# THE CHROMATIC INDEX OF GRAPHS WITH LARGE MAXIMUM DEGREE 

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Received 16 March, 1981
Revised 9 November 1982


#### Abstract

By Vizing's Theorem, any graph $G$ has chromatic index equal either to its maximum degree $\Delta(G)$ or $\Delta(G)+1$. A simple method is given for determining exactly the chromatic index of any graph with $2 s+2$ vertices and maximum degree $2 s$.


## 1. Introduction

We consider only undirected graphs without loops or multiple edges. The set of vertices of a graph is denoted by $V(G)$; the cardinality $p(G)$ of $V(G)$ is the order of $G$. Similarly the edge-set of $G$ is denoted by $E(G)$, and its cardinality, $q(G)$, is the size of $G$. Given a graph $G$ and a set $R \subset V(G)$, we let $\langle R\rangle$ denote the subgraph of $G$ induced by the vertices of $R$, and let $G-R$ denote the subgraph of $G$ induced by those vertices of $G$ which are not contained in $R$.

A coloring of a graph is an assignment of colors to its edges so that no two adjacent edges are assigned the same color. The chromatic index of a graph $G$, denoted $\chi^{\prime}(G)$, is the minimum number of colors used among all colorings of $G$. Vizing [11] has shown that for any graph $G, \chi^{\prime}(G)$ is either its maximum degree $\Delta(G)$ or $\Delta(G)+1$. If $\chi^{\prime}(G)=\Delta(G)$ then $G$ is in Class 1 ; otherwise $G$ is in Class 2. A vertex $v$ in a colored graph is said to miss a color $C$ (and similarly $C$ misses $v$ ) if no edge which is assigned color $C$ is incident with $v$. We follow [5] for most other terminology and notation.

Despite the tight bounds supplied by Vizing's Theorem, the problem of determining exactly the chromatic index of an arbitrary graph is an extremely difficult one; Holyer [6] has recently shown it to be NP-complete. Nevertheless, some good results have been obtained for graphs with additional properties.

Theorem A [8]. If $G$ is bipartite, then $G$ is in Class 1.
Theorem B [12]. The complete graph $K_{p}$ is in Class 1 if $p$ is even and in Class 2 if $p$ is odd.

Theorem $\mathbf{C}$ [1]. Let $G$ have order $2 s+1$ and maximum degree $r$. If the size of $G$ is at least $s r+1$, then $G$ is in Class 2.

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Theorem D [10]. If $G$ has order $2 s$ and maximum degree $2 s-1$, then $G$ is in Class 1. If $G$ has order $2 s+1$ and maximum degree $2 s$, then $G$ is in Class 2 if and only if the size of $G$ is at least $2 s^{2}+1$.

A non-increasing sequence $F=\left(f_{1}, \ldots, f_{n}\right)$ of non-negative integers is feasible for a graph $G$ if there exists an $n$-coloring of $G$ for which the cardinalities of the $n$ color classes are precisely $f_{1}, \ldots, f_{n}$. The following result was obtained independently by McDiarmid [9] and De Werra [2].

Theorem E. If the non-increasing sequence $F=\left(f_{1}, \ldots, f_{n}\right)$ is feasible for $G$, then so is any non-increasing sequence $F^{\prime}=\left(f_{1}^{\prime}, \ldots, f_{n}^{\prime}\right)$ such that

$$
\sum_{i=1}^{n} f_{i}^{\prime}=\sum_{i=1}^{n} f_{i} \quad \text { and } \quad \sum_{i=1}^{k} f_{i}^{\prime} \leqslant \sum_{i=1}^{k} f_{i} \quad \text { for } k=1, \ldots, n-1
$$

A coloring is equitable if the cardinalities of any two color classes differ by at most one. So if the graph $G$ can be $n$-colored, then by Theorem $E$ there is an equitable $n$-coloring of $G$.

For a non-increasing sequence $F=\left(f_{1}, \ldots, f_{n}\right)$ and non-negative integer $f$, let $f^{*}=\max \left\{i \mid f_{i} \geqslant f\right\}$. Thus $f^{*}$ is the number of elements in $F$ which are greater than or equal to $f$. We will use the following result, due to Folkman and Fulkerson.

Theorem F [3]. Assume $f_{1}=f_{2}=\cdots=f_{h} \geqslant f_{h+1}=\cdots=f_{n}$, and let $F=\left(f_{1}, \ldots, f_{n}\right)$. Let $B$ be a bipartite graph with size $\sum_{f=1}^{\infty} f^{*}$. Then $F$ is feasible for $B$ if and only if

$$
q(B-X) \geqslant \sum_{f=|X|+1}^{\infty} f^{*} \quad \text { for all } X \subset V(B)
$$

## 2. The chromatic index of graphs with order $2 s+2$ and maximum degree $2 s$

Theorem D seems to indicate that graphs with maximum degree 'close to' the order of the graph behave more predictably than arbitrary graphs. We extend those results with the following theorem.

Theorem 1. Let $G$ be a graph with order $2 s+2$ and maximum degree $2 s$. Then $G$ is in Class 2 if and only if it contains a vertex $v$ such that $G-v$ has size at least $2 s^{2}+1$.

Proof. If there is such a vertex $v$, then $\chi^{\prime}(G-v)=2 s+1$ by Theorem D. It then follows that $\chi^{\prime}(G)=2 s+1$, so that $G$ is in Class 2 .

The necessity 'half' of the proof is much more difficult. Suppose there is no such vertex $v$. Then $q(G)-\delta(G) \leqslant 2 s^{2}$, where $\delta(G)$ denotes the minimum degree of $G$. We want to construct a $2 s$-coloring of $G$.

Case 1. $q(G)-\delta(G)=2 s^{2}$.

Let $u$ be a vertex of minimum degree in $G$. By theorem $D, \chi^{\prime}(G-u)=2 s$. Consider any $2 s$-coloring of $G-u$. Since every color class in the coloring can have cardinality at most $s$, each must have cardinality exactly $s$; that is, each color misses exactly one vertex of $G-u$. Now to each edge $u w_{i}$ of $G$ incident with $u$ assign any color presently missing $w_{i}$ (there must be at least one color missing $w_{i}$, for otherwise $w_{i}$ has degree $2 s+1$ in $G$ ). This yields a valid $2 s$-coloring of $G$ since if $w_{i} \neq w_{j}$ then no color misses both $w_{i}$ and $w_{j}$.

Case 2. $q(G)-\delta(G)<2 s^{2}$.
If there is an edge in the complement of $G$ which can be added to $G$ without increasing $\Delta(G)$, we add it. Continue this process until a graph $H$ is obtained which has order $2 s+2$ and maximum degree $2 s$, contains $G$ as a subgraph, and additionally has one of the following two properties.
(i) $q(H)-\delta(H)=2 s^{2}$.
(ii) $q(H)-\delta(H)<2 s^{2}$ and no edge of $\bar{H}$ can be added to $H$ without increasing $\Delta(H)$.

If $q(H)-\delta(H)=2 s^{2}$, then by Case 1 the graph $H$ is $2 s$-colorable, so that $G$ is $2 s$-colorable.

In order to handle the second possibility we introduce some new terminology. A graph $H$ is saturated if no edge of $\bar{H}$ can be added to $H$ without increasing its maximum degree. A saturated graph $H$ with $2 s+2$ vertices is weakly saturated if $q(H-v)<s \cdot \Delta(H)$ for any vertex $v$ of $H$. Then by the previous discussion, to complete the proof it suffices to show that any weakly saturated graph with order $2 s+2$ and maximum degree $2 s$ is $2 s$-colorable. It should be noted that such weakly saturated graphs do exist; the complements of the five smallest are given in Table 1.

Table 1. The complements of the five smallest weakly saturated graphs with order $2 s+2$ and maximum degree $2 s$

| Order | Complement |
| :---: | :---: |
| 12 | $3 K_{1,3}$ |
| 12 | $4 K_{1,2}$ |
| 14 | $2 K_{1,4} \cup K_{1,3}$ |
| 14 | $2 K_{1,3} \cup 2 K_{1,2}$ |
| 14 | $4 K_{1,2} \cup K_{1,1}$ |

Since $H$ is saturated and $\Delta(H)=p(H)-2, \bar{H}$ is the disjoint union of stars. Let $\bar{H}$ be $K_{1, t_{1}} \cup K_{1, t_{2}} \cup \cdots \cup K_{1, t_{n}}$ where $t_{i} \geqslant t_{i+1}$ for $1 \leqslant i \leqslant n-1$. Note that $2 s+2=$ $n+\sum_{i=1}^{n} t_{i}$ and $\sum_{i \neq j} t_{i}>s$ for any $j$ because $H$ is weakly saturated. Let $z=\sum_{i=1}^{n} t_{i}$.

Subcase 1. $n$ is even.
Let $w_{i}$ denote the vertex with degree $t_{i}$ in component $i$ of $\bar{H}$ (if $t_{i}=1$ let $w_{i}$ be
either of the two vertices in component $i$ ). Let $W=\left\{w_{1}, \ldots, w_{n}\right\}$ and let $U=$ $V(G-W)=\left\{u_{1}, u_{2}, \ldots, u_{z}\right\}$. The edges of $H$ will be $2 s$-colored in three stages.

Stage 1. Coloring the edges of $\langle U\rangle$.
Notice that $\langle U\rangle$ is simply the complete graph with $z$ vertices. Since $n$ is even, $z$ must be also. It follows from Theorem B that $\langle U\rangle$ is $(z-1)$-colorable, that is, $\langle U\rangle$ has a factorization into $z-1$ disjoint 1 -factors. Color $n-1$ of these 1 -factors with the colors $C_{1}, C_{2}, \ldots, C_{n-1}$. Color $\frac{1}{2}(z-n)$ of the remaining 1 -factors equitably with the colors $M_{1}, \ldots, M_{(z-2) / 2}$; color the other $\frac{1}{2}(z-n) 1$-factors equitably with the colors $N_{1}, \ldots, N_{z / 2}$. Then letting $C=\left\{C_{1}, \ldots, C_{n-1}\right\}, M=\left\{M_{1}, \ldots, M_{(z-2) / 2}\right\}$ and $N=\left\{N_{1}, \ldots, N_{z / 2}\right\}$, it is easily verified that
(i) the colors of $C$ each miss 0 vertices of $\langle U\rangle$;
(ii) $\frac{1}{2}(z-n)$ colors of $M$ each miss $n-2$ vertices of $U$, while the remaining $\frac{1}{2}(n-2)$ colors of $M$ each miss $n$ vertices of $\langle U\rangle$;
(iii) the colors of $N$ each miss $n$ vertices of $\langle U\rangle$.

Stage 2. Coloring the edges joining $\langle U\rangle$ and $\langle W\rangle$.
We want to extend the coloring from Stage 1 to include all edges between $\langle U\rangle$ and $\langle W\rangle$. First we require the following observation.

Lemma 1. Let $H_{1}$ and $H_{2}$ be graphs, each constructed by beginning with the complete graph $K_{z}$ and $n$ isolated vertices, and adding edges from the isolated vertices to the vertices of $K_{z}$ in such a way that each vertex from $K_{z}$ is now adjacent to all but one of the previously isolated vertices. If the degree sequences of the previously isolated vertices are identical (except possibly for ordering) in $\mathrm{H}_{1}$ and $\mathrm{H}_{2}$, then $H_{1}$ and $H_{2}$ are isomorphic.

Proof of Lemma 1. The graphs $\bar{H}_{1}$ and $\bar{H}_{2}$ are both $K_{n}$ with $z$ vertices of degree 1 attached. Since the degree sequences of $\bar{H}_{1}$ and $\bar{H}_{2}$ are identical, there is an obvious isomorphism between these two graphs. Therefore, $H_{1}$ and $H_{2}$ are isomorphic.

Returning now to the proof of Theorem 1 , let $U_{i}$ be the set of colors missing vertex $u_{i}$ after Stage 1 for $i=1, \ldots, z$. Then each $U_{i}$ has cardinality $n-1$. Construct the bipartite graph $B$ with vertex set

$$
V(B)=\left\{U_{1}, \ldots, U_{z}, M_{1}, \ldots, M_{(z-2) / 2}, N_{1}, \ldots, N_{z / 2}\right\}
$$

and edge set

$$
E(B)=\left\{U_{i} M_{j} \mid M_{j} \in U_{i}\right\} \cup\left\{U_{i} N_{j} \mid N_{i} \in U_{i}\right\}
$$

By Lemma 1, extending the coloring from Stage 1 to include all edges between $U$ and $W$ is equivalent to finding an $n$-coloring of $B$ with colors $D_{n}, D_{n-1}, \ldots, D_{1}$ and cardinality sequence $\left(z-t_{n}, \ldots, z-t_{1}\right)$; coloring edge $U_{i} M_{j}$ with color $D_{k}$ indicates that an edge between $u_{i}$ and $w_{k}$ in $H$ should be assigned color $M_{j}$, and similarly coloring the edge $U_{i} N_{j}$ with color $D_{k}$ in $B$ corresponds to coloring an edge between $u_{i}$ and $w_{k}$ with the color $N_{j}$ in the graph $H$.

In order to determine the existence of such a coloring of $B$, it suffices to show that the cardinality sequence

$$
F=\left(z-1, \ldots, z-1, \frac{1}{2}(z+n-2), \frac{1}{2}(z+n-2)\right)
$$

of length $n$ is feasible for $B$. Then the desired sequence will be feasible by Theorem $E$, since $z-t_{i} \geqslant \frac{1}{2}(z+n-2)$ for any $i$, because otherwise the complement of $H-w_{i}$ has less than $\frac{1}{2}(z+n-2)=\frac{1}{2}(2 s)=s$ edges, a contradiction. It is straightforward to verify that the graph $B$ and the sequence $F$ together satisfy the conditions of Theorem $F$ : a subset $X$ of $V(B)$ with order at most $\frac{1}{2}(z+n-2)$ can be incident with at most $\frac{1}{2} n(z+n-2)$ edges of $B$ since $\Delta(B)=n$, and if $X$ has cardinality $r$ between $1+\frac{1}{2}(z+n-2)$ and $z-1$ it covers at most $\frac{1}{2} n(z+n-2)+$ $\frac{1}{2}(n-2)(2 r-z-n+2)$ edges of $B$ because each $U_{i}$ is adjacent to $\frac{1}{2} n$ of the $N_{i}$. Consequently, $F$ is a feasible sequence for $B$ and it follows by the preceding argument that the edges between $\langle U\rangle$ and $\langle W\rangle$ can be colored using only colors in $M$ and $N$.

Stage 3. Coloring $\langle W\rangle$.
In Stage 2, no colors of $C$ were assigned to any edges incident with vertices of $\langle W\rangle$. Thus, by Theorem B, we can complete the coloring of $H$ by coloring $\langle W\rangle$ with the colors $C_{1}, \ldots, C_{n-1}$. A $2 s$-coloring of $H$ is thereby obtained, and the proof of the theorem is complete for the subcase when $n$ is even.

Subcase 2. $n$ is odd.
If $n$ is odd, define $U$ and $W$ as in the proof of Subcase 1 . Choose any vertex $u$ in $\langle U\rangle$ which is not adjacent to $w_{1}$. Remove $u$ from $U$ and place it in $W$, renaming it $w_{n+1}$. A proof similar to, although somewhat more complicated than, that of Subcase 1 can now be used to obtain the theorem.

## 3. Final remarks

A connected graph $G$ is critical if $\chi^{\prime}(G)=\Delta(G)+1$ and $\chi^{\prime}(G-e)<\chi^{\prime}(G)$ for each edge $e$ of $G$. Several authors [1,7] conjectured that all critical graphs have an odd number of vertices. Goldberg [4] recently disproved this Critical Graph Conjecture. However, because of Theorem D, we can restate Theorem 1 in the following equivalent form.

Theorem 1'. There are no critical graphs with order $2 s+2$ and maximum degree $2 s$.

It is natural to ask if the predictability of the chromatic index of graphs with high maximum degree that is demonstrated in Theorems $D$ and 1 can be extended any further. We close with the following conjectures. Conjecture 1 is a special case of a conjecture due to Vizing [13].

Conjecture 1. A graph $G$ with order $2 s+1$ and maximum degree $2 s-1$ is in Class 2 if and only if it has size at least $2 s^{2}-s+1$.

Conjecture 2. A graph $G$ with order $2 s+2$ and maximum degree $2 s-1$ is in Class 2 if and only if $q(G)-\delta(G) \geqslant 2 s^{2}-s+1$.

## Acknowledgement

I would like to thank the referee for his many helpful suggestions.

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