An \(n\)-dimensional version of Steinhaus’ chessboard theorem

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Received 11 October 2006; accepted 25 July 2007

Abstract

The main result of this paper is an \(n\)-dimensional version of the Steinhaus’ chessboard theorem. Our theorem implies the Poincaré theorem as well as its parametric extension. But it is known that the Poincaré theorem is equivalent to the Brouwer Fixed-Point theorem.

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MSC: primary 54H25, 54-04; secondary 55M20, 54F55, 52B15

Keywords: Cubes; Simplex; Coloring function; Combinatorial cube; Minimal barrier; \(i\)-connected chain

The Steinhaus’ chessboard theorem was formulated in the book entitled “Kalejdoskop matematyczny” in this way:

Let some segments of the chessboard be mined. Assume that the king cannot go across the chessboard from the left edge to the right one without meeting a mined square. Then the rook can go from upper edge to the lower one moving exclusively on mined segments, see [8]. There exists several proofs of this theorem (see [4,7,9]). Similar problem was stated in Gale’s paper [1]. Moreover he proved that classical result of topology, the celebrated Brouwer Fixed-Point theorem, is a consequence of the fact that the Hex cannot end in a draw. The following version of the Steinhaus’ chessboard theorem will be proved:

For an arbitrary decomposition of \(n\)-dimensional cube \(I^n\) into \(k^n\) cubes and for arbitrary coloring function \(F: T(k) \rightarrow \{1, \ldots, n\}\) for some \(i \in \{1, \ldots, n\}\) there exists \(i\)th colored \(i\)-connected chain \(P_1, \ldots, P_r\) such that

\[ P_1 \cap I_i^+ \neq \emptyset \text{ and } P_r \cap I_i^- \neq \emptyset. \]

Where \(i\)-connectness is defined similarly as the road of king and rook.

Our theorem implies Poincaré theorem [6] as well as its parametric extension [3]. It is known that the Poincaré theorem is equivalent to the Brouwer Fixed-Point theorem (for more information see [2]).

Let \(I^n := \left[0, 1\right]^n\) be the \(n\)-dimensional cube in \(\mathbb{R}^n\). Its \(i\)th opposite faces are defined as follows:

\[ I_i^- := \{ x \in I^n : x(i) = 0 \}, \quad I_i^+ := \{ x \in I^n : x(i) = 1 \}. \]
Lemma 1
depending on whether or not it lies on $C$
Observation 2.
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where
Let
be the
boundary
map defined as follows
Observation 3.
Let
be the
$n$-simplex
Let
the decomposition of $I^n$.
Moreover
the set
be an arbitrary natural number. We call the family
for
Moreover
The map $F: T(k) \to \{1, \ldots, n\}$ is said to be a coloring function of the decomposition $T(k)$. The sequence $P_1, \ldots, P_r$ where $P_l \in T(k)$ for $l = 1, \ldots, r$ is said to be an $i$th colored chain, if for all $l \in \{1, \ldots, r\}$ $F(P_l) = i$ and $P_l \cap P_{l+1} \neq \emptyset$ for $j = 1, \ldots, r - 1$.
The set $C = \{-\frac{1}{2k}, -\frac{1}{k}, \ldots, 0, \frac{1}{k}, \frac{1}{2k}\}$ is said to be the $n$-dimensional combinatorial cube. Its $i$th opposite faces are defined as follows:
\[
C_i^- = \left\{ z \in C : z(i) = -\frac{1}{2k} \right\}, \quad C_i^+ = \left\{ z \in C : z(i) = 1 + \frac{1}{2k} \right\}.
\]
Let
\[
\partial C = C_1^- \cup C_1^+ = \bigcup_{i=1}^{n} C_i^- \cup C_i^+
\]
be the boundary of the $n$-dimensional combinatorial cube.
Let $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, $e_i(i) = 1$ be the $i$th basic vector. An ordered set $S = [z_0, \ldots, z_n] \subset C$ is said to be an $n$-simplex if there exists permutation $\alpha$ of set $\{1, \ldots, n\}$ such that $z_1 = z_0 + e_{a(1)} \ldots z_n = z_{n-1} + e_{a(n)}$. Any subset $[z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n] \subset S$, $i = 0, \ldots, n$, is said to be an $(n-1)$-face of the $n$-simplex $S$.

Observation 1. Let $S = [z_0, \ldots, z_n] \subset C$ be an $n$-simplex. Then for each $z_i \in S$ if $[z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n] \not\subseteq C_p^\epsilon$ for $p = 1, \ldots, n$, $\epsilon = +$ or $-$ then there exists exactly one $n$-simplex $S[i] \subset C$ such that $S \cap S[i] = [z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n]$ else there does not exist such $S[i] \subset C$.

Observation 2. Any $(n-1)$-face of an $n$-simplex $S \subset C$ is an $(n-1)$-face of exactly one or of two $n$-simplexes from $C$ depending on whether or not it lies on $C_p^\epsilon$ for $p = 1, \ldots, n$, $\epsilon = +$ or $-$.

Every map $\Phi: C \to \{1, \ldots, n\}$ is said to be a coloring map of $C$. The set $A \subset C$ we call $n'$-colored if $\Phi(A) = \{1, \ldots, n'\}$.

Observation 3. Let $S \subset C$ be an $n$-colored $n$-simplex then there exists exactly two $(n-1)$-faces which are $n$-colored.

Lemma 1 (on the existence of a linked system). Let $C$ be a combinatorial cube and let $\phi: C \to \{1, \ldots, n\}$ be a coloring map defined as follows:
\[
\phi(z) = \begin{cases} 
F(t) & \text{for } z \in t \in T(k), \\
1 & \text{for } z \in C_1^- \cup C_2^+, \\
j & \text{for } z \in (C_j^- \cup C_{j+1}^+) \setminus (\bigcup_{i=1}^{j-1} (C_i^- \cup C_{i+1}^+)) \text{ when } j = 2, 3, \ldots, n - 1, \\
n & \text{for } z \in (C_n^- \cup C_1^+) \setminus (\bigcup_{i=1}^{n-1} (C_i^- \cup C_{i+1}^+)). 
\end{cases}
\]
Then there exists a linked system i.e., a sequence of $n$-colored $n$-simplexes $S_1, \ldots, S_m \in C$ such that $\phi(S_p \cap S_{p+1}) = \{1, \ldots, n\}$ for $p = 1, \ldots, m - 1$.

Moreover
\[ S_1 \cap C_1^- \neq \emptyset, \]
\[ S_1 \cap \left( C_i^- \setminus \bigcup_{l=1}^{i-1} C_l^- \right) \neq \emptyset, \quad \text{for } i = 2, \ldots, n, \]
\[ S_m \cap \left( C_i^+ \setminus \bigcup_{l=1}^{i-1} C_l^+ \right) \neq \emptyset, \]
\[ S_m \cap C_2^+ \neq \emptyset, \]
\[ S_m \cap \left( C_i^+ \setminus \bigcup_{l=2}^{i-1} C_l^+ \right) \neq \emptyset, \quad \text{for } i = 3, \ldots, n. \]

**Proof.** Let us take \([z_0^1, \ldots, z_{n-1}^1] \) where \( z_0^1 = (-\frac{1}{2\pi}, -\frac{1}{2\pi}, \ldots, -\frac{1}{2\pi}) \), \( z_1^1 = z_0^1 + e_1, \ldots, z_{n-1}^1 = z_0^1 + e_{n-1} \). It is easy to see that \( \phi([z_0^1, \ldots, z_{n-1}^1]) = \{1, \ldots, n\} \) because \( z_0^1 \in C_1^- \cup C_2^+, \ldots, z_{n-1}^1 \in (C_1^- \cup C_2^+) \setminus (C_1^- \cup C_2^+) \). Moreover the face \([z_0^1, \ldots, z_{n-1}^1] \subset C_n^- \) and it is \((n - 1)\)-face of exactly one \( n \)-simplex which is a subset of \( C \) (Observation 2). This \( n \)-simplex is called \( S_1 = [z_0^1, \ldots, z_{n-1}^1] \) where \( z_n^1 = (\frac{1}{2\pi}, \frac{1}{2\pi}, \ldots, \frac{1}{2\pi}) = z_{n-1}^1 + e_n \). It is obvious that \( S_1 \cap C_1^- = z_1^1 \neq \emptyset \), \( S_1 \cap (C_i^- \setminus \bigcup_{p=1}^{i-1} C_p^-) = z_{i-1}^1 \) for \( i = 2, \ldots, n \). The \( S_1 \) has two \( n \)-colored \((n - 1)\)-faces (Observation 3). Let us take “an unused” \((n - 1)\)-face. If it is a subset of \( C_p^+ \) for \( p \in \{1, \ldots, n\} \) then end.

Otherwise there exists exactly one \( n \)-simplex different from \( S_1 \) which contains our \((n - 1)\)-face. Let us call it \( S_2 \). We obtain the sequence \( S_1, S_2 \) and \( \phi(S_1 \cap S_2) = \{1, \ldots, n\} \). Suppose the sequence \( S_1, \ldots, S_j, \phi(S_l \cap S_{l+1}) = \{1, \ldots, n\} \) for \( l = 1, \ldots, j - 1 \) has been defined. The \( n \)-simplex \( S_j \) has exactly one “unused” \( n \)-colored \((n - 1)\)-face. If this face is a subset of \( C_p^+ \) for some \( p \in \{1, \ldots, n\} \) then end. Otherwise exactly one \( n \)-simplex different from \( S_j \) and which contains our \((n - 1)\)-face can be built. We call it \( S_{j+1} \). We have the sequence \( S_1, S_2, \ldots, S_{j+1} \) and \( \phi(S_j \cap S_{j+1}) = \{1, \ldots, n\} \).

However, the combinatorial cube contains a finite number of elements, hence the number of \( n \)-simplexes is also finite. That is why our procedure has to be stopped. Let us denote the last \( n \)-colored \( n \)-simplex by \( S_m \). We have \( S_1, \ldots, S_m \subset C \) and \( \phi(S_l \cap S_{l+1}) = \{1, \ldots, n\} \) for \( l = 1, \ldots, m - 1 \).

The \( S_m \) must have the \( n \)-colored \((n - 1)\)-face which is a subset of \( C_p^+ \) for some \( p \in \{1, \ldots, n\} \). But the map \( \phi : C \to \{1, \ldots, n\} \) is defined such that the only \( n \)-colored \((n - 1)\)-faces which lie on one of the combinatorial cube faces are: \([z_0^m, \ldots, z_{n-1}^m] \) and \([z_1^m, \ldots, z_n^m] \) where \( z_1^m = (1 + \frac{1}{2\pi}, 1 - \frac{1}{2\pi}, \ldots, 1 - \frac{1}{2\pi}) \), \( z_2^m = z_0^1 + e_n \), \( z_3^m = z_2^m + e_{n-1} \), \( \ldots \), \( z_n^m = z_{n-1}^1 + e_2 \).

We have \( S_m = [z_0^m, z_1^m, \ldots, z_n^m] \) where \( z_0^m = (1 - \frac{1}{2\pi}, 1 - \frac{1}{2\pi}, \ldots, 1 - \frac{1}{2\pi}) \) and
\[ S_m \cap \left( C_i^+ \setminus \bigcup_{l=1}^{i-1} C_l^+ \right) \neq \emptyset, \]
\[ S_m \cap C_2^+ \neq \emptyset, \]
\[ S_m \cap \left( C_i^+ \setminus \bigcup_{l=2}^{i-1} C_l^+ \right) \neq \emptyset \quad \text{for } i = 3, \ldots, n. \]

This ends the proof of the lemma. □

**Theorem 1** (on the existence of a chain). For an arbitrary decomposition of \( n \)-dimensional cube \( I^n \) onto \( k^n \) cubes and an arbitrary coloring function \( F : T(k) \to \{1, \ldots, n\} \) for some natural number \( i \in \{1, \ldots, n\} \) there exists an \( i \) colored chain \( P_1, \ldots, P_r \) such that \( P_i \cap I_i^+ \neq \emptyset \) and \( P_r \cap I_r^- \neq \emptyset \).

**Proof.** Let us take the \( n \)-dimensional cube \( I^n \) divided into \( k^n \) cubes and the arbitrary coloring function \( F : T(k) \to \{1, \ldots, n\} \). Let us define the map \( \phi : C \to \{1, \ldots, n\} \) like in the lemma. It is obvious that the lemma assumptions are
satisfied, hence there exists the sequence $S_1, \ldots, S_m \subset C$ such that $\phi(S_l \cap S_{l+1}) = \{1, \ldots, n\}$ for $l = 1, \ldots, m - 1$ moreover

$$
S_1 \cap C_i^+ \neq \emptyset,
$$
$$
S_1 \cap \left( C_i^- \setminus \bigcup_{l=1}^{i-1} C_l^+ \right) \neq \emptyset, \quad \text{for } i = 2, \ldots, n,
$$
$$
S_m \cap \left( C_i^- \setminus \bigcup_{l=1}^{n-1} C_l^+ \right) \neq \emptyset,
$$
$$
S_m \cap C_i^+ \neq \emptyset,
$$
$$
S_m \cap \left( C_i^- \setminus \bigcup_{l=2}^{i-1} C_l^+ \right) \neq \emptyset \quad \text{for } i = 3, \ldots, n.
$$

Let us take the smallest index $l^1 \in \{1, 2, \ldots, m\}$ such that $S_{l^1} \cap C_i^+ \neq \emptyset$ for some $i \in \{1 \ldots n\}$, then let us find the biggest index $l^2 \in \{1, 2, \ldots, l^1\}$ for which $S_{l^2} \cap C_i^- \neq \emptyset$. These indexes exist because $S_m \cap C_i^+ \neq \emptyset$ and $S_1 \cap C_i^- \neq \emptyset$. $S_{l^2} \cap C_i^-$ is the set of one element. The $(n - 1)$-face proper to this point is a subset of $S_{l^2+1}$ and it is $n$-colored. From this face choose point $z_1$ which is $i$th colored, $z_1 \in \text{int}[C]$ and $z_1 - e_i \in C_i^-$. Then from the chain $S_{l^2+1}, \ldots, S_{l^1}$ choose successively points $z_1, z_2, \ldots, z_r$ in the way that $\phi(z_j) = i$ for $j = 1, 2, \ldots, r$ and $z_j \neq z_{j+1}$ for $j = 1, 2, \ldots, r - 1$, $z_r \in S_{l^1}$. Observe that $S_1 \cap \bigcup_{j=1}^r C_j^-$ is the set of one point and $z_r \notin S_1 \cap C_i^+$ that is why $z_r \in \text{int}[C]$ and $z_r + e_i \in C_i^+$.

We can see immediately that for every $j \in \{1, 2, \ldots, t - 1\}$ there exists $l(j) \in \{l^2+1, \ldots, l^1\}$ such that $z_j, z_{j+1} \in S_{l(j)}$ hence we have $z_{j+1} = z_j + \sum_{l=1}^{m} e_{a(t)}$ where $m \leq n$. It is clear that $\{z_1, z_2, \ldots, z_r\} \in \text{int}[C]$.

For the sequence $z_1, \ldots, z_r$ we have the chain $P_1, \ldots, P_r$ where $P_j \in T(k)$ and $z_j \in P_j$ for $j = 1, \ldots, r$. We choose the elements $z_1, \ldots, z_r$ in the way that: $P_1 \cap I_3^- \neq \emptyset$, $P_i \cap I_3^+ \neq \emptyset$ and $P_j \cap P_{j+1} \neq \emptyset$ for $j = 1, 2, \ldots, r - 1$ moreover $F(P_j) = i$ for $j = 1, 2, \ldots, r$. The proof is completed.  

We call the partition $B_{\text{min}} \subset \{t \in T(k): F(t) \in \{1, 2\}\}$ between $I_3^-$ and $I_3^+$ in $I^3$ the minimal barrier if for the coloring map

$$F'(t) = \begin{cases} F(t) & \text{for } t \in B_{\text{min}}, \\ 3 & \text{for } t \notin B_{\text{min}}. \end{cases}$$

there is not a third colored chain connecting $I_3^-$ and $I_3^+$, and the color change of arbitrary segment $t \in B_{\text{min}}$ into the third implies the existence of such a chain.

Let us define the extension of the family $T(k)$:

$$T'(k) := \left\{ \left[ \frac{i_1}{k}, \frac{i_1+1}{k} \right] \times \cdots \times \left[ \frac{i_{n-1}}{k}, \frac{i_{n-1}+1}{k} \right] \times \left[ \frac{i_n}{k}, \frac{i_n+1}{k} \right] : i_j \in \{-1, 0, \ldots, k\} \text{ for } j = 1, 2, \ldots, n \right\}.$$

The extension of the set $B_{\text{min}}$ is:

$$B'_{\text{min}} := B_{\text{min}} \cup \{ t' \in T'(k) : \exists t \in B_{\text{min}} \text{ dim}[t \cap t'] = 2 \text{ or dim}[t \cap t'] = 1 \}.$$  

Modification of $B'_{\text{min}}$.

Let $B$ be an emptyset.

Divide an arbitrary element $t \in B_{\text{min}}$ onto 27 cubes (in natural way), some of them are connected with $I_3^+$ by chains of cubes not belonging to $B'_{\text{min}}$. Add these cubes to $B$.

Repeat the procedure for each element of $B'_{\text{min}}$.

Divide each cube from $T'(k)$ onto 27 cubes.

Now $B$ is the minimal barrier (modified).

**Observation 4.** In $I^3$ for each cube $t \in B \cap I^3 \neq \emptyset$ let $F \subset B$ be a family of cubes such that $\dim[t \cap t'] = 2$ for each $t' \in F$. For each $t', t'' \in F$ such that $\dim[t' \cap t''] = 1$ and there exists exactly one $d' \in B$ such that $\dim[t' \cap d'] = 2$, $\dim[t'' \cap d'] = 2$, $\dim[t \cap d'] = 1$ there is a maximal indexing subfamily $\{t_0, \ldots, t_{r-1}\} \subset F$ with property $\{t', t''\} \subset \phi^{-1}(I_3^- \cap C_i^-)$.
\{t_0, \ldots, t_{r-1}\}. Let us observe that there exists $j \in \{0, \ldots, l^*-1\}$ such that \(\{t', t''\} = \{t_j, t_{(j+1) \mod r}\}\). Moreover we can choose indexing of \(\{t_0, \ldots, t_{r-1}\}\) in the way that:

\[
\dim[t_j \cap t_{(j+1) \mod r}] = 1 \quad \text{for} \quad j = 0, \ldots, l^*-1
\]

and if $t \cap (I_1^+ \cap I_2^+) = \emptyset$ and $t \cap (I_1^- \cap I_2^-) = \emptyset$ then for all $j \in \{0, \ldots, l^*-1\}$ there exists exactly one $t'(j) \in B$ such that

\[
\dim[t_j \cap t'(j)] = 2, \quad \dim[t_{(j+1) \mod r} \cap t'(j)] = 2, \quad \dim[t \cap t'(j)] = 1
\]

**Proof.** Take $I^3$ divided onto $3^3$ cubes. The “central” one is $t$. In our problem we have two sets, which are composted of the 3 colored cubes, separated by points from $B$. Remembering about the minimality of $B$ there are only few cases to consider. We leave it to the reader. (See Fig. 1.) \(\square\)

The $i$th colored chain is called $i$-connected if $\dim[P_j \cap P_{j+1}] = n - i$ for $j = 1, \ldots, r - 1$.

**Theorem 2** (on the existence of the $i$-connected chain). For an arbitrary decomposition of $n$-dimensional cube $I^n$ into $k^n$ cubes and arbitrary coloring function $F : T(k) \to \{1, \ldots, n\}$ for some natural number $i \in \{1, \ldots, n\}$ there exists $i$th colored $i$-connected chain $P_1, \ldots, P_r$ such that $P_1 \cap I_1^+ \neq \emptyset$ and $P_r \cap I_r^- \neq \emptyset$.

**Proof.** The proof will be shown for $n = 3$. If there exists a third colored chain then end. Otherwise the minimal barrier $B_{\min}$ and its extension $B'_{\min}$ must exist. Now we can create the modification $B$. Let us define the map $F^*: B \to \{1, 2\}$:

\[
F^*(t) = \begin{cases} 
F(t') & \text{for } t \subseteq t' \in B_{\min}, \\
1 & \text{for } t \subseteq t' \in B'_{\min} \setminus B_{\min}, t' \cap (I_1^- \cup I_2^-) \neq \emptyset, \text{ and} \\
\dim[t' \cap (I_2^- \cup I_1^+)] < 1, \\
2 & \text{for } t \subseteq t' \in B' \setminus B, t' \cap (I_2^- \cup I_1^+) \setminus (I_1^- \cup I_2^-) \neq \emptyset.
\end{cases}
\]

Let us take an element $t \in B$ such that $t \cap (I_1^- \cap I_2^-) \neq \emptyset, t \subset I^3$. From Observation 4 we obtain the sequence $\{t_0, \ldots, t_{r-1}\} \subset B$ such that $\dim[t \cap t_j] = 2$ for $j = 0, \ldots, l^*-1$.

Moreover there exists $j \in \{0, \ldots, l^*-1\}$ such that

\[
\dim[t_j \cap I_1^-] = 2 \quad \text{and} \quad \dim[t_{(j+1) \mod r} \cap I_2^-] = 2.
\]

We have $F^*(t_j) = 1$, $F^*(t_{(j+1) \mod r}) = 2$. If $t \cap (I_1^+ \cap I_2^+) \neq \emptyset$ then end. Otherwise we have two cases:

The first case $F^*(t_l) = 1$: starting from $t_{(j+1) \mod r}$ we search for the first index $l$ such that $F^*(t_l) = 1$ in opposite direction to the position of $t_j$. The previous cube to $t_j$ has color 2.

Let us create sequences:

- $T^1$—we take element $t_j$, the next is $t$, the last is $t_l$. 
- $T^2$—we take $t_{(j+1) \mod r}$, the next cubes are the ones we met during the $t_l$ searching (with respect to the order), the last one is $t_{(l-1) \mod r}$.
Theorem 4. From Theorem (on the existence of the chain) there exists a cube $t' \in B$ which is defined in Observation 4.

The map $F^n$ is defined in the way that $t' \cap I^3 \neq \emptyset$. So we can use observation 4 for $t'$. We get the sequence of the “nearest” neighbors $\{t'_0, \ldots, t'_j\} \subset B'$. Let us observe that $\{t^1_{\text{end}}, t^2_{\text{end}}\} \subset \{t_0, \ldots, t_{i-1}\}$ and there exists $j \in \{0, \ldots, l^n - 1\}$ such that $\{t^1_{\text{end}}, t^2_{\text{end}}\} = \{t'_j, t'_j + (j+1) \text{mod } t\}$. We can repeat the reasoning maintaining the condition: if the cube $t'$ appears in the sequence we do not change the indexing of its neighbors.

For every $t \in B$ used in reasoning the number of two colored $t_j, t_{j+1}$ is even, that is why this procedure is well defined and must end for point $t \in B$ such that $t \cap I^+_1 \cap I^+_2 \neq \emptyset$.

We obtain two sequences: 1 colored $T^1 = \{t^1_{\text{end}}\}$, such that $\dim |t^1_{\text{end}} \cap t_{i+1}| > 0$ and 2 colored $T^2 = \{t^2_{\text{end}}\}$ such that $\dim |t^2_{\text{end}} \cap t_{m+1}| > 0$. From one of them we can choose 1- or 2-connected subsequence $P_1, \ldots, P_r$ such that the intersection with proper opposite faces is not empty (see Theorem 1).

When we suppose that the 2-colored 2-connected chain does not exist in our barrier we have sequences $T^1, T^2$ and from $T^1$ we must find 1-colored chain connecting $I^+_1$ with $I^+_1$ $(P_1, \ldots, P_r)$. If it is 1-connected then end. Otherwise let us take cubes $P_j, P_{j+1}$ such that $\dim |P_j \cap P_{j+1}| = 1$. We have for them one unused in previous reasoning cube $t' \in B$ such that $\dim |P_j \cap t'| = 2$, $\dim |P_{j+1} \cap t'| = 2$. If $F^n(t) = 1$ then end, else $t'$ is treated as a central one.

The neighbors have color 1 we extend our chain by adding these neighbors, else we can start the procedure defined at the beginning of the proof. Let us see that the procedure must return to $t$, because all the possibilities to stop it have already been used. This is the moment where we stop. In this way we get the rings $T'^1, T'^2$. The condition that $T'^2$ cannot “touch” $I^+_2$ (follows from assumption) allows us to extend $T^1$. For every pair of cubes which are not 1-connected, we repeat the method. We can observe that the procedure is well defined.

Defined in this way chain could be extended to the chain in $B_m$ in the same property.

**Theorem 3** (Poincaré). Let $f : I^n \to R^n$, $f(x) = (f_1(x), \ldots, f_n(x))$ be a continuous map such that $f_i(I^+_1) \subset (-\infty, 0]$ and $f_i(I^-_1) \subset [0, \infty)$ for $i = 1, \ldots, n$ then there exists $x_0 \in I^n$ such that $f(x_0) = (0, \ldots, 0)$.

**Proof.** Let us assume, that for each $x \in I^n$ $f(x) \neq (0, \ldots, 0)$ and let us define sets:

$$U_i = \{x \in I^n : f_i(x) \neq 0\} \quad \text{for } i = 1, \ldots, n.$$ 

We have $I^n = U_1 \cup \cdots \cup U_n$, $U_i$ is open for $i = 1, \ldots, n$. In space $R^n$ let’s take metric: $\delta(x, y) = \max\{|x_i - y_i| : i = 1, \ldots, n\}$, from the Lebesgue lemma of covering we have: there exists $\epsilon > 0$ such that for every $k \in N$ \( \frac{1}{k} < \epsilon \) we have for every $t \in T(k)$ some $j \in \{1, \ldots, n\}$ such that $t \subset U_j$.

Let us define map $F : T(k) \to \{1, \ldots, n\}$ coloring cube $I^n$:

$$F(t) := \min\{j : t \subset U_j\}$$

From Theorem (on the existing of the chain) there exists $i$th colored sequence $P_1(k), \ldots, P_r(k)$ connecting $i$th opposite faces of the cube $I^n$. The set $W := \bigcup_{i=1}^{r(k)} P_i(k)$ is closed and connected. The intersections $W \cap I^-_i \neq \emptyset \neq W \cap I^+_i$ hence there exists $x, y \in W$ such that $f_i(x) < 0$ and $f_i(y) > 0$ but $f(x)$ is continuous map hence $f_i(W)$ is connected in $R$ i.e., $f_i(W)$ is an interval containing $[f_i(x), f_i(y)]$. Therefore there exists $c \in W$ such that $f_i(c) = 0$.

**Contradiction.**

**Lemma 2.** (See [5].) Let $\{A_m : m \in N\}$ be a sequence of connected subsets of a compact metric space $X$ such that some sequence $\{a_n : n \in N\}$ of points $a_n \in A_n$ is converging in $X$. Then the set $A := \text{cl}(A_n : n \in N)$ is closed and connected. Where $x \in \text{cl}(A_n : n \in N)$ if and only if there exists an infinite set $M \subset N$ of points $x_m \in A_m$ such that $x = \lim x_m : m \in M$.

**Theorem 4.** Let $\{(H^-_i, H^+_i) : i = 1, \ldots, n\}$ be a family of pairs of closed sets such that $I^+_i \times I \subset H^+_i$, $I^-_i \times I \subset H^+_i$ and $I^n \times I = H^-_i \cup H^+_i$. Then there exists a connected set $W \subset \bigcap_{i=1}^{n} H^-_i \cap H^+_i$ such that $W \cap (I^n \times \{0\}) \neq \emptyset \neq W \cap (I^n \times \{1\})$. 

Proof. Let us define a map $F^*: I^n \times I \to \{1, \ldots, n+1\}$:

$$F^*(x, t) := \max \left\{ j + 1 : (x, t) \in \bigcap_{i=0}^{j} F_i^+ \right\},$$

where $F_0^+ = I^n \times I$ and $F_i^+ = H_i^+ \setminus I_i^- \times I$ for each $i = 1, \ldots, n$. Let us take $(n+1)$-dimensional combinatorial cube:

$$C^* = \left\{ \frac{1}{k}, 0, \ldots, 1, 1 + \frac{1}{k} \right\}^n \times \left\{ \frac{1}{k}, 0, \ldots, 1, 1 + \frac{1}{k} \right\}$$

and define a map $\phi^*: C^* \to \{1, \ldots, n+1\}$:

$$\phi^*(z) = \begin{cases} F^*(x, t) & \text{for } z \in I^n \times I, \\ 1 & \text{for } z \in C_1^- \cup C_2^+, \\ j & \text{for } z \in (C_j^- \cup C_{j+1}^+) \setminus (\bigcup_{i=1}^{j-1} (C_i^- \cup C_i^+)) \text{ for } j = 2, 3, \ldots, n, \\ n+1 & \text{for } z \in (C_n^- \cup C_n^+) \setminus (\bigcup_{i=1}^{n-1} (C_i^- \cup C_i^+) ) \end{cases}$$

Repeating reasoning which we wrote in the proof of Lemma 1 we have the sequence $(n+1)$-colored $(n+1)$-simplexes connecting opposite faces of cube $C^* : S_0^k, \ldots, S_{m(k)}^k$. But for:

$$(x, t) \in I_i^- \times I, \quad F^*(x, t) < i + 1,$$

$$(x, t) \in I_i^+ \times I, \quad F^*(x, t) \neq i,$$

that is why

$$F^* (S_0^k \cap (C^* \times \{0\})) = \{1, \ldots, n+1\} = F^* (S_m^k \cap (C^* \times \{1\})).$$

Let $W_k := \bigcup_{i=0}^{m(k)} \text{conv } S_i^k \quad k = 2, 3, \ldots$. The set $I^n \times I$ with euclidean metric or equivalent is a compact space, so we can find infinite subset $M \subset N$ and convergent subsequence $\{w_m : m \in M\}$, $w_m \in W_m$. According to Lemma 2 the set $W := L \{W_m : m \in M\}$ is connected and closed. Obviously

$$W \cap (I^n \times \{0\}) \neq \emptyset \neq W \cap (I^n \times \{1\}).$$

Let us prove $W \subset \bigcap_{i=1}^{n} H_i^- \cap H_i^+$. Fix $x \in W$ and choose a subsequence $\{x_k : k \in K\}$, $K \subset M$, $x_k \in W_k$, such that $\lim x_k = x$. Next, take $(n+1)$-colored $(n+1)$-simplexes $\{S_k : k \in K\}$, such that $x_k \in \text{conv } S_k \subset W_k$. Since $\lim \text{diam} (\text{conv } S_k) = 0$ we infer that for arbitrary subsequence $\{y_l : l \in L\}$, $L \subset K$, $y_l \in \text{conv } S_l$ we have $x = \lim \{y_l : l \in L\}$.

The proof will be completed if we show that for each $i = 1, \ldots, n$ and for all $(n+1)$-colored $(n+1)$-simplex $S$ proceed $H_i^- \cap S \neq \emptyset \neq H_i^+ \cap S$. However taking $z \in S$ and $\phi^*(z) = i$ we have $z \in H_i^-$, because $z \notin H_i^+$ and $I^n \times I = H_i^- \cup H_i^+$, in case $z \in S$ and $\phi^*(z) = i + 1$ we have $z \in H_i^+$.

**Theorem 5** (Parametric extension of the Poincaré theorem). (See [3].) Let $f : I^n \times I \to \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$ be a continuous map such that for each $i \leq n$

$$f_i(I_i^- \times I) \subset (-\infty, 0] \quad \text{and} \quad f_i(I_i^+ \times I) \subset [0, \infty).$$

Then there exists a connected set $W \subset f^{-1}(0)$ such that

$$W \cap (I^n \times \{0\}) \neq \emptyset \neq W \cap (I^n \times \{1\}).$$

**Proof.** Letting $H_i^- := f_{i}^{-1}(-\infty, 0]$, $H_i^+ := f_{i}^{-1}[0, \infty)$ assumption of Theorem 4 are satisfied. □
References