Note

Proof of a conjecture in domination theory

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Abstract

A dominating set $D$ of a graph $G$ is a least dominating set (l.d.s) if $\gamma(\langle D \rangle) \leq \gamma(\langle D_1 \rangle)$ for any dominating set $D_1$ ($\gamma$ denotes domination number). The least domination number $\gamma_1(G)$ of $G$ is the minimum cardinality of a l.d.s. We prove a conjecture of Sampathkumar (1990) that $\gamma_1 \leq 3p/5$ for any connected graph $G$ of order $p \geq 2$. © 1998 Elsevier Science B.V. All rights reserved

We basically follow the standard terminology of [1]. Let $G = (V, E)$ be a graph. For a set $S \subseteq V$, let $\langle S \rangle$ be the subgraph of $G$ induced by $S$. A set $D \subseteq V$ is a dominating set if every point in $V - D$ is adjacent to some point in $D$. The domination number $\gamma(D)$ of $G$ is the minimum cardinality of a dominating set. A dominating set $D$ is a least dominating set (l.d.s.) if $\gamma(\langle D \rangle) \leq \gamma(\langle D_1 \rangle)$ for every dominating set $D_1$. The least domination number $\gamma_1(G)$ of $G$ is the minimum cardinality of a l.d.s.

Now we prove a conjecture stated in [2].

Theorem 1. For any connected graph $G$ with $p \geq 2$ points

$$\gamma_1 \leq 3p/5.$$ 

Proof. Denote by $D$ a minimum l.d.s. of $G$ with minimal possible number of isolates in $\langle D \rangle$. Choose a minimum dominating set $F$ of $\langle D \rangle$. Denote by $I$ the set of all isolates in $\langle D \rangle$. Clearly, $I \subseteq F$. For $A, B \subseteq V$ ($A \cap B = \emptyset$) and $a \in A$ put

$$q(a, A, B) = \{b \in B: N(b) \cap A = \{a\}\}.$$ 

First we show that $q(i, D, V - D) \neq \emptyset$ for any $i \in I$. Suppose it is not so, i.e., every point in $N(i)$ is adjacent to a point in $D - i$. Since $G$ is connected graph and $p \geq 2$, there is

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a point $i_1$ in $N(i)$. Clearly $i_1 \in V - D$. Then $D' = (D - i) \cup i_1$ is a minimum I.d.s. and the number of isolates in $\langle D' \rangle$ is less than that of in $\langle D \rangle$, a contradiction.

Now let $X = F - I$ and $Y = D - F$. Since $F$ is a minimum dominating set in $\langle D \rangle$ and $X \cap I = \emptyset$, then $q(x, F, Y) = \emptyset$ for any $x \in X$. Put

$$X_1 = \{x \in X: |q(x, F, Y)| = 1\}$$

and

$$X_2 = X - X_1 = \{x \in X: |q(x, F, Y)| \geq 2\}.$$

Clearly $q(y, D, V - D) \neq \emptyset$ for any $y \in Y$ (otherwise $y$ can be deleted from $D$). Similarly, $q(x_1, D, V - D) \neq \emptyset$ for any $x_1 \in X_1$. Indeed, by definition of $X_1$ the set $q(x_1, F, Y)$ consists of a single point $y_1$. If $q(x_1, D, V - D) = \emptyset$ than $D_1 = D - x_1$ is a dominating set of $G$ and

$$\gamma(\langle D_1 \rangle) = |(F - x_1) \cup y_1| = |F| = \gamma(\langle D \rangle),$$

which contradicts the minimality of $D$: $|D_1| = |D| - 1$. Thus, we have shown that $q(z, D, V - D) \neq \emptyset$ for any $z \in D - X_2$. Hence

$$|V - D| \geq |D - X_2|$$

or

$$p - \gamma_1 \geq \gamma_1 - |X_2|$$

or

$$\gamma_1 \leq (p + |X_2|)/2.$$

Further, by definition of $X_2$ we have

$$|Y| \geq 2|X_2|.$$

Then

$$p = |V - D| + |D|$$

$$\geq |D - X_2| + |D| \quad \text{(by (1))}$$

$$= 2|D| - |X_2|$$

$$\geq 2(|Y| + |X_2|) - |X_2|$$

$$= 2|Y| + |X_2|$$

$$\geq 5|X_2| \quad \text{(by (3))},$$

i.e.,

$$|X_2| \leq p/5.$$

Finally, substitute (4) into (2): $\gamma_1 \leq (p + p/5)/2 = 3p/5$. □

References
