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Generalized differentiation and fixed points sets behaviors with respect to Fisher convergence ☆

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ABSTRACT

We study the stability of a new concept of generalized differentiation for set-valued mappings involving positively homogeneous maps with respect to a variational convergence deriving from the proximal topology. This notion of convergence, well-suited for set-valued mappings, allows us as well to establish the continuous dependence of fixed points sets of set-valued contractions.

1. Introduction

In a seminal paper dealing with differential calculus of nondifferentiable mappings, Ioffe [17] introduced a general class of objects, for local approximation of nonsmooth single-valued function, consisting of set-valued mappings which are positively homogeneous and closed-valued. Such objects will be called *prederivatives.* His definition extended the framework of differential calculus to more general classes of functions. Usual Fréchet or strict Fréchet derivatives are, respectively, derivatives and strict prederivatives in the sense of Ioffe's definition. Moreover, derivative containers of Warga [28] and screens and fans of Halkin [16] belong to this class. More, the contingent derivative introduced by Aubin (see, *e.g.,* [2,3]) coincides with the weak derivative when it exists. For a comprehensive study of this topic, one may refer to [17] where the relationships between the above concepts and several other notions of generalized derivatives, such as generalized gradients of Clarke [8] and derivatives introduced by Mordukhovich [20], are investigated in detail.

Ever since, a growing literature on generalized differentiation attests the importance of this topic, especially in variational analysis where this kind of tools happen to be crucial. In particular, many efforts have been devoted to developing the theory of generalized differentiation of set-valued maps; for an overview of the State-of-the-Art, one may refer to the monograph by Mordukhovich [21].

Lately, Pang [23] extended the work of Ioffe by proposing a concept of generalized differentiation for set-valued mappings involving positively homogeneous maps. Here is the definition proposed by Pang.

Definition 1.1. (See Pang [23].) Let X and Y be Banach spaces. Let H be a positively homogeneous set-valued mapping from X into Y.

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(a) We say that a set-valued mapping S from X into Y is *outer H-differentiable* at $\bar{x}$ if for any $\delta > 0$, there exists a neighborhood V of $\bar{x}$ such that

$$S(x) \subset S(\bar{x}) + H(x - \bar{x}) + \delta\|x - \bar{x}\|\mathbb{B} \quad \text{for all } x \in V. \tag{1}$$

It is *inner H-differentiable* at $\bar{x}$ if the formula above is replaced by

$$S(\bar{x}) \subset S(x) - H(x - \bar{x}) + \delta\|x - \bar{x}\|\mathbb{B} \quad \text{for all } x \in V. \tag{2}$$

It is *H-differentiable* at $\bar{x}$ if it is both outer H-differentiable and inner H-differentiable.

(b) We say that a set-valued mapping S from X into Y is *strictly H-differentiable* at $\bar{x}$ if for any $\delta > 0$, there exists a neighborhood V of $\bar{x}$ such that

$$S(x) \subset S(x') + H(x - x') + \delta\|x - x'\|\mathbb{B} \quad \text{for all } x, x' \in V. \tag{3}$$

Obviously, when S is a single-valued map the definitions of outer H-differentiability, inner H-differentiability and H-differentiability in the above definition coincide. Note also that if S is strictly H-differentiable with respect to a homogeneous map H satisfying $H(x) \subset \kappa\|x\|\mathbb{B}$ for all $x \in X$ then S is Lipschitz continuous with constant κ. When S is outer H-differentiable with respect to such a map H, then it is calm at $\bar{x}$. For a comprehensive exposition of the concepts of Lipschitz continuity and calmness for set-valued mappings the reader could refer to [12]. Note also that recent developments regarding the H-differentiability are available in [10].

In this paper, we are primarily interested in studying the stability of the (strict) H-differentiability of set-valued maps. More precisely, we wonder whether the limit of a sequence of differentiable set-valued mappings (in the sense of Definition 1.1) is differentiable. In the very simple case of a sequence of real differentiable functions $f_n : \mathbb{R} \to \mathbb{R}$, we know that if f_n converges pointwise to f and f_n' (the sequence of derivatives) converges uniformly to a function g, then f is differentiable and $f' = g$. In the last decades, this question has been investigated by many authors who studied the stability of several concepts of derivatives or subdifferentials, see *e.g.,* [1,3,6,9,11,15]. Carrying such a study, in the framework of generalized differentiation of set-valued maps, is the purpose of this paper. To this end, in Section 2, we first have a closer look at positively homogeneous mappings since they play a central role in the concept of differentiation we are interested in. Then, in Section 3, we endeavor to propose a suitable topology from which we derive the notion of convergence – the so-called Fisher convergence – required for establishing our stability results. We will say that a sequence F_n of closed-valued mappings acting between two Banach spaces X and Y Fisher converges to a closed-valued mapping F if for all $x \in X$, $F(x) \subset \liminf_n F_n(x)$ and $\lim_n \sup_{x \in X} e(F_n(x), F(x)) = 0$, where $e(A, B)$ denotes the *excess* of the set A over the set B (see the notation at the end of the present section). Thanks to the Fisher convergence we are able to state our main result regarding the stability of the (strict) H-differentiability of set-valued maps. Given a converging sequence F_n of closed-valued mappings and a sequence H_n of homogeneous set-valued mappings, we prove that under some assumption of uniform H_n-differentiability of the sequence F_n, the Fisher limit F of F_n is (strictly) H-differentiable whenever H_n converges (in some sense) to a homogeneous mapping H with finite outer norm.

Finally, by considering another problem, namely the data dependence of fixed-points sets of set-valued mappings, we expand the scope of application of the convergence we introduced. The rest of the present section is devoted to collecting the notation we shall use in the sequel.

Notation. Let (E, d) be a metric space. If $x \in E$ and $\rho > 0$, then the open ball with center x and radius ρ is $\mathbb{B}_\rho(x) := \{z \in E \mid d(z, x) < \rho\}$ and the closed ball with center x and radius ρ is $\overline{\mathbb{B}}_\rho(x) := \{z \in E \mid d(z, x) \leqslant \rho\}$. The open (respectively closed) unit ball will be denoted by $\mathbb{B}$ (respectively by $\overline{\mathbb{B}}$). If $A \subset E$ and $\varepsilon > 0$, we denote by $\mathbb{B}_\varepsilon(A) := \bigcup_{a \in A} \mathbb{B}_\varepsilon(a)$ the *ε-enlargement* of A.

If A and B are two subsets of (E, d), the excess of A over B (with respect to d) is defined by the formula

$$e(A, B) = \sup_{a \in A} d(a, B).$$

It is clear that

$$e(A, B) = \inf\{\varepsilon > 0 \mid A \subset B + \varepsilon\mathbb{B}\} = \inf\{\varepsilon > 0 \mid A \subset \mathbb{B}_\varepsilon(B)\}.$$

We adopt the convention that $e(\emptyset, B) = 0$ when $B \neq \emptyset$ and $e(\emptyset, B) = \infty$ if $B = \emptyset$.

Throughout, X and Y stand for real Banach spaces. Let F be a set-valued mapping from X into the subsets of Y, indicated by $F : X \rightrightarrows Y$. Then, $\operatorname{gph} F = \{(x, y) \in X \times Y \mid y \in F(x)\}$ is the graph of F and the range of F is the set $\operatorname{rge} F = \{y \in Y \mid \exists x,\ F(x) \ni y\}$. The inverse of F, denoted by F^{-1}, is defined as $x \in F^{-1}(y) \Leftrightarrow y \in F(x)$.

We denote by $\mathcal{F}(X, Y)$ the space of all closed-valued mappings $F : X \rightrightarrows Y$ (*i.e.,* $F(x)$ is a closed subset of Y for all $x \in X$) while the space $\mathcal{H}(X, Y)$ consists of those closed-valued mappings $H : X \rightrightarrows Y$ which are positively homogeneous.

2. Preliminary results

The concept of generalized differentiation we are dealing with strongly rely on positively homogeneous set-valued mappings. This is the reason why we found it useful to present a few results regarding these particular mappings. We start by stating their definition.

Definition 2.1. Let $H : X \rightrightarrows Y$ be a set-valued mapping. It is called positively homogeneous if $H(0) \ni 0$ and $H(\lambda x) = \lambda H(x)$ for all $x \in X$ and $\lambda > 0$.

One can immediately note that a mapping is positively homogeneous if and only if its graph is a cone and that the inverse of a positively homogeneous mapping is another positively homogeneous mapping. Graphical derivatives of set-valued mappings, introduced by Aubin [2] (see also [3]), are positively homogeneous set-valued mappings and so are sublinear mappings (*i.e.,* set-valued mappings such that their graph is a convex cone). Let us recall that sublinear mappings have been considered by Rockafellar under the name of *convex processes* (see [26,27]). A practical example of homogeneous set-valued mappings (and, actually, sublinear mappings) is given by the constraint-type mappings, *i.e.,* the mappings H such that $H(x) = Ax - K$ where A is a linear mapping and K a closed convex cone.

To be able to work efficaciously with positively homogeneous mappings we need some tools to be available. One of them is the so-called *outer norm*.

Definition 2.2. Let $H : X \rightrightarrows Y$ be a positively homogeneous mapping. The outer norm of H is

$$|H|^+ = \sup_{\|x\| \leqslant 1} \sup_{y \in H(x)} \|y\|, \tag{4}$$

with the convention that $\sup_{y\in\emptyset} \|y\| = -\infty$.

Note that an equivalent (and useful) formulation of (4) is given by

$$|H|^+ = \inf\{\kappa > 0 \mid H(\overline{\mathbb{B}}) \subset \kappa \overline{\mathbb{B}}\}.$$

Positively homogeneous mappings having a finite outer norm are of undeniable interest and will play an important role in the results we state in the next section. For this reason, it is worth presenting a few necessary conditions for a positively homogeneous mapping to have this property. Such a question has already been meticulously studied by Robinson in a work dealing with convex processes (see [25]). The following proposition is from [12].

Proposition 2.3. *Let $H : X \rightrightarrows Y$ be a positively homogeneous mapping. Then*

$$|H|^+ < \infty \quad \Rightarrow \quad H(0) = \{0\},$$

with this implication becoming an equivalence when H has closed graph and $\dim X < \infty$.

Recall that a set-valued mapping $F : X \rightrightarrows Y$ is *open at $\bar{x}$ for $\bar{y}$*, where $\bar{y} \in F(\bar{x})$, if $\bar{x} \in \operatorname{int}(\operatorname{dom} F)$ and for every neighborhood U of $\bar{x}$, the set $F(U) = \bigcup_{x\in U} F(x)$ is a neighborhood of $\bar{y}$.

Proposition 2.4. *Let $H : X \rightrightarrows Y$ be a positively homogeneous mapping such that $0_X \in \operatorname{int}(\operatorname{dom} H)$. If $|H|^+ < \infty$ then H^{-1} is open at 0_Y for 0_X.*

Proof. Let V be a neighborhood of 0_Y in Y and let α be a positive constant such that $\mathbb{B}_\alpha(0_Y) \subset V$. We show that $H^{-1}(V)$ is a neighborhood of 0_X in X.

Since $|H|^+ < \infty$, one can find $\beta > 0$ such that $\beta|H|^+ < \alpha$ and, besides,

$$\text{for all } x \in X, \quad H(x) \subset |H|^+ \|x\| \overline{\mathbb{B}}.$$

Making β smaller if necessary, we can assume that $\mathbb{B}_\beta(0_X) \subset \operatorname{dom} H$ then, for any $x \in \mathbb{B}_\beta(0_X)$, there is an element $y \in H(x)$ such that $\|y\| \leqslant \beta|H|^+ < \alpha$, *i.e.,* $y \in \mathbb{B}_\alpha(0_Y)$.

We have thus proved that for all $x \in \mathbb{B}_\beta(0_X)$, there exists $y \in H(x) \cap \mathbb{B}_\alpha(0_Y)$, hence

$$\mathbb{B}_\beta(0_X) \subset H^{-1}\big(\mathbb{B}_\alpha(0_Y)\big) \subset H^{-1}(V),$$

and $H^{-1}(V)$ is a neighborhood of 0_X in X. □

3. Stability of generalized differentiation

To properly define the concept of convergence of set-valued mappings we will use in the sequel, it is necessary to introduce the topology from which it derives. Since the mappings we are dealing with are closed-valued it seems natural to consider a topology on the closed subsets of a metric (or normed linear) space; such a topological space is called a *hyperspace*. When E is a Hausdorff topological space we denote by $\mathrm{CL}(E)$ the class of nonempty closed subsets of E. Before presenting the topology we will consider here, we recall some usual notation. Let B be a subset of E, then the set B^- consists of those closed sets that *hit* B while the set B^{++} consists of those sets that are *strongly contained* in B; more precisely

$$B^- = \{A \in \mathrm{CL}(E)\colon\ A \cap B \neq \emptyset\}, \qquad B^{++} = \{A \in \mathrm{CL}(E)\colon\ \exists \varepsilon > 0,\ \mathbb{B}_\varepsilon(A) \subset B\}.$$

The following topology on $\mathrm{CL}(E)$ has been introduced by Beer et al. in [7].

Definition 3.1. Let (E, d) be a metric space. The *proximal topology* τ_{δ_d} on $\mathrm{CL}(E)$ has as a subbase all sets of the form V^-, where V is open in E, and all sets of the form W^{++}, where W is open in E.

A base for the proximal topology consists of all finite intersections of elements of its subbase. As mentioned in [7], since

$$(V \cap W)^{++} = V^{++} \cap W^{++},$$

a base for this topology consists of all the sets of the form

$$V^{++} \cap V_1^- \cap V_2^- \cap \cdots \cap V_n^-;$$

where $V, V_1, \ldots, V_n$ are open in E. It follows that a local base for the proximal topology at $A \in \mathrm{CL}(E)$ consists of all the sets of the form

$$\mathbb{B}_\varepsilon(A)^{++} \cap \mathbb{B}_\varepsilon(a_1)^- \cap \mathbb{B}_\varepsilon(a_2)^- \cap \cdots \cap \mathbb{B}_\varepsilon(a_n)^-;$$

where the points $a_1, a_2, \ldots, a_n$ lie in A and ε is a positive number. Thanks to this description, one can infer that the proximal topology is compatible with Fisher convergence, the definition of which reads as follows.

Definition 3.2 *(Fisher convergence).* Let (E, d) be a metric space. A sequence A_n in $\mathrm{CL}(E)$ converges to $A \in \mathrm{CL}(E)$ in the sense of Fisher if

(1) $A \subset \liminf_n A_n$;
(2) $\lim_n e(A_n, A) = 0$.

A sequence A_n of closed subsets of E satisfying assertion (1) (respectively, assertion (2)) above will be said to be *lower Fisher convergent* (respectively, *upper Fisher convergent*) to the subset A. Recall that the lower limit of a sequence A_n of subsets of a normed space, with unit ball $\mathbb{B}$, is defined by:

$$\liminf_n A_n := \bigcap_{\varepsilon>0} \bigcup_{N>0} \bigcap_{n \geqslant N} (A_n + \varepsilon\mathbb{B}).$$

A useful alternative formulation (in normed spaces) is given by:

$$\begin{aligned}\liminf_n A_n &= \Big\{x \in E \;\Big|\; \limsup_{n\to\infty} d(x, A_n) = 0\Big\} \\ &= \{x \in E \mid \exists x_n \in A_n \text{ with } x_n \to x\}.\end{aligned}$$

The convergence in the sense of Fisher was introduced in [13] and then, studied in several works (see *e.g.*, [5,7,14]). One can prove (see for instance [6]) that a sequence A_n converges to a subset A in $(\mathrm{CL}(E), \tau_{\delta_d})$ if and only if it is convergent to A in the sense of Fisher.

Adapting these concepts to our framework we obtain the following convergence for set-valued maps that we will also call Fisher convergence.

Definition 3.3. Let X and Y be Banach spaces and let F_n be a sequence in $\mathcal{F}(X, Y)$. We say that F_n Fisher converges to $F \in \mathcal{F}(X, Y)$, and we write $F_n \xrightarrow{F} F$, if

(a) $F(x) \subset \liminf_n F_n(x)$ for all $x \in X$;
(b) $\lim_n \sup_{x\in X} e(F_n(x), F(x)) = 0$.

If the mappings F_n and F satisfy assertion (a) in Definition 3.3 then we say that F_n *lower Fisher converges* to F while relation (b) corresponds to the *upper Fisher convergence* of the sequence F_n to F. For the sake of simplicity, we chose not to mention in the terminology we adopt here the "uniformity" of the convergence in (b).

Remark. When the mappings $F_n := f_n$ (for all n) and $F := f$ are single-valued then assertion (a) in Definition 3.3 reduces to the convergence of the sequence $f_n(x)$ to $f(x)$ for all $x \in X$ while assertion (b) is nothing but the uniform convergence of the sequence f_n to f.

From now on, and until the end of the present section, we work in the finite dimensional setting. The following proposition states that the positive homogeneity of set-valued mappings is stable with respect to Fisher convergence.

Proposition 3.4. *Let H_n be a sequence in $\mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$ and let $H \in \mathcal{F}(\mathbb{R}^m, \mathbb{R}^p)$ be such that $H_n \xrightarrow{F} H$. Then H is a positively homogeneous mapping.*

Proof. Take an element $x \in \mathbb{R}^m$ and a positive scalar λ. We first prove that $H(\lambda x) \subset \lambda H(x)$. If $H(\lambda x) = \emptyset$, there is nothing to prove. Otherwise, let $y \in H(\lambda x)$. Since $H(\lambda x) \subset \liminf H_n(\lambda x)$ there is a sequence y_n, the elements of which are in $H_n(\lambda x)$ for each n, converging to y. It follows that

$$y_n/\lambda \in H_n(x) \quad \text{for all } n. \tag{5}$$

Moreover, $\sup_{z \in \mathbb{R}^m} e(H_n(z), H(z)) \to 0$ then

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ n \geqslant N \quad \Rightarrow \quad \forall z \in \mathbb{R}^m, \quad e\big(H_n(z), H(z)\big) < \varepsilon.$$

Hence,

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ n \geqslant N \quad \Rightarrow \quad \forall z \in \mathbb{R}^m, \quad H_n(z) \subset H(z) + \varepsilon \mathbb{B}.$$

It follows from (5) that

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ n \geqslant N \quad \Rightarrow \quad y_n/\lambda \in H(x) + \varepsilon \mathbb{B}.$$

Passing to the limit over n we obtain that for all $\varepsilon > 0$, $y/\lambda \in \overline{H(x) + \varepsilon \mathbb{B}}$. Since $H(x)$ is a closed subset of $\mathbb{R}^p$, together with the fact that $\varepsilon \mathbb{B}$ is compact, the set $H(x) + \varepsilon \mathbb{B}$ is closed. Hence, $y/\lambda \in \bigcap_{\varepsilon > 0}(H(x) + \varepsilon \mathbb{B})$. Consequently, $y/\lambda \in \overline{H(x)} = H(x)$ and we get

$$H(\lambda x) \subset \lambda H(x). \tag{6}$$

Conversely, let $x \in \mathbb{R}^m$ and let $\lambda > 0$. If $H(x) = \emptyset$ then there is nothing more to do. Otherwise, take $y \in \lambda H(x)$; since $H(z) \subset \liminf H_n(z)$, for all $z \in \mathbb{R}^m$, there is a sequence y_n in $\mathbb{R}^p$ such that $y_n \in H_n(x)$ for $n = 0, 1, \ldots$ and $y_n \to y/\lambda$. Then the sequence λy_n converges to y and we have $\lambda y_n \in \lambda H_n(x) = H_n(\lambda x)$ for $n = 0, 1, \ldots$.

Thanks to the upper Fisher convergence of the sequence H_n to H we get

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ n \geqslant N \quad \Rightarrow \quad H_n(\lambda x) \subset H(\lambda x) + \varepsilon \mathbb{B}.$$

Thus,

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ n \geqslant N \quad \Rightarrow \quad \lambda y_n \in H(\lambda x) + \varepsilon \mathbb{B}.$$

Then, $\forall \varepsilon > 0$, $y \in \overline{H(\lambda x) + \varepsilon \mathbb{B}} = H(\lambda x) + \varepsilon \mathbb{B}$. Therefore, $y \in H(\lambda x)$ and we thus get

$$\lambda H(x) \subset H(\lambda x). \tag{7}$$

Combining inclusions (6) and (7) we obtain that $H(\lambda x) = \lambda H(x)$ for all $x \in \mathbb{R}^m$ and $\lambda > 0$. To complete the proof, it remains to show that $0 \in H(0)$. Thanks to the upper Fisher convergence of the sequence H_n to H, for all positive ε, we have

$$H_n(0) \subset H(0) + \varepsilon \mathbb{B}, \text{ eventually.}$$

Since $0 \in H_n(0)$ for all n, we infer that, for all positive ε, $0 \in H(0) + \varepsilon \mathbb{B}$. It follows that $0 \in \overline{H(0)} = H(0)$ and we have completed the proof. □

Now we are interested in investigating the relationships between a converging sequence of positively homogeneous mappings, having a finite norm, and the finiteness of the norm of its limit. Propositions 3.5, 3.6 and 3.7 below provide us with some answers.

Proposition 3.5. *Let H_n be a sequence in $\mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$ and let $H \in \mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$. Assume that the sequence H_n is uniformly bounded,* i.e., *there is $\kappa > 0$ such that $|H_n|^+ \leqslant \kappa$ for $n = 0, 1, \ldots$. If the sequence H_n lower Fisher converges to H then $|H|^+ \leqslant \kappa$.*

Proof. From the uniform boundedness of the sequence H_n we obtain the existence of a positive scalar κ such that

$$H_n(x) \subset \kappa \|x\| \overline{\mathbb{B}}, \quad \forall x \in \mathbb{R}^m. \tag{8}$$

Let $x \in \operatorname{dom} H$ and let $y \in H(x)$. Since $H(x) \subset \liminf H_n(x)$, there is a sequence y_n converging to y, such that $y_n \in H_n(x)$ for $n = 0, 1, \ldots$. It follows from (8) that $y_n \in \kappa \|x\| \overline{\mathbb{B}}$, $\forall n \in \mathbb{N}$. Passing to the limit over n we get $y \in \kappa \|x\| \overline{\mathbb{B}}$ and therefore $H(x) \subset \kappa \|x\| \overline{\mathbb{B}}$. This clearly yields $|H|^+ \leqslant \kappa$. □

A more precise statement is available in Proposition 3.6 where the convergence of the sequence H_n is strengthened.

Proposition 3.6. *Let H_n be a sequence in $\mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$ and let $H \in \mathcal{F}(\mathbb{R}^m, \mathbb{R}^p)$. Assume that there is a positive constant κ such that $|H_n|^+ = \kappa$ for $n = 0, 1, \ldots$. If $H_n \xrightarrow{F} H$ then $H \in \mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$ and $|H|^+ = \kappa$.*

Proof. From Proposition 3.4 we have $H \in \mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$ while Proposition 3.5 yields $|H|^+ \leqslant \kappa$. If $|H|^+ < \kappa$ then there is a positive scalar $\tilde{\kappa} < \kappa$ such that $|H|^+ < \tilde{\kappa}$. Let $\varepsilon > 0$ be such that $\tilde{\kappa} + \varepsilon < \kappa$. It follows, from the upper Fisher convergence of the sequence H_n to H, that there is an integer N such that for all $n \geqslant N$ and $x \in \mathbb{R}^m$, $H_n(x) \subset H(x) + \varepsilon \mathbb{B}$. Since $|H|^+ < \tilde{\kappa}$, using the convexity of the unit ball $\mathbb{B}$, we obtain that for all x in $\overline{\mathbb{B}}$

$$H_n(x) \subset (\tilde{\kappa} + \varepsilon) \overline{\mathbb{B}}, \text{ eventually.}$$

Consequently, $|H_n|^+ < \kappa$, eventually. This contradicts our assumption that $|H_n|^+ = \kappa$ for $n = 0, 1, \ldots$ and terminates the proof. □

Proposition 3.7. *Let H_n be a sequence in $\mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$ and let $H \in \mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$ be such that the sequence H_n upper Fisher converges to H. If $|H|^+ < \infty$ then the sequence H_n is eventually uniformly bounded.*

Proof. Since $|H|^+ < \infty$ there is a constant $\kappa \geqslant 0$ such that $H(x) \subset \kappa \overline{\mathbb{B}}$ for all $x \in \overline{\mathbb{B}}$. Fix $\varepsilon > 0$; there is an integer N such that for any integer $n \geqslant N$ one has

$$\sup_{x \in X} e\big(H_n(x), H(x)\big) \leqslant \varepsilon.$$

In particular, for all $n \geqslant N$, $x \in \overline{\mathbb{B}}$ we can write

$$H_n(x) \subset H(x) + \varepsilon \overline{\mathbb{B}} \subset \kappa \overline{\mathbb{B}} + \varepsilon \overline{\mathbb{B}} = (\kappa + \varepsilon) \overline{\mathbb{B}}.$$

It follows that the sequence H_n is eventually uniformly bounded. □

In the light of Propositions 3.5 and 3.7 we can state the following result, the proof of which is straightforward.

Proposition 3.8. *Let H_n be a sequence in $\mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$ and let $H \in \mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$ be such that $H_n \xrightarrow{F} H$. Then $|H|^+ < \infty$ if and only if the sequence H_n is eventually uniformly bounded.*

Definition 3.9. Let F_n be a sequence in $\mathcal{F}(\mathbb{R}^m, \mathbb{R}^p)$ and let H_n be a sequence in $\mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$. We say that the sequence F_n is uniformly strictly H_n-differentiable at $\bar{x}$ if for any $\delta > 0$, there exists a uniform neighborhood V of $\bar{x}$ (*i.e.*, which does not depend on n) such that for $n = 0, 1, \ldots$

$$F_n(x) \subset F_n\big(x'\big) + H_n\big(x - x'\big) + \delta \big\|x - x'\big\| \mathbb{B} \quad \text{for all } x, x' \in V.$$

Of course, a slight and obvious modification of Definition 3.9 allows us to define the concepts of uniform outer H_n-differentiability, uniform inner H_n-differentiability and uniform H_n-differentiability.

Theorem 3.10. *Consider a sequence F_n in $\mathcal{F}(\mathbb{R}^m, \mathbb{R}^p)$ along with two mappings $F \in \mathcal{F}(\mathbb{R}^m, \mathbb{R}^p)$ and $H \in \mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$. Let H_n be a sequence in $\mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$. We make the following assumptions*:

(1) *The sequence F_n is uniformly strictly H_n-differentiable at $\bar{x} \in \mathbb{R}^m$;*
(2) *The sequence F_n Fisher converges to F;*

(3) *The sequence H_n upper Fisher converges to H;*
(4) $|H|^+ < \infty$.

Then, the mapping F is strictly H-differentiable at $\bar{x}$.

Proof. Assertion (1) above means that for any $\delta > 0$, there is a neighborhood V of $\bar{x}$ such that

$$F_n(x) \subset F_n(x') + H_n(x - x') + \delta \|x - x'\| \mathbb{B}, \quad \forall x, x' \in V,\ n \in \mathbb{N}. \tag{9}$$

Fix $\varepsilon > 0$. Since the sequence F_n Fisher converges to F there is an integer N_ε^1 such that for all $n \geqslant N_\varepsilon^1$, $\sup_{x \in \mathbb{R}^m} e(F_n(x), F(x)) < \varepsilon/2$. It follows that

$$\forall n \geqslant N_\varepsilon^1, \quad F_n(x') \subset F(x') + \varepsilon/2\mathbb{B}, \quad \text{for all } x' \in \mathbb{R}^m. \tag{10}$$

In a very similar manner one can prove that the upper Fisher convergence of the sequence H_n to H yields the existence of an integer N_ε^2 such that

$$\forall n \geqslant N_\varepsilon^2, \quad H_n(x - x') \subset H(x - x') + \varepsilon/2\mathbb{B}, \quad \text{for all } x, x' \in \mathbb{R}^m. \tag{11}$$

From the uniform strict H_n-differentiability of the sequence F_n at $\bar{x}$, together with inclusions (10) and (11), we infer that for all $\delta > 0$, there is a neighborhood V of $\bar{x}$ such that

$$F_n(x) \subset F(x') + H(x - x') + \varepsilon\mathbb{B} + \delta \|x - x'\| \mathbb{B}, \quad \forall x, x' \in V,\ n \geqslant N_\varepsilon;$$

where $N_\varepsilon := \max\{N_\varepsilon^1, N_\varepsilon^2\}$. Then,

$$\liminf_n F_n(x) \subset \overline{F(x') + H(x - x') + (\varepsilon + \delta\|x - x'\|)\mathbb{B}}, \quad \forall x, x' \in V. \tag{12}$$

Since $|H|^+ < \infty$, $H(x - x')$ is compact. Moreover, the sets $F(x')$ and $(\varepsilon + \delta\|x - x'\|)\mathbb{B}$ are, respectively, closed and compact, consequently $F(x') + H(x - x') + (\varepsilon + \delta\|x - x'\|)\mathbb{B}$ is a closed subset of $\mathbb{R}^p$ and relation (12), together with the Fisher convergence of the sequence F_n to F, yields

$$F(x) \subset F(x') + H(x - x') + (\varepsilon + \delta\|x - x'\|)\mathbb{B}, \quad \forall x, x' \in V.$$

Thus, we have proved so far that, for all $\delta > 0$ there is a neighborhood V of $\bar{x}$ such that for all $\varepsilon > 0$ one has

$$F(x) \subset F(x') + H(x - x') + \delta\|x - x'\|\mathbb{B} + \varepsilon\mathbb{B}, \quad \forall x, x' \in V.$$

Since the set $F(x') + H(x - x') + \delta\|x - x'\|\mathbb{B}$ is closed it follows that for all $\delta > 0$ there is a neighborhood V of $\bar{x}$ such that

$$F(x) \subset F(x') + H(x - x') + \delta\|x - x'\|\mathbb{B}, \quad \forall x, x' \in V. \tag{13}$$

Then the mapping F is strictly H-differentiable at $\bar{x}$. □

Obviously, by considering in Theorem 3.10 a sequence F_n which is uniformly outer H_n-differentiable at $\bar{x}$ (respectively, uniformly inner H_n-differentiable at $\bar{x}$ or uniformly H_n-differentiable at $\bar{x}$) we obtain, as a conclusion, the outer H-differentiability (respectively, the inner H-differentiability or the H-differentiability) of the mapping F at $\bar{x}$.

The following result is a straightforward consequence of the above theorem. It asserts that, by strengthening the convergence of the sequence H_n to H, one can remove most of the assumptions regarding the mapping H made in Theorem 3.10; namely, the positive homogeneity of H as well as the finiteness of its outer norm.

Corollary 3.11. *Consider a sequence F_n in $\mathcal{F}(\mathbb{R}^m, \mathbb{R}^p)$ along with two mappings F and H in $\mathcal{F}(\mathbb{R}^m, \mathbb{R}^p)$. Let H_n be a sequence in $\mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$. We make the following assumptions:*

(1) *The sequence F_n is uniformly strictly H_n-differentiable at $\bar{x} \in \mathbb{R}^m$;*
(2) *The sequences F_n and H_n Fisher converge respectively to F and H;*
(3) *The sequence H_n is eventually uniformly bounded.*

Then, the mapping F is strictly H-differentiable at $\bar{x}$.

Proof. Assertions (1) and (2) above clearly yield assumptions (1) to (3) of Theorem 3.10. Moreover, thanks to Proposition 3.4 we have that $H \in \mathcal{H}(\mathbb{R}^m, \mathbb{R}^p)$ while Proposition 3.8 gives us $|H|^+ < \infty$. It remains to apply Theorem 3.10 to obtain the desired conclusion. □

Remark. To the best of our knowledge there are no similar results, dealing with the variational stability of the main concepts of generalized differentiation for set-valued mappings, namely, the contingent derivative of Aubin (see, *e.g.*, [3]) and the coderivative of Mordukhovich (see *e.g.*, [21]). This issue is of course related to the stability of the contingent cone (for the contingent derivative) and to the stability of the normal cone with respect of a set (for the coderivative). Some answers about the stability of these cones can be found in [3,21], nevertheless properly speaking, there are no works regarding the variational stability of these concepts we are aware of.

4. Continuous dependence of fixed points sets

In this last section we study the stability of fixed points sets of set-valued contractions. Such studies have already been carried out by Markin (see [19]) who proved a stability theorem in Hilbert spaces for closed- and convex-valued mappings while Lim [18] established a few years later a similar result for closed-valued mappings defined in a complete metric space. In both these works, a uniform Hausdorff-type convergence was needed to guarantee the good behavior of fixed points sets of a sequence of set-valued contractions. For additional developments on this topic, including applications to the dependence of solutions to differential inclusions or partial differential equations, one may refer also to [4,22,24]. Here, we prove that if a sequence of set-valued mappings T_n upper Fisher converges to a set-valued contraction T then the sequence of fixed points of T_n upper Fisher converges to a fixed point of T. Before going further we recall two useful definitions and present the results of Lim which are the very inspiration behind our investigations.

Let us now recall the definitions of the Pompeiu–Hausdorff distance and set-valued contractions.

Definition 4.1. The Pompeiu–Hausdorff distance between two subsets A and B of a metric space is the quantity

$$h(A,B)=\max\big\{e(A,B);e(B,A)\big\}.$$

Equivalently, it can be expressed by

$$h(A,B)=\inf\{\varepsilon>0 \mid A\subset B+\varepsilon\mathbb{B},\ B\subset A+\varepsilon\mathbb{B}\}.$$

Definition 4.2. Let (X,d) and (Y,δ) be two metric spaces and let $T:X\rightrightarrows Y$ be a closed-valued mapping. We say that T is a λ-contraction if there is a constant $\lambda\in(0,1)$ such that

$$h\big(T(x),T(y)\big)\leqslant \lambda d(x,y),\quad \forall x,y\in X.$$

In [18], Lim proved the following result establishing the continuous dependence of fixed points sets of set-valued contractions.

Proposition 4.3. *Let X be a complete metric space and T_1 and T_2 be λ-contractions from X into* $\mathrm{CL}(X)$. *Then,*

$$h\big(\phi(T_1),\phi(T_2)\big)\leqslant \frac{1}{1-\lambda}\sup_{x\in X} h\big(T_1(x),T_2(x)\big),$$

where $\phi(T_1)$ and $\phi(T_2)$ denote, respectively, the fixed points sets of the mappings T_1 and T_2.

Next comes a straightforward consequence of Proposition 4.3, namely a result regarding the stability of the fixed points of a uniformly convergent sequence of set-valued contractions.

Proposition 4.4. *Let X be a complete metric space and $T_i:X\to \mathrm{CL}(X)$ a sequence of λ-contractions, $i=0,1,2,\ldots$ If $\lim_{n\to\infty} h(T_i(x), T_0(x))=0$ uniformly for all x in X, then $\lim_{n\to\infty} h(\phi(T_i),\phi(T_0))=0$.*

An adaptation of the proof of Proposition 4.3 leads us to the following result, the assumptions of which are more general than the ones made by Lim since we do not need the two set-valued mappings involved in the statement to be λ-contractions.

Proposition 4.5. *Let X be a complete metric space. Let T_0 be a λ-contraction from X into* $\mathrm{CL}(X)$ *and let $T:X\rightrightarrows X$ be any set-valued mapping. Then,*

$$e\big(\phi(T),\phi(T_0)\big)\leqslant \frac{1}{1-\lambda}\sup_{x\in X} e\big(T(x),T_0(x)\big);$$

where $\phi(T)$ and $\phi(T_0)$ denote, respectively, the fixed points sets of the mappings T and T_0.

Proof. If the quantity $\sup_{x\in X} e(T(x), T_0(x)) = \infty$ there is nothing to prove; therefore, we may assume that $M := \sup_{x\in X} e(T(x), T_0(x)) < \infty$.

Moreover, if $\phi(T) = \emptyset$ then according to the convention we adopted in Section 1, $e(\phi(T), \phi(T_0)) = 0$ and we are done. Otherwise, $\phi(T) \neq \emptyset$ and we take $x_0 \in \phi(T)$, *i.e.*, $x_0 \in \phi(x_0)$.

Fix $\varepsilon > 0$, clearly $e(T(x_0), T_0(x_0)) < M + \varepsilon$ thus, $d(x_0, T_0(x_0)) < M + \varepsilon$ and, consequently, there exists $x_1 \in T_0(x_0)$ such that $d(x_0, x_1) < M + \varepsilon$.

The mapping T_0 being a λ-contraction we have $e(T_0(x_0), T_0(x_1)) \leqslant \lambda d(x_0, x_1)$. It follows that $d(x_1, T_0(x_1)) \leqslant \lambda d(x_0, x_1)$. Hence

$$d\big(x_1, T_0(x_1)\big) < \lambda d(x_0, x_1) + \lambda\varepsilon_1, \tag{14}$$

where $\varepsilon_1 := c\varepsilon/(1-\lambda)$, c being any positive constant such that

$$c\sum_{n=1}^{\infty} n\lambda^n < 1. \tag{15}$$

From (14) we infer the existence of an element $x_2 \in T_0(x_1)$ such that

$$d(x_1, x_2) < \lambda d(x_0, x_1) + \lambda\varepsilon_1.$$

Since T_0 is a λ-contraction we have again

$$e\big(T_0(x_1), T_0(x_2)\big) \leqslant \lambda d(x_1, x_2) < \lambda d(x_1, x_2) + \lambda^2\varepsilon_1.$$

Hence, there exists $x_3 \in T_0(x_2)$ such that $d(x_2, x_3) < \lambda d(x_1, x_2) + \lambda^2\varepsilon_1$. The construction process is now clear and we are thus able to define a sequence x_n such that

$$\begin{cases} x_{n+1} \in T_0(x_n); \\ d(x_{n+1}, x_n) \leqslant \lambda d(x_n, x_{n-1}) + \lambda^n\varepsilon_1 \quad (n \geqslant 1). \end{cases}$$

Then,

$$\begin{aligned} d(x_{n+1}, x_n) &\leqslant \lambda d(x_n, x_{n-1}) + \lambda^n\varepsilon_1 \\ &\leqslant \lambda^2 d(x_{n-1}, x_{n-2}) + 2\lambda^n\varepsilon_1 \\ &\;\;\vdots \\ &\leqslant \lambda^n d(x_1, x_0) + n\lambda^n\varepsilon_1. \end{aligned}$$

Hence, for all integers p and q such that $p \geqslant q$, one has

$$d(x_p, x_q) \leqslant \sum_{i=q}^{\infty} d(x_{i+1}, x_i) \leqslant \frac{\lambda^q}{1-\lambda} d(x_1, x_0) + \sum_{i=q}^{\infty} i\lambda^i\varepsilon_1,$$

and using (15) we get $\lim_{q\to\infty} \sum_{i=q}^{\infty} d(x_{i+1}, x_i) = 0$. Therefore, x_n is a Cauchy sequence and the space X being complete x_n converges to some point $x \in X$. Before going further, we need the following result:

Claim. *Since T_0 is a closed-valued λ-contraction, it has closed graph.*

Indeed, let (x_n, y_n) be a sequence in $\operatorname{gph} T$ such that (x_n, y_n) converges to some $(\bar{x}, \bar{y}) \in X \times X$. Since T is a λ-contraction we have $h(T(x_n), T(\bar{x})) \leqslant \lambda d(x_n, \bar{x})$ which yields $d(y_n, T(\bar{x})) \leqslant \lambda d(x_n, \bar{x})$. The sequence x_n converging to $\bar{x}$, we get $\lim_{n\to\infty} d(y_n, T(\bar{x})) = d(\bar{y}, T(\bar{x})) = 0$. And because T is closed-valued it follows that $\bar{y} \in T(\bar{x})$ which completes the proof of the claim.

Since for all $n \in \mathbb{N}$, $x_{n+1} \in T_0(x_n)$, thanks to the claim we get $x \in T_0(x)$, *i.e.*, $x \in \phi(T_0)$. Moreover,

$$\begin{aligned} d(x_0, x) &\leqslant \sum_{n=0}^{\infty} d(x_{n+1}, x_n) \\ &\leqslant \frac{1}{1-\lambda} d(x_1, x_0) + \sum_{n=1}^{\infty} n\lambda^n\varepsilon_1 \\ &\leqslant \frac{1}{1-\lambda}\big(d(x_1, x_0) + \varepsilon\big) \\ &\leqslant \frac{1}{1-\lambda}(M + 2\varepsilon). \end{aligned}$$

Since $d(x_0, \phi(T_0)) \leqslant d(x_0, x)$ we obtain

$$d\big(x_0, \phi(T_0)\big) \leqslant \frac{1}{1-\lambda}(M + 2\varepsilon). \tag{16}$$

The last inequality being valid for any $x_0 \in \phi(T)$ we get

$$e\big(\phi(T), \phi(T_0)\big) \leqslant \frac{1}{1-\lambda}(M + 2\varepsilon), \tag{17}$$

letting ε go to zero we complete the proof. □

From Proposition 4.5, we derive the following stability result and its corollary.

Proposition 4.6. *Let X be a Banach space. Let T be a λ-contraction from X into $\mathrm{CL}(X)$ and let $T_n : X \rightrightarrows X$ be a sequence in $\mathcal{F}(X, X)$ upper Fisher converging to T. Then, the sequence $\phi(T_n)$ upper Fisher converges to $\phi(T)$.*

Proof. From Proposition 4.5, we get

$$e\big(\phi(T_n), \phi(T)\big) \leqslant \frac{1}{1-\lambda} \sup_{x \in X} e\big(T_n(x), T(x)\big) \quad \text{for } n = 0, 1, 2, \ldots.$$

Moreover, thanks to the upper Fisher convergence of the sequence T_n to T, we obtain

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ n \geqslant N \quad \Rightarrow \quad \sup_{x \in X} e\big(T_n(x), T(x)\big) < (1-\lambda)\varepsilon.$$

Combining these two relations, we thus get

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ n \geqslant N \quad \Rightarrow \quad e\big(\phi(T_n), \phi(T)\big) < \varepsilon,$$

which gives the upper Fisher convergence of the sequence $\phi(T_n)$ to $\phi(T)$ and completes the proof. □

Corollary 4.7. *Let $T : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ be a closed-valued λ-contraction. Let T_n be a sequence in $\mathcal{F}(\mathbb{R}^m, \mathbb{R}^m)$ upper Fisher converging to T. If, for $n = 0, 1, 2, \ldots, x_n$ is a fixed point of T_n and the sequence x_n converges to some $\bar{x} \in \mathbb{R}^m$ then $\bar{x}$ is a fixed point of T.*

Proof. Let $\varepsilon > 0$. From Proposition 4.6, the sequence $\phi(T_n)$ upper Fisher converges to the set $\phi(T)$. Therefore, there is an integer N such that for all $n \geqslant N$ one has

$$\phi(T_n) \subset \phi(T) + \varepsilon\overline{\mathbb{B}}.$$

Since $x_n \in \phi(T_n)$, for $n = 0, 1, 2, \ldots$ and $x_n \to \bar{x}$ we have $\bar{x} \in \liminf_n \phi(T_n)$. Thanks to the claim established in the proof of Proposition 4.5 we know that the graph of T is closed, it follows that the set $\phi(T)$ is a closed subset of $\mathbb{R}^m$. Consequently,

$$\bar{x} \in \overline{\phi(T) + \varepsilon\overline{\mathbb{B}}} = \phi(T) + \varepsilon\overline{\mathbb{B}}.$$

Since ε is an arbitrary positive number we get $\bar{x} \in \phi(T)$. □

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