,2}^i, 2), \ldots, (b_{j,k}^i, k), (b_{j,k-1}^i, k+1), (b_{j,k}^i, k+2)), \quad 1 \leq i, j \leq p.$$

Let $F_i = \bigcup_{j=1}^p M_j^i$, $1 \leq i \leq p$. It is easy to check each F_i is a 2-factor of $C_{k+2} \wr \overline{K_p}$. Thus, $C_{k+2} \wr \overline{K_p}$ is $(k+2)$-resolvable. $\square$

Lemma 2.5. $C_k \wr \overline{K_t}$ *is k-resolvable for all positive t and all* $k \geq 3$ *with the definite exceptions* $(t, k) = (6, 3)$ *and* $(t, k) \in \{(2, m) : m \geq 3 \text{ is odd}\}$.

Proof. The conclusion is trivial for $t = 1$. According to the definition, it is easy to see that $C_3 \wr \overline{K_6}$ is not 3-resolvable, and $C_m \wr \overline{K_2}$ ($m \geq 3$ is odd) is not m-resolvable.

For even $k \geq 4$ and all $t > 1$, we can start from a 1-factorization of $K_2(t, t)$ and use the same method of Construction 2.4 to obtain that $C_4 \wr \overline{K_t}$ is 4-resolvable. Then $C_k \wr \overline{K_t}$ is k-resolvable for all even $k \geq 4$ and all $t > 1$ by Construction 2.4.

For odd $k \geq 3$, $t > 1$ and $t \neq 6$, we have a $(3, 1)$-RCGDD of type t^3 by Theorem 1.1 which indicates that $C_3 \wr \overline{K_t}$ is 3-resolvable. Thus, $C_k \wr \overline{K_t}$ is k-resolvable for all odd $k \geq 3$ and $t \neq 6$ by Construction 2.4.

Now, we only need to decide whether $C_k \wr \overline{K_6}$ is k-resolvable for all odd $k \geq 5$. This case can be solved if $C_5 \wr \overline{K_6}$ is 5-resolvable by Construction 2.4. Let the vertex set of $C_5 \wr \overline{K_6}$ be $I_6 \times I_5$. A 5-resolvable $C_5 \wr \overline{K_6}$ is constructed below.

$\{(1_1, 1_2, 1_3, 1_4, 1_5), (2_1, 2_2, 2_3, 2_4, 2_5), (3_1, 3_2, 3_3, 3_4, 3_5), (4_1, 4_2, 4_3, 4_4, 4_5), (5_1, 5_2, 5_3, 5_4, 5_5), (6_1, 6_2, 6_3, 6_4, 6_5)\}$
$\{(1_1, 2_2, 1_3, 2_4, 3_5), (2_1, 1_2, 2_3, 1_4, 4_5), (3_1, 4_2, 3_3, 4_4, 1_5), (4_1, 3_2, 4_3, 5_4, 6_5), (5_1, 6_2, 5_3, 6_4, 2_5), (6_1, 5_2, 6_3, 3_4, 5_5)\}$
$\{(1_1, 3_2, 1_3, 3_4, 2_5), (2_1, 4_2, 2_3, 4_4, 5_5), (3_1, 5_2, 4_3, 6_4, 4_5), (4_1, 6_2, 3_3, 2_4, 1_5), (5_1, 1_2, 5_3, 1_4, 6_5), (6_1, 2_2, 6_3, 5_4, 3_5)\}$
$\{(1_1, 4_2, 1_3, 5_4, 4_5), (2_1, 5_2, 2_3, 3_4, 6_5), (3_1, 6_2, 4_3, 2_4, 5_5), (4_1, 1_2, 6_3, 1_4, 3_5), (5_1, 2_2, 3_3, 6_4, 1_5), (6_1, 3_2, 5_3, 4_4, 2_5)\}$
$\{(1_1, 5_2, 1_3, 4_4, 6_5), (2_1, 6_2, 2_3, 6_4, 3_5), (3_1, 1_2, 3_3, 5_4, 2_5), (4_1, 2_2, 4_3, 1_4, 5_5), (5_1, 3_2, 6_3, 2_4, 4_5), (6_1, 4_2, 5_3, 3_4, 1_5)\}$
$\{(1_1, 6_2, 1_3, 6_4, 5_5), (2_1, 3_2, 2_3, 5_4, 1_5), (3_1, 2_2, 5_3, 2_4, 6_5), (4_1, 5_2, 3_3, 1_4, 2_5), (5_1, 4_2, 6_3, 4_4, 3_5), (6_1, 1_2, 4_3, 3_4, 4_5)\}$ $\square$

Now, we are in the position to give our recursive constructions for cycle frames. The proofs of the following constructions are similar to some well-known recursive constructions for frames or group divisible designs, see [2,4,7,11,20]. So, we just state them without proof.

Construction 2.6 (*Fundamental Cycle Frame Construction*). *If there exists a* $(K, 1)$-*GDD of type* g^u *and a* (k, λ)-*cycle frame of type* h^m *for each* $m \in K$, *then there is a* (k, λ)-*cycle frame of type* $(hg)^u$.

Construction 2.7 (*Filling in Holes*). *Suppose there exists a* (k, λ)-*cycle frame with groups of sizes from* $T = \{t_1, \ldots, t_n\}$ *and* $\varepsilon \geq 0$. *For* $1 \leq i \leq n$, *suppose there exists a* (k, λ)-*cycle frame with groups of sizes from* $T_i \cup \{\varepsilon\}$, *where* $\sum_{t \in T_i} t = t_i$. *Then there exists a* (k, λ)-*cycle frame with groups of sizes from* $(\bigcup_{i=1}^n T_i) \cup \{\varepsilon\}$.

Construction 2.8. *Suppose there is a (K, λ)-cycle frame of type g^u and $C_k \wr \overline{K_h}$ is m-resolvable for each $k \in K$, then there exists a (m, λ)-cycle frame of type $(hg)^u$.*

Construction 2.9. *Suppose there is a (K, λ_1)-frame of type g^u and a (m, λ_2)-RCGDD of type h^k for each $k \in K$, then there exists a $(m, \lambda_1\lambda_2)$-cycle frame of type $(hg)^u$.*

Construction 2.10. *If there exists a (k, λ_i)-cycle frame of type g^u for $i = 1, 2$, then there is a $(k, \lambda_1 + \lambda_2)$-cycle frame.*

3. (4, λ)-cycle frames

In this section, we deal with the existence of a $(4, \lambda)$-cycle frame of type g^u.

Lemma 3.1. *There exist $(4, 1)$-cycle frames of type 2^u for $u \in \{3, 5\}$ and type 4^u for $u \in \{3, 4, 5, 6, 8\}$.*

Proof. Take a $(2, 1)$-frame of type 1^3 or 1^5 from Theorem 2.2. Applying Construction 2.9, we can obtain a cycle frame of type 2^3 or 2^5, where the input design $(4, 1)$-RCGDD of type 2^2 comes from Theorem 1.1. Similarly, start from a $(2, 1)$-frame of type 2^u for $u \in \{3, 4, 5, 6, 8\}$ to obtain a cycle frame of type 4^u. □

Theorem 3.2. *There exists a $(4, 1)$-cycle frame of type g^u if $u \geq 3$, $g \equiv 0 \pmod 2$, and $g(u-1) \equiv 0 \pmod 4$.*

Proof. We distinguish the necessary conditions into the following two cases.

(1) $g \equiv 0 \pmod 4$ and $u \geq 3$. Let $g = 4m$. We first prove the case $m = 1$. There exists a $(4, 1)$-cycle frame of type 4^u for $u \in \{3, 4, 5, 6, 8\}$ by Lemma 3.1. For other values of u, take a PBD$(\{3, 4, 5\}, 1, u)$ from Theorem 2.1. Apply Construction 2.6 to obtain a $(4, 1)$-cycle frame of type 4^u. Further, applying Construction 2.8 with a $C_4 \wr \overline{K_m}$ which is 4-resolvable by Lemma 2.5, we can obtain a $(4, 1)$-cycle frame of type $(4m)^u$ for any $m \geq 2$.

(2) $g \equiv 2 \pmod 4$, $u \geq 3$ and $u \equiv 1 \pmod 2$. Let $g = 4k + 2$. Applying Construction 2.6 with a PBD$(\{3, 5\}, 1, u)$ from Theorem 2.1, we can obtain a $(4, 1)$-cycle frame of type 2^u, where the input designs $(4, 1)$-cycle frames of types 2^3 and 2^5 exist by Lemma 3.1. Then, we can use Construction 2.8 to obtain the required $(4, 1)$-cycle frame of type g^u since $C_4 \wr \overline{K_{2k+1}}$ is 4-resolvable by Lemma 2.5. □

Theorem 3.3. *There exists a $(4, 2)$-cycle frame of type g^u if $u \geq 3$ and $g(u-1) \equiv 0 \pmod 4$.*

Proof. The necessary conditions are distinguished into three cases: (1) $u \geq 5$, $u \equiv 1 \pmod 4$ and $g \equiv 1 \pmod 2$, (2) $u \geq 4$ and $g \equiv 0 \pmod 4$, and (3) $u \geq 3$, $u \equiv 1 \pmod 2$ and $g \equiv 2 \pmod 4$. The last two cases can be obtained from a $(4, 1)$-cycle frame of type g^u from Theorem 3.2 by Construction 2.10. Start with a $(4, 2)$-cycle frame of type 1^u by Theorem 1.4. Applying Construction 2.8 with a $C_4 \wr \overline{K_g}$ which is 4-resolvable by Lemma 2.5, we can obtain the required $(4, 2)$-cycle frame of type g^u in case (1). □

The following conclusion comes from Construction 2.10, Theorems 3.2 and 3.3.

Theorem 3.4. *There exists a $(4, \lambda)$-cycle frame of type g^u if and only if $u \geq 3$, $\lambda g \equiv 0 \pmod 2$ and $g(u-1) \equiv 0 \pmod 4$.*

4. (5, λ)-cycle frames

In this section, we investigate the existence of $(5, \lambda)$-cycle frames of type g^u. First we construct some cycle frames which will be used as input designs for recursive constructions.

Lemma 4.1. *There exists a $(5, 1)$-cycle frame of type g^6 for $g \in \{2, 10\}$.*

Proof. For $g = 2$, let the point set be $V = (Z_5 \times Z_2) \cup \{a, b\}$ and the group set be $\{\{i_0, i_1\} : 0 \leq i \leq 4\} \cup \{a, b\}$. One holey 2-factor is composed of the two 5-cycles $(0_0, 2_0, 4_0, 1_0, 3_0)$ and $(0_1, 1_1, 2_1, 3_1, 4_1)$. The other holey 2-factors will be generated from the two 5-cycles $(a, 3_0, 2_0, 4_1, 1_1)$ and $(b, 1_0, 2_1, 4_0, 3_1)$ by $(+1 \bmod 5, -)$. The case $g = 10$ can be solved by applying Construction 2.8 since $C_5 \wr \overline{K_5}$ is 5-resolvable. □

Lemma 4.2. *There exist $(5, 1)$-cycle frames of types 2^{11}, 10^{11} and 10^{23}.*

Proof. Let the point set be $V = Z_{11} \times Z_2$ and the group set be $\{\{i_0, i_1\} : 0 \leq i \leq 10\}$. The required 11 holey 2-factors of a $(5, 1)$-cycle frame of type 2^{11} can be obtained from the following initial holey 2-factor $\{(1_0, 2_0, 4_0, 1_1, 2_1), (6_0, 3_0, 8_0, 3_1, 10_1), (4_1, 9_1, 7_0, 6_1, 10_0), (7_1, 5_1, 8_1, 5_0, 9_0)\}$ by $(+1 \bmod 11, -)$ Further, we can apply Construction 2.8 to obtain a $(5, 1)$-cycle frame of type 10^{11} since $C_5 \wr \overline{K_5}$ is 5-resolvable.

For a $(5, 1)$-cycle frame of type 10^{23}, let the point set be $V = Z_{23} \times Z_{10}$ and the group set G_i be $\{\{i\} \times Z_{10} : 0 \leq i \leq 22\}$. The five holey 2-factors of group G_0 can be obtained from four cycles in F_0 by $(\times 5^{2j} \bmod 23, +2 \bmod 10)$ for $j = 0, 1, 2, 3, 4$, where $F_0 = \{(1_0, 5_0, 6_2, 4_3, 11_2), (1_1, 5_1, 6_4, 20_8, 2_7), (5_3, 1_6, 2_9, 7_4, 9_8), (6_5, 5_6, 10_9, 7_5, 20_7)\}$. All the other holey 2-factors can be obtained from the five holey 2-factors of group G_0 by $(+1 \bmod 23, -)$. □

Lemma 4.3. *There exists a* $(5, 1)$*-cycle frame of type* g^u *where* g *and* u *are*

(1) $g = 2$ *and* $u = 16$;
(2) $g = 4$ *and* $u \in \{6, 11, 16\}$;
(3) $g = 10$ *and* $u \in \{4, 5, 7, 8, 10, 19\}$;
(4) $g = 20$ *and* $u \in \{4, 6, 7, 10, 19\}$.

Proof. Let the point set be $V = Z_{gu}$ and the group set be $\{\{i, i+u, \dots, i+(g-1)u\} : 0 \le i \le u-1\}$. First, we can obtain an initial holey 2-factor F_0 from $g(u-1)/10$ 5-cycles in the following table by $+gu/2$ mod gu. Then, all the other required $gu/2 - 1$ holey 2-factors can be generated from F_0 by $+i$ mod gu, $i = 1, 2, \dots, gu/2 - 1$.

$g = 2, u = 16$:	(1, 2, 4, 7, 11)	(3, 8, 21, 10, 28)	(6, 14, 29, 9, 15)
$g = 4, u = 6$:	(1, 2, 5, 3, 10)	(4, 8, 21, 7, 23)	
$g = 4, u = 11$:	(3, 8, 12, 5, 19)	(6, 14, 39, 10, 37)	(9, 18, 38, 20, 43)
	(1, 2, 4, 7, 13)		
$g = 4, u = 16$:	(6, 13, 21, 10, 23)	(9, 27, 51, 25, 50)	(12, 31, 60, 29, 56)
	(15, 30, 52, 22, 58)	(3, 8, 14, 5, 17)	(1, 2, 4, 7, 11)
$g = 10, u = 5$:	(3, 11, 18, 14, 37)	(6, 17, 34, 16, 47)	(8, 21, 49, 23, 44)
	(1, 2, 4, 7, 13)		
$g = 10, u = 8$:	(10, 25, 58, 19, 61)	(1, 11, 63, 2, 78)	(6, 13, 26, 15, 33)
	(12, 75, 29, 60, 37)	(7, 28, 30, 31, 67)	(4, 34, 79, 36, 62)
	(3, 9, 14, 5, 17)		

Start from a $(5, 1)$-cycle frame of type 4^6 and apply Construction 2.8 with a 5-resolvable $C_5 \wr \overline{K_5}$ to obtain a cycle frame of type 20^6. For each $u \in \{4, 7, 10, 19\}$, take a $(3, 1)$-cycle frame of type 2^u or 4^u from Theorem 1.3. Then we can apply Construction 2.8 with a $C_3 \wr \overline{K_5}$ which is 5-resolvable by Theorem 1.1 to obtain a cycle frame of type 10^u or 20^u. □

Lemma 4.4. *There exist* $(\{3, 5\}, 1)$*-cycle frames of type* 2^u *for* $u \in \{9, 12\}$ *and type* 4^u *for any* $u \in \{5, 8, 9, 11, 12, 14, 15, 18, 23\}$.

Proof. Let the point set be $V = Z_{gu}$ and the group set be $\{\{ i + mu : 0 \le m \le g-1\} : 0 \le i \le u-1\}$. First, we can obtain an initial holey 2-factor F_0 from $g(u-1)/10$ cycles in the following table by $+gu/2$ mod gu. Then, all the other required $gu/2 - 1$ holey 2-factors can be generated from F_0 by $+i$ mod gu, $i = 1, 2, \dots, gu/2 - 1$. □

$g = 2,\ u = 9$:	(1, 2, 4, 7, 15)	(3, 8, 14)	
$g = 2,\ u = 12$:	(1, 2, 4, 7, 20)	(5, 9, 23)	(3, 10, 18)
$g = 4,\ u = 5$:	(1, 2, 8, 6, 9)	(3, 7, 14)	
$g = 4,\ u = 8$:	(6, 11, 26, 12, 31)	(3, 7, 13)	(5, 14, 25)
	(1, 2, 4)		
$g = 4,\ u = 9$:	(3, 8, 28, 14, 33)	(1, 2, 4, 7, 11)	(5, 13, 34)
	(6, 17, 30)		
$g = 4,\ u = 11$:	(9, 15, 36, 19, 39)	(5, 13, 20)	(6, 16, 32)
	(8, 21, 40)	(3, 7, 12)	(1, 2, 4)
$g = 4,\ u = 12$:	(9, 23, 45, 20, 40)	(5, 10, 18)	(1, 2, 4)
	(8, 19, 38, 11, 41)	(3, 7, 13)	(6, 15, 22)
$g = 4,\ u = 14$:	(9, 27, 50, 24, 43)	(5, 11, 18, 6, 16)	(1, 2, 4)
	(8, 21, 13, 45, 25)	(10, 26, 47, 20, 51)	(3, 7, 12)
$g = 4,\ u = 15$:	(3, 7, 12, 5, 11)	(6, 16, 25, 8, 19)	(13, 27, 47, 26, 48)
	(9, 21, 44, 20, 53)	(10, 29, 58, 24, 52)	(1, 2, 4)
$g = 4,\ u = 18$:	(16,29, 64, 22, 67)	(15, 35, 60, 27, 53)	(1, 2, 4)
	(6, 13, 23, 8, 25)	(9, 20, 32, 10, 33)	(3, 7, 12)
	(14, 30, 70, 26, 57)	(5, 11, 19)	
$g = 4,\ u = 23$:	(17,41, 83, 28, 82)	(21, 42, 75, 40, 81)	(3, 7, 12)
	(25, 45, 79, 27, 89)	(6, 14, 24, 8, 20)	(1, 2, 4)
	(9, 26, 15, 30, 56)	(13, 31, 62, 19, 38)	(5, 11, 18)
	(22, 44, 80, 32, 85)		

Lemma 4.5. *There exists a* $(\{3,5\},1)$*-cycle frame of type* 2^u *for* $u \in \{14, 18\}$.

Proof. Let the point set be $V = Z_{u/2} \times Z_4$ and the group set be $\{\{i_0, i_2\} : 0 \le i \le u/2-1\} \cup \{\{i_1, i_3\} : 0 \le i \le u/2-1\}$. The required u holey 2-factors will be generated from the two initial holey 2-factors F_0^u and F_1^u by $(+1 \bmod u/2, -)$. □

F_0^{14}:	$(1_1, 2_2, 3_3)$	$(6_2, 2_1, 0_3, 3_2, 5_0)$	$(4_3, 1_3, 2_0, 4_2, 1_2)$
	$(4_0, 5_1, 1_0)$	$(6_1, 0_1, 5_2, 3_1, 2_3)$	$(5_3, 3_0, 6_3, 6_0, 4_1)$
F_1^{14}:	$(0_0, 6_2, 4_3)$	$(3_3, 2_0, 4_1, 4_0, 2_3)$	$(5_1, 4_2, 6_3, 0_2, 1_2)$
	$(2_2, 5_0, 6_0)$	$(0_3, 3_1, 3_2, 5_2, 5_3)$	$(1_0, 0_1, 2_1, 6_1, 3_0)$

F_0^{18}:	$(1_1, 2_2, 3_3)$	$(6_2, 0_1, 4_1)$	$(7_3, 1_2, 3_0)$	$(2_3, 6_3, 8_1)$	$(4_2, 1_0, 3_2, 6_0, 4_3)$
	$(5_2, 1_3, 8_2)$	$(7_0, 3_1, 0_3)$	$(2_0, 2_1, 7_2)$	$(4_0, 5_1, 8_0)$	$(5_3, 5_0, 8_3, 7_1, 6_1)$
F_1^{18}:	$(0_0, 7_0, 8_3)$	$(2_2, 4_1, 4_2)$	$(3_3, 6_3, 5_3)$	$(4_0, 5_2, 0_3)$	$(0_1, 7_1, 2_3, 3_2, 8_2)$
	$(7_3, 0_2, 1_0)$	$(8_0, 2_0, 6_1)$	$(6_2, 3_1, 7_2)$	$(5_1, 3_0, 2_1)$	$(1_2, 6_0, 5_0, 8_1, 4_3)$

Lemma 4.6. *There exists a* $(\{3,5\},1)$*-cycle frame of type* 2^u *for* $u \in \{15, 23\}$.

Proof. Let the point set be $V = Z_{2u}$ and the group set be $\{\{i, i+u\} : 0 \le i \le u-1\}$. The required u holey 2-factors will be generated from the initial holey 2-factor F_0^u by $+2 \bmod 2u$. □

F_0^{15}:	(12, 17, 25)	(4, 6, 9, 13, 18)	(11, 23, 14, 21, 27)
	(1, 2, 3, 5, 8)	(7, 20, 24, 16, 26)	(10, 22, 28, 19, 29)
F_0^{23}:	(20, 37, 41)	(19, 36, 43)	(11, 17, 27, 39, 26)
	(10, 13, 15)	(1, 33, 38)	(5, 31, 34, 16, 35)
	(12, 24, 32)	(4, 29, 40)	(2, 42, 28, 21, 3)
	(14, 25, 44)	(9, 18, 22)	(6, 8, 7, 45, 30)

Lemma 4.7. *There exist* $(5,1)$*-cycle frames of type* 10^u *for each* $u \in \{9, 12, 14, 15, 18\}$ *and type* 20^u *for each* $u \in \{5, 8, 9, 11, 12, 14, 15, 18, 23\}$.

Proof. Take a $(\{3,5\},1)$-cycle frame of type 2^u for each $u \in \{9, 12, 14, 15, 18\}$ or type 4^u for each $u \in \{5, 8, 9, 11, 12, 14, 15, 18, 23\}$ from Lemmas 4.4–4.6. Applying Construction 2.8 with a $C_3 \wr \overline{K_5}$ and a $C_5 \wr \overline{K_5}$ which are both 5-resolvable, we can obtain a cycle frame of type 10^u or 20^u. □

Lemma 4.8. *There exists a* $(5,2)$*-cycle frame of type* 5^{3k+1} *for* $k \ge 1$.

Proof. Take a $(3,2)$-cycle frame of type 1^{3k+1} for $k \ge 1$ from Theorem 1.3 and apply Construction 2.8 to obtain a $(5,2)$-cycle frame of type 5^{3k+1}. □

Lemma 4.9. *There exists a* $(5,2)$*-cycle frame of type* 5^u *for* $u \in \{5, 6, 8, 11\}$.

Proof. For $u = 5, 8$, let the point set be $V = Z_{5u}$ and the group set be $\{\{i + mu : 0 \le m \le 4\} : 0 \le i \le u-1\}$. The required $5u$ holey 2-factors will be generated from the initial holey 2-factor F_0^u by $+1 \bmod 5u$.

F_0^5:	(1, 2, 3, 6, 4)	(7, 9, 16, 22, 18)	(8, 14, 23, 11, 19)	(12, 21, 13, 17, 24)
F_0^8:	(4, 9, 6, 10, 13)	(11, 15, 20, 14, 21)	(12, 19, 38, 23, 34)	(18, 37, 28, 17, 35)
	(1, 2, 29, 27, 7)	(3, 30, 5, 33, 31)	(22, 36, 26, 25, 39)	

Take a $(5,2)$-cycle frame of type 1^6 or type 1^{11} from Theorem 1.4. Apply Construction 2.8 to obtain a cycle frame of type 5^6 or 5^{11}. □

Lemma 4.10. *There exists a* $(\{3,5\},2)$*-cycle frame of type* 1^u *for* $u \in \{9, 12, 15, 23\}$.

Proof. Let the point set be $V = Z_u$ and the group set be $\{\{i\} : 0 \le i \le u-1\}$. The required u holey 2-factors will be generated from the initial holey 2-factor F_0^u by $+1 \bmod u$. □

F_0^9:	(3, 5, 8)	(1, 2, 4, 7, 6)		
F_0^{12}:	(4, 6, 9)	(5, 8, 10)	(1, 2, 3, 11, 7)	
F_0^{15}:	(4, 7, 12)	(6, 9, 13)	(1, 2, 3, 5, 11)	(8, 10, 14)
F_0^{23}:	(4, 6, 9)	(5, 10, 17)	(8, 18, 15, 11, 19)	
	(1, 2, 3)	(7, 14, 20)	(12, 16, 22, 13, 21)	

Lemma 4.11. *There exists a* $(\{3, 5\}, 2)$*-cycle frame of type* 1^u *for* $u \in \{14, 18\}$.

Proof. Let the point set be $V = Z_u$ and the group set be $\{\{i\} : 0 \le i \le u-1\}$. The required u holey 2-factors will be generated from the initial two holey 2-factors F_0^u and F_1^u by $+2 \bmod u$. □

F_0^{14}:	(9, 11, 13)	(1, 2, 3, 4, 5)	(6, 8, 10, 7, 12)		
F_1^{14}:	(7, 10, 13)	(0, 4, 9, 2, 6)	(3, 8, 11, 5, 12)		
F_0^{18}:	(6, 8, 10)	(7, 12, 15)	(9, 13, 16)	(11, 14, 17)	(1, 2, 3, 4, 5)
F_1^{18}:	(2, 7, 14)	(3, 9, 11)	(5, 10, 16)	(6, 13, 15)	(0, 4, 12, 17, 8)

Lemma 4.12. *There exists a* $(5, 2)$*-cycle frame of type* 5^u *for* $u \in \{9, 12, 14, 15, 18, 23\}$.

Proof. Take a $(\{3, 5\}, 2)$-cycle frame of type 1^u from Lemmas 4.10 and 4.11. Then, apply Construction 2.8 to obtain a $(5, 2)$-cycle frame of type 5^u. □

Theorem 4.13. *There exists a* $(5, 1)$*-cycle frame of type* g^u *if* $u \ge 4$, $g \equiv 0 \pmod 2$, *and* $g(u-1) \equiv 0 \pmod 5$.

Proof. We distinguish the necessary conditions into the following two cases.

(1) $g \equiv 0 \pmod{10}$ and $u \ge 4$.

Let $g = 10m$. We start with $m = 1, 2$. For $u \in \{4\text{–}12, 14, 15, 18, 19, 23\}$, there exists a $(5, 1)$-cycle frame of type g^u by Lemmas 4.1–4.3 and 4.7. For other values of u, we start from a PBD$(\{4, 5, 6\}, 1, u)$ from Theorem 2.1. Then apply Construction 2.6 to obtain a $(5, 1)$-cycle frame of type g^u. For $m \ge 3$ and $m \not\equiv 2 \pmod 4$, we start from a $(5, 1)$-cycle frame of type 10^u. Applying Construction 2.8 with a $C_5 \wr \overline{K_m}$ which is 5-resolvable by Lemma 2.5, we can obtain a $(5, 1)$-cycle frame of type $(10m)^u$. For $m \ge 3$ and $m \equiv 2 \pmod 4$, let $m = 4e + 2$. Take a $(5, 1)$-cycle frame of type 20^u. We can use Construction 2.8 to obtain a $(5, 1)$-cycle frame of type $(10m)^u$ since $C_5 \wr \overline{K_{2e+1}}$ is 5-resolvable.

(2) $g \equiv 2, 4, 6, 8 \pmod{10}$, $u \ge 4$ and $u \equiv 1 \pmod 5$.

Let $g = 2m$ and $u = 5k + 1$. We begin with $m = 1, 2$. There exist $(5, 1)$-cycle frames of types g^6, g^{11} and g^{16} from Lemmas 4.1–4.3. For other values of u, we start from a cycle frame of type $(5g)^k$ for $k \ge 4$ in case (1). Applying Construction 2.7, we get a $(5, 1)$-cycle frame of type g^{5k+1}. For $m \ge 3$ and $m \not\equiv 2 \pmod 4$, we start from a $(5, 1)$-cycle frame of type 2^{5k+1}. We can apply Construction 2.8 to obtain a $(5, 1)$-cycle frame of type $(2m)^{5k+1}$ since $C_5 \wr \overline{K_m}$ is 5-resolvable. For $m \ge 3$ and $m \equiv 2 \pmod 4$, let $m = 4e+2$, we start from a $(5, 1)$-cycle frame of type 4^{5k+1}. Applying Construction 2.8 with a $C_5 \wr \overline{K_{2e+1}}$ which is 5-resolvable, we can obtain a $(5, 1)$-cycle frame of type $(2m)^{5k+1}$. □

Theorem 4.14. *There exists a* $(5, 2)$*-cycle frame of type* g^u *if* $u \ge 4$ *and* $g(u-1) \equiv 0 \pmod 5$.

Proof. We distinguish the necessary conditions into the following two cases.

(1) $g \equiv 0 \pmod 5$ and $u \ge 4$.

Let $g = 5m$. For $m = 2$, take a $(5, 1)$-cycle frame of type 10^u in case (1) from Theorem 4.13. Apply Construction 2.10 to obtain a $(5, 2)$-cycle frame of type 10^u. Further, we consider the case $m = 1$. For $u \in \{4\text{–}12, 14, 15, 18, 19, 23\}$, there exists a $(5, 2)$-cycle frame of type 5^u by Lemmas 4.8, 4.9 and 4.12. For other values of u, use Construction 2.6 with a PBD$(\{4, 5, 6\}, 1, u)$ to get a $(5, 2)$-cycle frame of type 5^u. For $m \ge 3$ and $m \not\equiv 2 \pmod 4$, we start from a $(5, 2)$-cycle frame of type 5^u constructed above. Then, apply Construction 2.8 to obtain a $(5, 2)$-cycle frame of type $(5m)^u$. For $m \ge 3$ and $m \equiv 2 \pmod 4$, let $m = 4e+2$. Take a $(5, 2)$-cycle frame of type 10^u and apply Construction 2.8 to obtain a $(5, 2)$-cycle frame of type $(5m)^u$.

(2) $g \ge 1$, $u \ge 6$ and $u \equiv 1 \pmod 5$.

Let $u = 5k + 1$. We start with $g = 1, 2$. There exists a $(5, 2)$-cycle frame of type 1^u by Theorem 1.4. Take a $(5, 1)$-cycle frame of type 2^{5k+1} in case (2) from Theorem 4.13. Apply Construction 2.10 to obtain a $(5, 2)$-cycle frame of type 2^{5k+1}. For $g \ge 3$ and $g \not\equiv 2 \pmod 4$, we can obtain a $(5, 2)$-cycle frame of type g^{5k+1} by applying Construction 2.8 with a $(5, 2)$-cycle frame of type 1^{5k+1}. For $g \ge 6$ and $g \equiv 2 \pmod 4$, let $g = 4e + 2$, thus $e \ge 1$. Apply Construction 2.8 with a $(5, 2)$-cycle frame of type 2^{5k+1} to get the required designs. □

By Construction 2.10, Theorems 4.13 and 4.14, we have the following theorem.

Theorem 4.15. *There exists a* $(5, \lambda)$*-cycle frame of type* g^u, *if and only if* $u \ge 4$, $\lambda g \equiv 0 \pmod 2$, *and* $g(u-1) \equiv 0 \pmod 5$.

5. $(6, \lambda)$-cycle frames

In this section, we deal with the existence of a $(6, \lambda)$-cycle frame of type g^u. We begin with some direct constructions for small designs.

Lemma 5.1. *There exists a* $(6, 1)$*-cycle frame of type* 2^u *for* $u \in \{4, 7, 10, 19\}$.

Proof. For $u = 4$, let the point set be $V = Z_6 \cup \{a, b\}$ and the group set be $\{\{i, i+3\} : 0 \le i \le 2\} \bigcup \{a, b\}$. One holey 2-factor is $(0, 1, 2, 3, 4, 5)$ and the other holey 2-factors will be generated from the holey 2-factor $(a, 0, 4, b, 1, 3)$ by $+2 \bmod 6$.

For $u = 7, 19$, let the point set be $V = Z_{2u}$ and the group set be $\{\{i, i+u\} : 0 \le i \le u-1\}$. The required u holey 2-factors will be generated from the initial holey 2-factor F_0^u by $+2 \bmod 2u$. □

F_0^7:	(4, 8, 10, 13, 9, 12)	(1, 2, 3, 5, 11, 6)	
F_0^{19}:	(1, 2, 23, 5, 3, 9)	(4, 7, 22, 8, 33, 11)	(17, 21, 28, 29, 34, 31)
	(6, 18, 26, 20, 30, 35)	(13, 24, 37, 27, 15, 36)	(10, 14, 25, 16, 32, 12)

For $u = 10$, let the point set be $V = Z_{20}$ and the group set be $\{\{i, i+10\} : 0 \le i \le 9\}$. The required 10 holey 2-factors will be generated from the two initial holey 2-factors F_0 and F_1 by $+4 \bmod 20$.

F_0:	(1, 2, 3, 4, 5, 7)	(6, 8, 11, 13, 16, 12)	(9, 17, 14, 19, 15, 18)
F_1:	(10, 2, 6, 19, 4, 17)	(12, 14, 3, 9, 13, 18)	(15, 7, 16, 5, 0, 8)

Lemma 5.2. *There exists a* $(6, 1)$*-cycle frame of type* 6^u *for* $u \in \{5, 6, 9, 11, 15\}$.

Proof. Let the point set be $V = Z_{6u}$ and the group set be $\{\{i, i+u, i+2u, i+3u, i+4u, i+5u\} : 0 \le i \le u-1\}$. For $u = 6$, the required 18 holey 2-factors will be generated from the two initial holey 2-factors F_0 and F_1 by $+4 \bmod 36$. □

F_0:	(7, 9, 11, 14, 10, 15)	(13, 17, 22, 20, 16, 23)	(19, 29, 26, 35, 21, 32)
	(25, 33, 28, 31, 27, 34)	(1, 2, 3, 4, 5, 8)	
F_1:	(27, 11, 34, 23, 2, 16)	(29, 9, 24, 5, 32, 15)	(30, 17, 10, 21, 6, 22)
	(33, 14, 4, 18, 35, 20)	(26, 0, 8, 3, 12, 28)	

For other values of u, the required $3u$ holey 2-factors will be generated from the initial holey 2-factor F_0^u by $+2 \bmod 6u$.

F_0^5:	(13, 24, 17, 21, 19, 27)	(8, 12, 26, 14, 23, 16)	(9, 18, 29, 11, 28, 22)
	(1, 2, 3, 6, 4, 7)		
F_0^9:	(49, 30, 53, 24, 50, 35)	(20, 28, 8, 12, 7, 10)	(2, 43, 22, 15, 17, 34)
	(23, 29, 48, 46, 40, 51)	(16, 5, 42, 21, 25, 47)	(26, 14, 38, 37, 32, 33)
	(44, 4, 19, 39, 31, 6)	(1, 13, 3, 41, 11, 52)	
F_0^{11}:	(64, 37, 65, 46, 62, 32)	(60, 23, 21, 42, 39, 45)	(57, 41, 29, 38, 56, 52)
	(26, 1, 49, 63, 48, 7)	(50, 59, 28, 35, 36, 53)	(34, 8, 10, 20, 40, 17)
	(58, 30, 54, 47, 12, 6)	(25, 15, 61, 19, 27, 4)	(2, 3, 16, 24, 51, 14)
	(5, 9, 43, 13, 18, 31)		
F_0^{15}:	(50, 87, 59, 85, 53, 86)	(63, 62, 78, 39, 49, 43)	(5, 12, 73, 57, 79, 61)
	(21, 18, 51, 64, 16, 80)	(55, 74, 81, 89, 13, 67)	(56, 24, 1, 41, 68, 77)
	(4, 54, 29, 71, 72, 23)	(19, 84, 17, 38, 3, 14)	(69, 20, 47, 36, 65, 26)
	(11, 7, 44, 88, 33, 9)	(25, 42, 37, 83, 31, 34)	(28, 52, 32, 46, 8, 6)
	(2, 58, 40, 48, 76, 82)	(10, 22, 35, 66, 70, 27)	

Lemma 5.3. *There exists a* $(6, 1)$*-cycle frame of type* 6^u *for* $u \in \{8, 12\}$.

Proof. Let the point set be $V = Z_{6u}$ and the group set be $\{\{i, i+u, i+2u, i+3u, i+4u, i+5u\} : 0 \le i \le u-1\}$. The initial holey 2-factor F_0^u will be generated from $u/2$ cycles by $+3u \bmod 6u$ in F^u. All the required $3u$ holey 2-factors will be generated from the initial holey 2-factor F_0^u by $+i \bmod 6u$ for $0 \le i \le 3u-1$. □

F^8:	(5, 11, 6, 17, 37, 18)	(9, 19, 44, 14, 47, 21)	(1, 2, 4, 7, 3, 10)
	(12, 39, 22, 36, 15, 46)		
F^{12}:	(5, 11, 19, 6, 15, 25)	(9, 20, 34, 13, 28, 46)	(1, 2, 4, 7, 3, 8)
	(17, 58, 35, 54, 27, 59)	(14, 30, 52, 26, 69, 31)	(21, 65, 32, 57, 29, 68)

Lemma 5.4. *There exists a* $(6, 1)$*-cycle frame of type* 6^u *for* $u \in \{14, 18\}$.

Proof. Let the point set be $V = (Z_{3u-3} \times Z_2) \cup \{\infty_1, \infty_2, \ldots, \infty_6\}$ and the group set be $\{\{i_0, (i+u-1)_0, (i+2u-2)_0, i_1, (i+u-1)_1, (i+2u-2)_1\} : 0 \le i \le u-2\} \cup \{\infty_1, \infty_2, \ldots, \infty_6\}$. Three holey 2-factors of the group $\{\infty_1, \infty_2, \ldots, \infty_6\}$ will be generated from the F_0^u by $+1 \bmod 3u-3$. All the required holey 2-factors of other groups will be generated from an initial holey 2-factor F_1^u by $+1 \bmod 3u-3$. □

F_0^{14}:	$(0_0, 1_1, 2_0, 0_1, 4_0, 8_1)$		
F_1^{14}:	$(8_0, 22_0, 30_0, 27_0, 31_0, 15_0)$	$(2_1, 3_1, 6_1, 18_1, 24_1, 4_1)$	$(5_1, 27_1, 22_1, 33_1, 25_1, 21_1)$
	$(3_0, 35_1, 23_0, 11_1, 4_0, 14_1)$	$(1_0, 15_1, 10_0, 1_1, 25_0, 7_1)$	$(17_0, 20_1, 34_0, 36_1, 20_0, 37_1)$
	$(9_0, 31_1, 7_0, 30_1, 2_0, 38_1)$	$(14_0, 16_0, 33_0, 18_0, 24_0, 35_0)$	$(5_0, 6_0, 11_0, 21_0, 12_0, 32_0)$
	$(9_1, 19_1, 10_1, 34_1, 16_1, 23_1)$	$(19_0, 28_1, 36_0, \infty_1, 12_1, \infty_4)$	$(28_0, 8_1, 29_0, \infty_2, 17_1, \infty_5)$
	$(37_0, 32_1, 38_0, \infty_3, 29_1, \infty_6)$		
F_0^{18}:	$(0_0, 1_1, 2_0, 0_1, 4_0, 8_1)$		
F_1^{18}:	$(4_1, 22_1, 42_1, 13_1, 15_1, 18_1)$	$(2_0, 7_1, 46_0, 5_1, 45_0, 27_1)$	$(6_0, 48_1, 20_0, 1_1, 29_0, 49_1)$
	$(23_0, 28_0, 37_0, 18_0, 25_0, 19_0)$	$(40_0, 41_0, 1_0, 13_0, 48_0, 15_0)$	$(30_0, 25_1, 22_0, 6_1, 44_0, 8_1)$
	$(9_0, 40_1, 47_0, \infty_2, 31_1, \infty_5)$	$(3_1, 16_1, 9_1, 20_1, 43_1, 33_1)$	$(4_0, 44_1, 38_0, \infty_1, 23_1, \infty_4)$
	$(12_1, 36_1, 45_1, 41_1, 47_1, 28_1)$	$(7_0, 36_0, 16_0, 39_0, 31_0, 21_0)$	$(12_0, 27_0, 24_0, 11_0, 35_0, 14_0)$
	$(42_0, 10_1, 3_0, 24_1, 49_0, 39_1)$	$(8_0, 26_1, 50_0, 29_1, 43_0, 30_1)$	$(11_1, 19_1, 14_1, 2_1, 38_1, 37_1)$
	$(5_0, 21_1, 33_0, 35_1, 26_0, 50_1)$	$(10_0, 46_1, 32_0, \infty_3, 32_1, \infty_6)$	

Lemma 5.5. *There exists a* $(6, 1)$*-cycle frame of type* 6^{23}.

Proof. Let the point set be $V = Z_{69} \times Z_2$ and the group set be $\{\{i_0, (i+23)_0, (i+46)_0, i_1, (i+23)_1, (i+46)_1\} : 0 \le i \le 22\}$. One holey 2-factor F_0^{23} will be generated from $(1_0, 2_0, 5_0, 12_0, 1_1, 61_1)$ and $(5_1, 3_1, 2_1, 10_0, 15_1, 66_0)$ by $(\times 4j \bmod 69, -)$ for $j = 0, \ldots, 10$. The required 69 holey 2-factors will be generated from the holey 2-factor F_0^{23} by $(+1 \bmod 69, -)$. □

Lemma 5.6. *There exists a* $(6, 2)$*-cycle frame of type* 3^u *for* $u \in \{3, 5\}$.

Proof. Let the point set be $V = Z_{3u}$ and the group set be $\{\{i, i+u, i+2u\} : 0 \le i \le u-1\}$. The required $3u$ holey 2-factors will be generated from the initial holey 2-factor F_0^u by $+1 \bmod 3u$, where $F_0^3 = \{(1, 2, 4, 8, 7, 5)\}$ and $F_0^5 = \{(1, 2, 3, 6, 4, 12), (7, 13, 9, 11, 8, 14)\}$. □

Lemma 5.7. *There exists a* $(6, \lambda)$*-cycle frame of type* 6^3 *for* $\lambda = 2, 3$.

Proof. Let the point set be $V = Z_{18}$ and the group set be $\{\{i, i+3, i+6, i+9, i+12, i+15\} : 0 \le i \le 2\}$. For $\lambda = 2$, the required 18 holey 2-factors will be generated from the initial holey 2-factor $F_0 = \{(1, 2, 4, 5, 13, 8), (7, 11, 16, 14, 10, 17)\}$ by $+1 \bmod 18$. For $\lambda = 3$, the required 27 holey 2-factors will be generated from the initial three holey 2-factors F_0, F_1 and F_2 by $+2 \bmod 18$. □

F_0:	(1, 2, 4, 5, 7, 8)	(10, 11, 13, 14, 16, 17)
F_1:	(1, 5, 10, 17, 7, 14)	(11, 13, 2, 4, 8, 16)
F_2:	(2, 7, 11, 1, 14, 10)	(4, 8, 16, 5, 13, 17)

Theorem 5.8. *There exists a* $(6, 1)$*-cycle frame of type* g^u *if* $u \ge 3, g \equiv 0 \pmod 2$*, and* $g(u-1) \equiv 0 \pmod 6$*, except for* $(g, u) = (6, 3)$.

Proof. We distinguish the necessary conditions into the following two cases.

(1) $g \equiv 2, 4 \pmod 6$, $u \ge 4$ and $u \equiv 1 \pmod 3$.

Let $g = 2m$. We start with $m = 1$. There exists a $(6, 1)$-cycle frame of type 2^u for $u \in \{4, 7, 10, 19\}$ by Lemma 5.1. For other values of u, we start from a PBD$(\{4, 7\}, 1, v)$ from Theorem 2.1. Then, a $(6, 1)$-cycle frame of type 2^u can be obtained by

Construction 2.6. Further, we can apply Construction 2.8 to obtain a (6, 1)-cycle frame of type g^u since $C_6 \wr \overline{K_m}$ is 6-resolvable for $m \geq 2$ by Lemma 2.5.

(2) $g \equiv 0 \pmod 6$, $u \geq 3$.

Let $g = 6m$. We begin with $u = 3$. There does not exist a (6, 1)-cycle frame of the type 6^3 since the three holey 2-factors of one hole form a (6, 1)-RCGDD of type 6^2 which does not exist by Theorem 1.1. For $m \geq 2$, a (6, 1)-cycle frame of type $(6m)^3$ can be obtained by using Construction 2.9 with a (2, 1)-frame of type 1^3 and a (6, 1)-RCGDD of type $(6m)^2$ from Theorem 1.1.

Now, we suppose $u > 3$. For $u = 5, 6$, we have a (6, 1)-cycle frame of type 6^u by Lemma 5.2. For $u = 4, 7, 10, 19$, there exists a (6, 1)-cycle frame of type 2^u by case (1). Applying Construction 2.8 with a $C_6 \wr \overline{K_3}$ which is 6-resolvable by Lemma 2.5, we can obtain a (6, 1)-cycle frame of type 6^u. Further, there exists a (6, 1)-cycle frame of type 6^u for any $u \in \{8, 9, 11, 12, 14, 15, 18, 23\}$ by Lemmas 5.2–5.4. For other values of u, we apply Construction 2.6 with a PBD($\{4, 5, 6\}, 1, u$) to obtain a (6, 1)-cycle frame of type 6^u. Thus, there exists a (6, 1)-cycle frame of type 6^u for any $u \geq 4$. Finally, we can apply Construction 2.8 to obtain a (6, 1)-cycle frame of type g^u since $C_6 \wr \overline{K_m}$ is 6-resolvable for $m \geq 2$. □

Theorem 5.9. *There exists a (6, 2)-cycle frame of type g^u if $u \geq 3$ and $g(u-1) \equiv 0 \pmod 6$.*

Proof. We distinguish the necessary conditions into the following four cases.

(1) $g \equiv 0 \pmod 6$ and $u \geq 3$.

Let $g = 6m$, $m \geq 1$. For $u = 3$, we have a (6, 2)-cycle frame of type 6^3 by Lemma 5.7. Applying Construction 2.8 with a $C_6 \wr \overline{K_m}$ which is 6-resolvable by Lemma 2.5, we can obtain a (6, 2)-cycle frame of type $(6m)^3$ for $m \geq 2$. For $u > 3$, we take a (6, 1)-cycle frame of type $(6m)^u$ from Theorem 5.8 and apply Construction 2.10 to obtain a (6, 2)-cycle frame of type g^u.

(2) $g \equiv 1$ or $5 \pmod 6$ and $u \equiv 1 \pmod 6$.

Let $g = 6m+1$ or $6m+5$, $m \geq 0$. Suppose $u = 6k+1$, $k \geq 1$. There exists a (6, 2)-cycle frame of type 1^u by Theorem 1.4. We can apply Construction 2.8 with a $C_6 \wr \overline{K_g}$ which is 6-resolvable by Lemma 2.5 to obtain a (6, 2)-cycle frame of type g^u.

(3) $g \equiv 3 \pmod 6$ and $u \equiv 1 \pmod 2$.

Let $g = 6m + 3$ and $u = 2k + 1$, $m \geq 0$, $k \geq 1$. A (6, 2)-cycle frame of type g^u can be obtained from a (6, 2)-cycle frame of type 3^u by applying Construction 2.8 with a $C_6 \wr \overline{K_{2m+1}}$ which is 6-resolvable by Lemma 2.5. So we only need to construct a (6, 2)-cycle frame of type 3^u. For $k = 1, 2$, a (6, 2)-cycle frame of type 3^5 exists by Lemma 5.6. For $k = 3$, we take a (6, 2)-cycle frame of type 1^7 from Theorem 1.4. Then, apply Construction 2.8 with a $C_6 \wr \overline{K_3}$ which is 6-resolvable by Lemma 2.5 to obtain a (6, 2)-cycle frame of type 3^7. For $k \geq 4$, we take a (6, 1)-cycle frame of type 6^k from Theorem 5.8 and apply Construction 2.10 to obtain a (6, 2)-cycle frame of type 6^k. Further, applying Construction 2.7 with a (6, 2)-cycle frame of type 3^3, we can obtain a (6, 2)-cycle frame of type 3^{2k+1}.

(4) $g \equiv 2$ or $4 \pmod 6$ and $u \equiv 1 \pmod 3$.

This case can be proved by applying Construction 2.10 with a (6, 1)-cycle frame of type g^u from Theorem 5.8. □

Theorem 5.10. *There exists a $(6, \lambda)$-cycle frame of type g^u if and only if $u \geq 3$, $\lambda g \equiv 0 \pmod 2$, and $g(u-1) \equiv 0 \pmod 6$, except for $(\lambda, g, u) = (1, 6, 3)$.*

Proof. There does not exist a (6, 1)-cycle frame of type 6^3 by Theorem 5.8. For $(\lambda, g, u) = (2k+1, 6, 3)$, $k \geq 1$, a $(6, 2k+1)$-cycle frame of type 6^3 can be obtained from a $(6, \lambda)$-cycle frame of type 6^3 ($\lambda = 2, 3$) from Lemma 5.7 by Construction 2.10. For $(\lambda, g, u) \neq (2k+1, 6, 3)$, $k \geq 0$, we apply Construction 2.10 with a (6, 1)-cycle frame of type g^u from Theorem 5.8 and a (6, 2)-cycle frame of type g^u from Theorem 5.9 to get the required designs. □

Combining Theorems 3.4, 4.15 and 5.10, we have proved Theorem 1.5.

6. Constructions for ARCS

In this section, we shall will use cycle frames to construct almost resolvable k-cycle systems for $k \leq 10$. We first present a general construction for an almost resolvable k-cycle system which can also be applied when k is odd.

Construction 6.1. *Suppose there exists a $(k, 1)$-cycle frame of type $(2k)^t$ and a k-ARCS$(2k + 1)$. Then there exists a k-ARCS$(2kt + 1)$.*

Proof. Let $(V, \mathcal{G}, \mathcal{C})$ be a $(k, 1)$-cycle frame of type $(2k)^t$. Suppose the point set be $V = Z_{2kt}$ and $\mathcal{G} = \{G_0, G_1, \ldots, G_{t-1}\}$, $G_i = \{i, i+t, \ldots, i+(2k-1)t\}$, $0 \leq i \leq t-1$. We denote these k holey parallel classes for the group G_i by Q_i^j, $1 \leq j \leq k$. For each group G_i, we construct a k-ARCS$(2k+1)$ on the point set $G_i \cup \{\infty\}$. It has k almost parallel classes denoted by M_i^j, $1 \leq j \leq k$, and a half-parallel class denoted by H_i. It should be mentioned that the point ∞ does not appear in H_i. Let $P = \{P_i^j \mid P_i^j = Q_i^j \cup M_i^j, 1 \leq j \leq k, 0 \leq i \leq t-1\}$ and $H = \{H_i : 0 \leq i \leq t-1\}$. Then P is the set of these required kt almost parallel classes of a k-ARCS$(2kt + 1)$ and H is the required half-parallel class. □

Lemma 6.2. *There exists a $(k, 1)$-cycle frame of type $(2k)^t$ for $k \equiv 0 \pmod 2$, $k \geq 8$ and $t \geq 3$.*

Proof. Starting from a (2, 1)-frame of type 2^t from Theorem 2.2, we can obtain a $(k, 1)$-cycle frame of type $(2k)^t$ by Construction 2.9, where the input design $(k, 1)$-RCGDD of type k^2 comes from Theorem 1.1. □

From Construction 6.1 and the above lemma, we have obtained the same conclusion as Theorem 1.7. Furthermore, we shall prove the existence of an almost resolvable k-cycle system for $k \in \{5, 7, 8, 9\}$. We start with $k = 8$. By Theorem 1.7, we only need to construct a 8-ARCS(17) for recursive constructions.

Theorem 6.3. *There exists a* 8-ARCS$(16t + 1)$ *for* $t = 1$ *and for all* $t \geq 3$.

Proof. For $t = 1$, let the point set be $V = Z_{17}$. The cycle set $\mathcal{C}$ of a 8-ARCS$(16t + 1)$ contains the following 8 almost parallel classes and one half-parallel class.

(0, 16, 2, 12, 8, 13, 1, 14)	(10, 5, 9, 7, 11, 6, 15, 3)
(0, 1, 16, 10, 9, 15, 8, 11)	(4, 3, 12, 6, 7, 13, 14, 5)
(0, 3, 6, 1, 5, 2, 7, 4)	(8, 10, 14, 9, 12, 15, 11, 16)
(0, 5, 8, 1, 7, 3, 9, 13)	(2, 6, 10, 15, 16, 4, 11, 14)
(0, 6, 4, 8, 2, 9, 1, 10,)	(3, 11, 5, 12, 7, 15, 13, 16)
(0, 9, 4, 10, 2, 11, 1, 12)	(3, 8, 6, 13, 5, 16, 7, 14)
(0, 2, 4, 12, 16, 14, 8, 7)	(1, 3, 13, 11, 9, 6, 5, 15)
(0, 8, 9, 16, 6, 14, 4, 15)	(2, 3, 5, 7, 10, 11, 12, 13)
(1, 2, 15, 14, 12, 10, 13, 4)	

For $t \geq 3$, we start with a (8, 1)-cycle frame of type 16^t from Lemma 6.2. Then apply Construction 6.1 with a 8-ARCS(17) to get a 8-ARCS$(16t + 1)$. □

For $k = 5, 7, 9$, we begin with the direct constructions for some designs of small orders. We also construct some $(k, 1)$-cycle frames for later use.

Lemma 6.4. *There exists a* k-ARCS$(2kt + 1)$ *for* $k \in \{5, 7, 9\}$ *and* $t \in \{1, 2, 3\}$.

Proof. Let $X = \{\infty\} \cup Z_{2kt}$. The half-parallel class can be generated from $(0, 2t, 4t, \ldots, 2(k-1)t)$ by $+2 \bmod 2kt$. The required kt almost parallel classes can be generated from the following initial base cycles by $+2 \bmod 2kt$ for $t \in \{1, 3\}$ or $+4 \bmod 2kt$ for $t = 2$. □

$k = 5, t = 1$:	(0, 1, 2, 6, 3)	(∞, 4, 9, 5, 7)	
$t = 2$:	(11, 14, 19, 12, 17)	(∞, 0, 1, 2, 3)	(4, 6, 8, 5, 7)
	(9, 13, 10, 16, 15)		
	(5, 13, 11, 19, 15)	(∞, 1, 6, 0, 10)	(3, 8, 17, 4, 12)
	(2, 9, 18, 7, 14)		
$t = 3$:	(11, 19, 28, 13, 24)	(6, 14, 23, 10, 22)	(3, 5, 8, 12, 17)
	(∞, 15, 25, 29, 18)	(9, 16, 26, 21, 27)	(0, 1, 2, 4, 7)

$k = 7, t = 1$:	(∞, 4, 10, 6, 13, 7, 11)	(0, 1, 2, 5, 3, 12, 9)	
$t = 2$:	(∞, 8, 21, 13, 27, 20, 10)	(4, 15, 6, 23, 19, 25, 16)	(5, 9, 7, 17, 22, 12, 18)
	(0, 1, 2, 3, 11, 26, 14)		
	(∞, 9, 20, 15, 24, 16, 19)	(5, 11, 14, 6, 12, 27, 17)	(7, 8, 13, 22, 25, 18, 23)
	(0, 2, 4, 1, 3, 10, 21)		
$t = 3$:	(∞, 15, 29, 41, 30, 23, 32)	(5, 37, 22, 36, 20, 39, 31)	(8, 28, 16, 34, 9, 26, 35)
	(11, 14, 24, 13, 19, 38, 33)	(6, 10, 17, 12, 25, 27, 40)	(0, 1, 2, 4, 7, 3, 21)

$k = 9, t = 1$:	(0, 1, 2, 5, 3, 6, 10, 15, 7)	(∞, 4, 14, 8, 17, 11, 16, 9, 13)
$t = 2$:	(∞, 9, 31, 20, 34, 10, 29, 12, 32)	(0, 27, 19, 25, 1, 14, 3, 30, 22)
	(2, 8, 18, 35, 4, 17, 26, 11, 15)	(5, 21, 13, 28, 7, 23, 33, 24, 6)
	(∞, 18, 34, 28, 17, 31, 25, 35, 23)	(4, 6, 8, 13, 12, 11, 10, 20, 27)
	(14, 21, 26, 19, 22, 33, 15, 32, 29)	(0, 3, 1, 5, 2, 7, 9, 16, 24)
$t = 3$:	(13, 24, 14, 22, 31, 19, 29, 18, 32)	(17, 34, 53, 38, 50, 35, 44, 51, 37)
	(12, 28, 49, 27, 43, 20, 47, 21, 45)	(6, 11, 16, 9, 15, 33, 41, 10, 36)
	(∞, 23, 48, 30, 52, 39, 26, 46, 42)	(0, 1, 2, 4, 7, 3, 5, 8, 25)

Theorem 6.5. *There exists a* $(7, 1)$*-cycle frame of type* 14^u *for any* $u \geq 4$*.*

Proof. For $u = 5$, let the point set be $V = Z_{70}$ and the group set be $\{\{i, i+5, \ldots, i+65\} : 0 \leq i \leq 4\}$. First, we can obtain an initial holey 2-factor F_0 from the four cycles: $(11, 24, 47, 26, 64, 18, 62)$, $(6, 13, 22, 8, 19, 31, 49)$, $(1, 2, 4, 7, 3, 9, 17)$, $(16, 33, 67, 28, 69, 21, 58)$ by $+35 \mod 70$. Then, all the other 34 holey 2-factors can be generated from F_0 by $+i \mod 70$, $i = 1, 2, \ldots, 34$.

For $u \in \{4, 7, 10, 19\}$, we begin with a $(3, 1)$-cycle frame of type 2^u from Theorem 1.3. Applying Construction 2.8 with a $C_3 \wr \overline{K_7}$ which is 7-resolvable by Lemma 2.3, we can obtain a $(7, 1)$-cycle frame of type 14^u. For $u \in \{6, 11\}$, we start with a $(5, 1)$-cycle frame of type 2^u from Theorem 4.15. Then we apply Construction 2.8 with a $C_5 \wr \overline{K_7}$ which is 7-resolvable by Lemma 2.3 to obtain a $(7, 1)$-cycle frame of type 14^u.

For $u = 8$, we first construct a $(7, 1)$-cycle frame of type 2^8. Let the point set be $V = Z_{16}$ and the group set be $\{\{i, i+8\} : 0 \leq i \leq 7\}$. The required 8 holey 2-factors will be generated from the initial holey 2-factor $\{(1, 2, 4, 7, 3, 13, 6), (9, 10, 12, 15, 11, 5, 14)\}$ by $+i \mod 16$, $i = 0, 1, 2, 3, 4, 5, 6, 7$. Thus, a $(7, 1)$-cycle frame of type 14^8 can be obtained by applying Construction 2.8 since $C_7 \wr \overline{K_7}$ is 7-resolvable. For $u \in \{9, 12, 14, 15, 18, 23\}$, we take a $(\{3, 5\}, 1)$-cycle frame of type 2^u from Lemmas 4.4–4.6. Then we apply Construction 2.8 with a $C_5 \wr \overline{K_7}$ and a $C_3 \wr \overline{K_7}$ which are both 7-resolvable by Lemma 2.3 to obtain a $(7, 1)$-cycle frame of type 14^u.

For other values of u, we start from a PBD$(\{4, 5, 6\}, 1, u)$ from Theorem 2.1. Applying Construction 2.6, we can obtain a $(7, 1)$-cycle frame of type 14^u, where the input designs $(7, 1)$-cycle frames of types 14^4, 14^5 and 14^6 have been constructed above. □

Theorem 6.6. *There exists a* $(9, 1)$*-cycle frame of type* 18^u *for any* $u \geq 4$*.*

Proof. Take a $(3, 1)$-cycle frame of type 6^u from Theorem 1.3. Applying Construction 2.8 with a $(9, 1)$-RCGDD of type 3^3 from Theorem 1.1, we can get a $(9, 1)$-cycle frame of type 18^u. □

Theorem 6.7. *There exists a* k-ARCS$(2kt + 1)$ *for* $k \in \{5, 7, 9\}$ *and* $t \geq 1$*.*

Proof. For $t \in \{1, 2, 3\}$, these designs exist by Lemma 6.4. For $t \geq 4$, we start from a $(k, 1)$-cycle frame of type $(2k)^t$ which exists by Theorems 4.15, 6.5 and 6.6. Applying Construction 6.1 with a k-ARCS$(2k + 1)$ from Lemma 6.4, we can get a k-ARCS$(2kt + 1)$. □

Combining Theorems 1.6, 6.3 and 6.7, we have proved Theorem 1.8.

7. Concluding remarks

In this paper, we have solved the existence of a (k, λ)-cycle frame with type g^u for $4 \leq k \leq 6$. We also obtain some results for $k \geq 7$. These results lead to some progress on the existence of a k-ARCS$(2k + 1)$. Cycle frames also play a significant role in the construction for the well-known Oberwolfach Problems, see [6,14,15]. Suppose $m_i \geq 3$ and $\alpha_i \geq 1 (1 \leq i \leq t)$ are integers. Let $n = \sum_{i=1}^{t} m_i \alpha_i$. The Oberwolfach Problem OP$(m_1^{\alpha_1}, m_2^{\alpha_2}, \ldots, m_t^{\alpha_t})$ is to determine whether the edges of K_n (for n odd) or K_n minus a 1-factor (for n even) can be partitioned into isomorphic 2-factors such that each 2-factor consists of exactly α_i cycles of length m_i. The problem was formulated by Ringel at a graph theory conference in 1967. With these cycle frames constructed in this paper, we can obtain some new OPs by using the recursive constructions in [6,14,15]. For example, we have solved the existence of an OP$(5^a, s^1)$ for $3 \leq s \leq 7$ completely. Thus, it is necessary to research the existence of cycle frames. Currently, we just get some results with small cycle size k. The general problem is to show the existence of a (k, λ)-cycle frames for any $k \geq 7$ and $\lambda \geq 1$.

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