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Superfilters, Ramsey theory, and van der Waerden's Theorem

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Superfilters are generalizations of ultrafilters, and capture the underlying concept in Ramsey-theoretic theorems such as van der Waerden's Theorem. We establish several properties of superfilters, which generalize both Ramsey's Theorem and its variants for ultrafilters on the natural numbers. We use them to confirm a conjecture of Kočinac and Di Maio, which is a generalization of a Ramsey-theoretic result of Scheepers, concerning selections from open covers. Following Bergelson and Hindman's 1989 Theorem, we present a new simultaneous generalization of the theorems of Ramsey, van der Waerden, Schur, Folkman–Rado–Sanders, Rado, and others, where the colored sets can be much smaller than the full set of natural numbers.

1. A unified Ramsey Theorem

It is a simple observation that when each element of an infinite set is colored by one of finitely many colors, the set must contain an infinite monochromatic subset. When replacing *infinite* by *containing arithmetic progressions of arbitrary length*, we obtain van der Waerden's Theorem [28]. Some of the best references for many beautiful theorems of this kind, together with applications, are the classical [11], the monumental [12], the elegant Protasov [19], and the more recent [17]. These results lead naturally to the concept of superfilter.

Definition 1.1. For a set S, $[S]^n = \{F \subseteq S\colon |F| = n\}$, and $[S]^\infty$ is the family of infinite subsets of S.

A nonempty family $\mathcal{S} \subseteq [\mathbb{N}]^\infty$ is a *superfilter* if for all $A, B \subseteq \mathbb{N}$:

(1) If $A \in \mathcal{S}$ and $B \supseteq A$, then $B \in \mathcal{S}$.
(2) If $A \cup B \in \mathcal{S}$, then $A \in \mathcal{S}$ or $B \in \mathcal{S}$.

Superfilters were identified at least as early as in Berge's 1959 monograph [3] (under the name *grille*). They were also considered under the name *coideal* (e.g., [8]). Superfilters are large types of Banakh and Zdomskyy's *semifilters* and *unsplit* semifilters [1].

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Recall that a *nonprincipal ultrafilter* is a family as in Definition 1.1 which is also closed under finite intersections.[2] For brevity, by *ultrafilter* we always mean a nonprincipal one.

Example 1.2.

(1) Every ultrafilter is a superfilter.
(2) Every union of a family of ultrafilters is a superfilter.
(3) $[\mathbb{N}]^\infty$ is a superfilter which is not an ultrafilter.

In fact, one can show that every superfilter is a union of a family of ultrafilters, but we will not use this here.

Definition 1.3. AP is the family of all subsets of $\mathbb{N}$ containing arbitrarily long arithmetic progressions.

Clearly, AP is not an ultrafilter. The finitary version of van der Waerden's Theorem implies the following.

Theorem 1.4 *(van der Waerden). AP is a superfilter.*

Definition 1.5. $\mathcal{S} \to (\mathcal{S})^n_k$ is the statement: For each $A \in \mathcal{S}$ and each coloring $c : [A]^n \to \{1, 2, \dots, k\}$, there is $M \subseteq A$ such that $M \in \mathcal{S}$ and c is constant on $[M]^n$. The set M is called *monochromatic* for the coloring c.

Thus for upwards-closed $\mathcal{S} \subseteq [\mathbb{N}]^\infty$, the following are equivalent:

(1) $\mathcal{S}$ is a superfilter.
(2) $\mathcal{S} \to (\mathcal{S})^1_2$.
(3) $\mathcal{S} \to (\mathcal{S})^1_k$ for all k.

The assertion $\mathcal{S} \to (\mathcal{S})^n_k$ becomes stronger when n or k is increased.

Definition 1.6. A superfilter $\mathcal{S}$ is:

(1) *Ramsey* if $\mathcal{S} \to (\mathcal{S})^n_k$ holds for all n and k.
(2) *Strongly Ramsey* if for all pairwise disjoint $A_1, A_2, \dots$ with $\bigcup_{n \geqslant m} A_n \in \mathcal{S}$ for all m, there is $A \subseteq \bigcup_n A_n$ such that $A \in \mathcal{S}$ and $|A \cap A_n| \leqslant 1$ for all n.
(3) *Weakly Ramsey* if for all pairwise disjoint $A_1, A_2, \dots \notin \mathcal{S}$ with $\bigcup_n A_n \in \mathcal{S}$, there is $A \subseteq \bigcup_n A_n$ such that $A \in \mathcal{S}$ and $|A \cap A_n| \leqslant 1$ for all n.

Clearly, strongly Ramsey superfilters are weakly Ramsey. We will soon show that Ramsey is sandwiched between strongly Ramsey and weakly Ramsey. Before doing so, we give examples showing that converse implications cannot be proved.

Example 1.7. Fix a partition $\mathbb{N} = \bigcup_n I_n$ with each I_n infinite. Let $\mathcal{S}$ be the upwards closure of $\bigcup_n [I_n]^\infty$. It is easy to see that $\mathcal{S}$ is a superfilter.

$\mathcal{S}$ is Ramsey: Let $A \in \mathcal{S}$, and $c : [A]^n \to \{1, \dots, k\}$ be a coloring of A. Pick m such that $A \cap I_m$ is infinite, and use Ramsey's Theorem 1.14 for the coloring $c : [A \cap I_m]^2 \to \{1, \dots, k\}$ to obtain an infinite $M \subseteq A \cap I_m$ which is monochromatic for c.

$\mathcal{S}$ is not strongly Ramsey: For each m, $\bigcup_{n \geqslant m} I_n \in \mathcal{S}$, but if $|A \cap I_n| \leqslant 1$ for all n, then $A \notin \mathcal{S}$.

Example 1.8. Following is an example of a weakly Ramsey superfilter which is not Ramsey. Essentially the same example was, independently, found by Filipów, Mrożek, Recław, and Szuca [9].

Let $\mathbb{N}^*$ be the set of all finite sequences of natural numbers. For $\sigma, \rho \in \mathbb{N}^*$, write $\sigma \supseteq \rho$ if the sequence ρ is a prefix of σ. As $\mathbb{N}^*$ is countable, we may use it instead of $\mathbb{N}$ to define our superfilter. Say that a set $D \subseteq \mathbb{N}^*$ is *somewhere dense* if there is $\rho \in \mathbb{N}^*$ such that for each $\sigma \in \mathbb{N}^*$ with $\sigma \supseteq \rho$, there is $\eta \supseteq \sigma$ such that $\eta \in D$. Let $\mathcal{S}$ be the family of all somewhere dense subsets of $\mathbb{N}^*$.

It is not difficult to see that $\mathcal{S}$ is a superfilter, and that it is weakly Ramsey. To see that it is not Ramsey, define a coloring $c : [\mathbb{N}^*]^2 \to \{1, 2\}$ by $c(\sigma, \eta) = 1$ if one of σ, η is a prefix of the other, and 2 otherwise. If $M \subseteq \mathbb{N}^*$ is monochromatic of color 1, then M is a branch in $\mathbb{N}^*$, and thus $M \notin \mathcal{S}$. On the other hand, if M is somewhere dense, then it must contain at least two elements, one of which a prefix of the other. Thus, M is not monochromatic of color 2, either.

[2] Definition 1.1 does not change if we assume that A, B are disjoint in (2). But if, in addition, we replace there *or* by *exclusive or*, we obtain a characterization of ultrafilter. That is, the assumption about intersections need not be stated explicitly.

Examples 1.7 and 1.8 show that some hypothesis is required to make the Ramseyan notions coincide. We suggest a rather mild one.

Definition 1.9. A superfilter $\mathcal{S}$ is *shrinkable* if, for all pairwise disjoint $A_1, A_2, \dots$ with $\bigcup_{n\geqslant m} A_n \in \mathcal{S}$ for all m, there are $B_n \subseteq A_n$ such that $B_n \notin \mathcal{S}$ and $\bigcup_n B_n \in \mathcal{S}$.

Remark 1.10 *(Thuemmel).* A superfilter $\mathcal{S}$ is shrinkable if, and only if, for each sequence $S_1 \supseteq S_2 \supseteq \cdots$ of element of $\mathcal{S}$, there is $S \in \mathcal{S}$ such that for each n, $S \setminus S_n \notin \mathcal{S}$. To see this, identify S_m with $\bigcup_{n\geqslant m} A_n$ for each $m \in \mathbb{N}$, and S with $\bigcup_n B_n$.

All ultrafilters are shrinkable, for a trivial reason: If a disjoint union $\bigcup_n A_n$ is in the ultrafilter, and some A_m is in the ultrafilter, then $\bigcup_{n>m} A_n$ is not in the ultrafilter.

The superfilters in Examples 1.7 and 1.8 are not shrinkable. For shrinkable superfilters, we have a complete characterization of being Ramsey.

Theorem 1.11. *For superfilters $\mathcal{S}$, the following are equivalent*:

(1) *$\mathcal{S}$ is strongly Ramsey.*
(2) *$\mathcal{S}$ is Ramsey and shrinkable.*
(3) *$\mathcal{S} \to (\mathcal{S})^2_2$, and $\mathcal{S}$ is shrinkable.*
(4) *$\mathcal{S}$ is weakly Ramsey and shrinkable.*

Proof. (1) $\Rightarrow$ (2) As singletons do not belong to superfilters, strongly Ramsey implies shrinkable. It therefore suffices to prove the following.

Lemma 1.12. *Every strongly Ramsey superfilter is Ramsey.*

Proof. Let $\mathcal{S}$ be a strongly Ramsey superfilter, $A \in \mathcal{S}$, and $c : [A]^d \to \{1, \dots, k\}$. The proof is by induction on d, with $d = 1$ following from $\mathcal{S}$ being a superfilter.

Induction step: We repeatedly apply the following fact. For each $A \in \mathcal{S}$ and each $n \in A$, there is $M \subseteq A \setminus \{n\}$ such that $M \in \mathcal{S}$, and a color $i \in \{1, \dots, k\}$, such that for each $F \in [M]^{d-1}$, $c(\{n\} \cup F) = i$. Indeed, we can define a coloring $c_n : [A \setminus \{n\}]^{d-1} \to \{1, \dots, k\}$ by $c_n(F) = c(\{n\} \cup F)$ and use the induction hypothesis.

Enumerate $A = \{a_n\colon\ n \in \mathbb{N}\}$. Choose $A_{a_1} \subseteq A \setminus \{a_1\}$ and a color i_{a_1} such that $A_{a_1} \in \mathcal{S}$ and for each $F \in [A_{a_1}]^{d-1}$, $c(\{a_1\} \cup F) = i_{a_1}$. In a similar manner, choose inductively for each $n > 1$, $A_{a_n} \subseteq A_{a_{n-1}} \setminus \{a_n\}$ and a color i_{a_n} such that $A_{a_n} \in \mathcal{S}$ and for each $F \in [A_{a_n}]^{d-1}$, $c(\{a_n\} \cup F) = i_{a_n}$.

As $a_n \notin A_{a_n}$ for all n, $\bigcap_n A_{a_n} = \emptyset$. Let $B_0 = A \setminus A_{a_1}$ and for each $n > 0$, let $B_n = A_{a_n} \setminus A_{a_{n+1}}$. The sets B_n are pairwise disjoint, $\bigcup_n B_n = A$, and $\bigcup_{n\geqslant m} B_n = A_{a_m} \in \mathcal{S}$ for all m. As $\mathcal{S}$ is strongly Ramsey, there is $B \subseteq A$ such that $B \in \mathcal{S}$ and $|B \cap B_n| \leqslant 1$ for all n. Fix a color i such that $C = \{n \in B\colon\ i_n = i\} \in \mathcal{S}$.

Let $c_1 = \min C$. Inductively, for each $n > 1$ choose $c_n \in C$ such that $c_n > c_{n-1}$ and $C \setminus [1, c_n) \subseteq A_{c_{n-1}}$.[3] For each n, $C \cap [c_n, c_{n+1})$ is finite and thus not a member of $\mathcal{S}$. As $\bigcup_n (C \cap [c_n, c_{n+1})) = C \in \mathcal{S}$ and $\mathcal{S}$ is weakly Ramsey, there is $D \in \mathcal{S}$ such that $D \subseteq C$ and $|D \cap [c_n, c_{n+1})| \leqslant 1$ for all n. As

$$D = \left(D \cap \bigcup_{n\in\mathbb{N}} [c_{2n}, c_{2n+1})\right) \cup \left(D \cap \bigcup_{n\in\mathbb{N}} [c_{2n-1}, c_{2n})\right),$$

there is $l \in \{0, 1\}$ such that $M = D \cap \bigcup_n [c_{2n-l}, c_{2n+1-l}) \in \mathcal{S}$. Let $m_1 < m_2 < \cdots < m_d$ be members of M. Let n be minimal such that $m_1 < c_n$. Then

$$m_2, \dots, m_d \in C \setminus [1, c_{n+1}) \subseteq A_{c_n} \subseteq A_{m_1},$$

and thus $c(\{m_1, \dots, m_d\}) = c(\{m_1\} \cup \{m_2, \dots, m_d\}) = i_{m_1} = i$. $\square$

(2) $\Rightarrow$ (3) Trivial.
(3) $\Rightarrow$ (4) In fact, the following holds.

Lemma 1.13. *If $\mathcal{S} \to (\mathcal{S})^2_2$, then $\mathcal{S}$ is weakly Ramsey.*

Proof. Let $A_1, A_2, \dots$ be as in the definition of weakly Ramsey. Let $D = \bigcup_n A_n$, and define a coloring $c : [D]^2 \to \{1, 2\}$ by

$$c(m, k) = \begin{cases} 1 & (\exists n)\ m, k \in A_n, \\ 2 & \text{otherwise.} \end{cases}$$

[3] For example, let $k = |C \setminus A_{c_{n-1}}| + 1$ and let c_n be the kth element of C.

As $\mathcal{S}$ is Ramsey, there is a monochromatic $A \subseteq D$ with $A \in \mathcal{S}$. If all elements of $[A]^2$ have color 1, then $A \subseteq A_n$ for some n, and thus $A_n \in \mathcal{S}$, a contradiction. Thus, all elements of $[A]^2$ have color 2, which means that $|A \cap A_n| \leqslant 1$ for all n. □

(4) ⇒ (1) Let $A_1, A_2, \ldots$ be as in the definition of strongly Ramsey. As $\mathcal{S}$ is shrinkable, there are $B_n \subseteq A_n$ such that $B_n \notin \mathcal{S}$ and $B = \bigcup_n B_n \in \mathcal{S}$. As $\mathcal{S}$ is weakly Ramsey, there is a subset A of B such that $A \in \mathcal{S}$ and $|A \cap B_n| \leqslant 1$ for all n. As $B_n \subseteq A_n$ for all n and the sets A_n are pairwise disjoint, $|A \cap A_n| \leqslant 1$ for all n.

This completes the proof of Theorem 1.11. □

Corollary 1.14 *(Ramsey [21]).* $[\mathbb{N}]^\infty \to ([\mathbb{N}]^\infty)^n_k$ *for all n and k.*

Proof. Clearly, $[\mathbb{N}]^\infty$ is strongly Ramsey. □

Corollary 1.15 *(Booth–Kunen [5]). An ultrafilter is weakly Ramsey if, and only if, it is Ramsey.*

Proof. Ultrafilters are shrinkable. □

The following definition and subsequent result will be useful later.

Definition 1.16 *(Scheepers [23]).* $\mathsf{S}_1(\mathcal{S}, \mathcal{S})$ is the statement: Whenever $S_1, S_2, \ldots \in \mathcal{S}$, there are $s_n \in S_n$, $n \in \mathbb{N}$, such that $\{s_n\colon n \in \mathbb{N}\} \in \mathcal{S}$.

Theorem 1.17. *For superfilters* $\mathcal{S}$:

(1) *If $\mathcal{S}$ is strongly Ramsey, then $\mathsf{S}_1(\mathcal{S}, \mathcal{S})$ holds.*
(2) *$\mathsf{S}_1(\mathcal{S}, \mathcal{S})$ implies that $\mathcal{S}$ is shrinkable.*

Proof. (1) We first observe that, in the definition of strongly Ramsey, there is no need for the sets A_n to be pairwise disjoint.

Lemma 1.18. *If a superfilter $\mathcal{S}$ is strongly Ramsey, then for all nonempty $A_1, A_2, \ldots$ with $\bigcup_{n \geqslant m} A_n \in \mathcal{S}$ for all m, there are $a_n \in A_n$, $n \in \mathbb{N}$, such that $A = \{a_n\colon n \in \mathbb{N}\} \in \mathcal{S}$.*

Proof. Assume that $\bigcup_{n \geqslant m} A_n \in \mathcal{S}$ for all m. Let

$$L = \bigcap_{m \in \mathbb{N}} \bigcup_{n \geqslant m} A_n.$$

If $L \in \mathcal{S}$, enumerate $L = \{l_n\colon n \in \mathbb{N}\}$. Pick m_1 such that $a_{m_1} := l_1 \in A_{m_1}$. For each $n > 1$, there is $m_n > m_{n-1}$ such that $a_{m_n} := l_n \in A_{m_n}$. For $m \notin \{m_n\}_{n \in \mathbb{N}}$, pick any $a_m \in A_m$. Then we obtain a sequence as required.

Thus, assume that $L \notin \mathcal{S}$. Taking $B_n = A_n \setminus L$ for all n, we have that

$$\bigcup_{n \geqslant m} B_n = \left(\bigcup_{n \geqslant m} A_n\right) \setminus L \in \mathcal{S}$$

for all m. Now, $\bigcap_m \bigcup_{n \geqslant m} B_n = \emptyset$, that is, each $k \in \bigcup_n B_n$ belongs to only finitely many B_n. For each n, let

$$C_n = B_n \setminus \bigcup_{m > n} B_m.$$

The sets C_n are pairwise disjoint, and for each m, $\bigcup_{n \geqslant m} C_n = \bigcup_{n \geqslant m} B_n \in \mathcal{S}$. As $\mathcal{S}$ is strongly Ramsey, we obtain $A \subseteq \bigcup_n C_n$ such that $A \in \mathcal{S}$ and $|A \cap C_n| \leqslant 1$ for all n. For each n, let $a_n \in A \cap C_n$ if $|A \cap C_n| = 1$, and an arbitrary element of A_n otherwise. Then the sequence $\{a_n\}_{n \in \mathbb{N}}$ is as required. □

Thus, assume that $A_1, A_2, \ldots \in \mathcal{S}$. Clearly, they are all nonempty, and $\bigcup_{n \geqslant m} A_n \in \mathcal{S}$ for all m. By Lemma 1.18, there are $a_n \in A_n$, $n \in \mathbb{N}$, such that $\{a_n\colon n \in \mathbb{N}\} \in \mathcal{S}$.

(2) Apply $\mathsf{S}_1(\mathcal{S}, \mathcal{S})$ to the sequence $\bigcup_{n \geqslant m} A_n$, $m \in \mathbb{N}$, and recall that finite sets do not belong to superfilters. □

As Ramsey does not imply strongly Ramsey (Example 1.7), but does for shrinkable superfilters (Theorem 1.11(4)), we have that the converse of Theorem 1.17(2) is false. Unfortunately, we do not have a concrete example for the following.

Conjecture 1.19. *There is a superfilter $\mathcal{S}$ such that $\mathsf{S}_1(\mathcal{S}, \mathcal{S})$ holds, but $\mathcal{S}$ is not strongly* (*equivalently, by Theorem* 1.17(2)*, weakly*) *Ramsey.*

2. An application to topological selection principles

Our initial motivation for studying superfilters came from an attempt to provide a (mainly) combinatorial proof of a major Ramsey-theoretic result of Scheepers, concerning selections from open covers. The general theory has connections and applications far beyond Ramsey theory, and the interested reader is referred to the survey papers [24,14,27]. The Ramsey-theoretic aspect of this theory is surveyed in [15]. Here, we present only the concepts which are necessary for the present paper.

Fix a topological space X. A family $\mathcal{U}$ of subsets of X is a *cover* of X if $X \notin \mathcal{U}$ but $X = \bigcup \mathcal{U}$. A cover $\mathcal{U}$ of X is an *ω-cover* if for each finite $F \subseteq X$, there is $U \in \mathcal{U}$ such that $F \subseteq U$. Let $\Omega = \Omega(X)$ denote the family of all open ω-covers of X. According to Definition 1.5, the statement $\Omega \to (\Omega)^2_2$ makes sense, and it is natural to ask what is required from X for this statement to be true. Say that X is *Ω-Lindelöf* if each element of Ω contains a countable element of Ω. The following result is essentially proved in [23], using an auxiliary result from [13]. In the general form stated here, it is proved in [16].

Theorem 2.1 *(Scheepers [23,13,16]). For Ω-Lindelöf spaces, the following are equivalent*:

(1) $\mathsf{S}_1(\Omega, \Omega)$.
(2) $\Omega \to (\Omega)^2_2$.
(3) $\Omega \to (\Omega)^n_k$ *for all n, k.*

We proceed in a general manner that will prove, in addition to Scheepers's Theorem, a conjecture of Di Maio, Kočinac, and Meccariello from [6], and a subsequent one of Di Maio and Kočinac from [7].

Let $C(X)$ denote the space of continuous real-valued functions of X. ω-covers arise when considering the closure operator in $C(X)$, with the topology of pointwise convergence [10]. When considering the compact-open topology, *k-covers* arise, which are covers such that each compact set is contained in a member of the cover (e.g., [6] and references therein). In [6] it is conjectured that Scheepers's Theorem also holds when ω-covers are replaced by k-covers.

A natural generalization of these topologies on $C(X)$ gives rise to the following notion. An *abstract boundedness* is a family $\mathbb{B}$ of nonempty closed subsets of X which is closed under taking finite unions and closed subsets, and contains all singletons [7]. A cover $\mathcal{U}$ is a *$\mathbb{B}$-cover* if each $B \in \mathbb{B}$ is contained in some member of $\mathcal{U}$. In [7] it is conjectured that Scheepers's Theorem holds in general, when ω-covers are replaced by $\mathbb{B}$-covers for any abstract boundedness notion $\mathbb{B}$.

Closing an abstract boundedness notion $\mathbb{B}$ downwards will not change the notion of $\mathbb{B}$-covers. Thus, for simplicity we use a more familiar notion. A nonempty family $\mathcal{I}$ of subsets of X is an *ideal* on X if $X \notin \mathcal{I}$, $\{x\} \in \mathcal{I}$ for all $x \in X$, and for all $A, B \in \mathcal{I}$, $A \cup B \in \mathcal{I}$.

Definition 2.2. Fix an ideal $\mathcal{I}$ on X. $\mathcal{U}$ is an $\mathcal{I}$-cover of X if $X \notin \mathcal{U}$, and for each $B \in \mathcal{I}$ there is $U \in \mathcal{U}$ such that $B \subseteq U$. $\mathcal{O}_\mathcal{I}$ is the family of all open $\mathcal{I}$-covers of X.

Lemma 2.3.

(1) *If* $\mathcal{U}_1 \cup \mathcal{U}_2 \in \mathcal{O}_\mathcal{I}$, *then* $\mathcal{U}_1 \in \mathcal{O}_\mathcal{I}$ *or* $\mathcal{U}_2 \in \mathcal{O}_\mathcal{I}$.
(2) *Each* $\mathcal{U} \in \mathcal{O}_\mathcal{I}$ *is infinite.*

Proof. (1) Assume that $B_1, B_2 \in \mathcal{I}$ witness that $\mathcal{U}_1, \mathcal{U}_2 \notin \mathcal{O}_\mathcal{I}$, respectively. Then no element of $\mathcal{U}_1 \cup \mathcal{U}_2$ contains $B_1 \cup B_2$.
(2) $\mathcal{O}_\mathcal{I} \subseteq \Omega$. □

Let $\mathcal{U} \in \mathcal{O}_\mathcal{I}$. If $\mathcal{U}$ is countable, we may use it as an index set instead of $\mathbb{N}$, and consider superfilters on $\mathcal{U}$.

Definition 2.4. $\mathcal{U}_\mathcal{I} = \{\mathcal{V} \subseteq \mathcal{U}\colon\ \mathcal{V} \in \mathcal{O}_\mathcal{I}\} = P(\mathcal{U}) \cap \mathcal{O}_\mathcal{I}$.

Lemma 2.3 implies the following.

Corollary 2.5. *For each countable* $\mathcal{U} \in \mathcal{O}_\mathcal{I}$, $\mathcal{U}_\mathcal{I}$ *is a superfilter.*

$\mathcal{U}_\mathcal{I}$ cannot be assumed to be an ultrafilter when proving Scheepers's Theorem 2.1: If $\mathsf{S}_1(\Omega, \Omega)$ holds, then each $\mathcal{U} \in \Omega$ can be split into two disjoint elements of Ω [23].

We are now ready to prove the general statement. Say that X is *$\mathcal{O}_\mathcal{I}$-Lindelöf* if each element of $\mathcal{O}_\mathcal{I}$ contains a countable element of $\mathcal{O}_\mathcal{I}$.

Theorem 2.6. *Let* $\mathcal{I}$ *be an ideal on X. For* $\mathcal{O}_\mathcal{I}$*-Lindelöf spaces, the following are equivalent*:

(1) $\mathsf{S}_1(\mathcal{O}_\mathcal{I}, \mathcal{O}_\mathcal{I})$.
(2) *For all disjoint* $\mathcal{U}_1, \mathcal{U}_2, \ldots \notin \mathcal{O}_\mathcal{I}$ *with* $\bigcup_n \mathcal{U}_n \in \mathcal{O}_\mathcal{I}$, *there is* $\mathcal{V} \subseteq \bigcup_n \mathcal{U}_n$ *such that* $\mathcal{V} \in \mathcal{O}_\mathcal{I}$ *and* $|\mathcal{V} \cap \mathcal{U}_n| \leqslant 1$ *for all n.*

(3) $\mathcal{O}_\mathcal{I} \to (\mathcal{O}_\mathcal{I})^2_2$.
(4) $\mathcal{O}_\mathcal{I} \to (\mathcal{O}_\mathcal{I})^n_k$ *for all n, k.*

Proof. Using $\mathcal{O}_\mathcal{I}$-Lindelöfness, we may restrict attention to countable $\mathcal{I}$-covers in all of our arguments. More precisely, we prove the stronger assertion, where $\mathcal{O}_\mathcal{I}$ is replaced with the family of *countable* open $\mathcal{I}$-covers, and no assumption is posed on the space X.

(4) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (2) Let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be as in (2). Set $\mathcal{U} = \bigcup_n \mathcal{U}_n$. Then $\mathcal{U} \in \mathcal{O}_\mathcal{I}$, and by Corollary 2.5, $\mathcal{U}_\mathcal{I}$ is a superfilter. By (3), we have in particular $\mathcal{U}_\mathcal{I} \to (\mathcal{U}_\mathcal{I})^2_2$. By Lemma 1.13, $\mathcal{U}_\mathcal{I}$ is weakly Ramsey. As $\mathcal{U}_1, \mathcal{U}_2, \ldots \notin \mathcal{U}_\mathcal{I}$ and $\bigcup_n \mathcal{U}_n = \mathcal{U} \in \mathcal{U}_\mathcal{I}$, there is $\mathcal{V} \in \mathcal{U}_\mathcal{I} \subseteq \mathcal{O}_\mathcal{I}$ as required.

(2) $\Rightarrow$ (1) Assume that $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O}_\mathcal{I}$. Fix $\{U_n\colon n \in \mathbb{N}\} \in \mathcal{O}_\mathcal{I}$. For each n, let

$$\mathcal{V}_n = \{U_n \cap U\colon U \in \mathcal{U}_n\}.$$

Then

$$\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n \in \mathcal{O}_\mathcal{I}.$$

By Corollary 2.5, $\mathcal{U}_\mathcal{I}$ is a superfilter. By (2), $\mathcal{U}_\mathcal{I}$ is weakly Ramsey. Now, $\bigcup_n \mathcal{V}_n = \mathcal{U} \in \mathcal{U}_\mathcal{I}$, and for each n, $\mathcal{V}_n \notin \mathcal{U}_\mathcal{I}$. By thinning out the sets $\mathcal{V}_n$ if necessary, we may assume that they are disjoint. Thus, there is $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \in \mathcal{U}_\mathcal{I}$ and $|\mathcal{V} \cap \mathcal{V}_n| \leqslant 1$ for all n.

For each n, if $|\mathcal{V} \cap \mathcal{V}_n| = 1$, take the $U \in \mathcal{U}_n$ such that $U_n \cap U \in \mathcal{V}$, and otherwise take an arbitrary $U \in \mathcal{U}_n$. We obtain an $\mathcal{I}$-cover of X with one element from each $\mathcal{U}_n$.

(1) $\Rightarrow$ (4) Let $\mathcal{U} \in \mathcal{O}_\mathcal{I}$. Let $\mathcal{V}$ be the closure of $\mathcal{U}$ under finite intersections. $\mathcal{V}$ is countable, and $\mathcal{U} \in \mathcal{V}_\mathcal{I} \subseteq \mathcal{O}_\mathcal{I}$.

Consider the superfilter $\mathcal{V}_\mathcal{I}$. By $\mathsf{S}_1(\mathcal{O}_\mathcal{I}, \mathcal{O}_\mathcal{I})$, we have $\mathsf{S}_1(\mathcal{V}_\mathcal{I}, \mathcal{V}_\mathcal{I})$. By Theorem 1.17, $\mathcal{V}_\mathcal{I}$ is shrinkable. By Theorem 1.11, it remains to prove that $\mathcal{V}_\mathcal{I}$ is weakly Ramsey.

Let $\mathcal{V}_1, \mathcal{V}_2, \ldots \notin \mathcal{V}_\mathcal{I}$ be pairwise disjoint with $\bigcup_{n\geqslant m} \mathcal{V}_n \in \mathcal{V}_\mathcal{I}$ for all m. For each n, let

$$\mathcal{U}_n = \left\{ \bigcap_{m\in I} V_m\colon\ I \subseteq \mathbb{N},\ |I| = n\ (\forall m \in I)\ V_m \in \mathcal{V}_m \right\}.$$

Claim 2.7. $\mathcal{U}_n \in \mathcal{V}_\mathcal{I}$.

Proof. As $\mathcal{V}$ is closed under finite intersections, $\mathcal{U}_n \subseteq \mathcal{V}$. Assume that there is $B \in \mathcal{I}$ not contained in any member of $\mathcal{U}_n$. Let $I = \{m\colon (\exists U \in \mathcal{V}_m)\ B \subseteq U\}$. Then $|I| < n$. For each $m \in I$ choose $B_m \in \mathcal{I}$ witnessing that $\mathcal{V}_m \notin \mathcal{O}_\mathcal{I}$. Then $B \cup \bigcup_{m\in I} B_m$ is not covered by any $U \in \bigcup_n \mathcal{V}_n$, a contradiction. $\square$

Apply $\mathsf{S}_1(\mathcal{V}_\mathcal{I}, \mathcal{V}_\mathcal{I})$ to the sequence $\mathcal{U}_n$, $n \in \mathbb{N}$, to obtain elements $U_n \in \mathcal{U}_n$ with $\{U_n\colon n \in \mathbb{N}\} \in \mathcal{V}_\mathcal{I}$. Let m_1 be such that $V_{m_1} := U_1 \in \mathcal{V}_{m_1}$. Inductively, for each $n > 1$, U_n is an intersection of elements from n many $\mathcal{V}_m$'s, and thus there are m_n distinct from $m_1, \ldots, m_{n-1}$, and an element $V_{m_n} \in \mathcal{V}_{m_n}$, such that $U_n \subseteq V_{m_n}$. Then $\mathcal{A} = \{V_{m_n}\colon n \in \mathbb{N}\} \in \mathcal{V}_\mathcal{I}$. $\mathcal{A} \subseteq \bigcup_n \mathcal{V}_n$, and $|\mathcal{A} \cap \mathcal{V}_n| \leqslant 1$ for all n. $\square$

At the price of a slightly less combinatorial proof, we can weaken the restriction of $\mathcal{O}_\mathcal{I}$-Lindelöfness substantially.

Theorem 2.8. *Assume that X has a countable open $\mathcal{I}$-cover. Then the four items of Theorem* 2.6 *are equivalent.*

Proof. The proof is the same as that of Lemma 1.13, but we argue directly in some of its steps. We do this briefly.

(1) $\Rightarrow$ (4) By (1), X is $\mathcal{O}_\mathcal{I}$-Lindelöf, and the argument in the proof of Theorem 2.6 applies.

(3) $\Rightarrow$ (2) Let $\mathcal{U}_1, \mathcal{U}_2, \ldots \notin \mathcal{O}_\mathcal{I}$ be disjoint with $\bigcup_n \mathcal{U}_n \in \mathcal{O}_\mathcal{I}$. Set $\mathcal{U} = \bigcup_n \mathcal{U}_n$. Define a coloring $c : [\mathcal{U}]^2 \to \{1, 2\}$ by

$$c(U, V) = \begin{cases} 1 & (\exists n)\ U, V \in \mathcal{U}_n, \\ 2 & \text{otherwise}. \end{cases}$$

By (3), there is a monochromatic $\mathcal{V} \subseteq \mathcal{U}$ with $\mathcal{V} \in \mathcal{O}_\mathcal{I}$. It is easy to see that $\mathcal{V}$ is as required in (2).

(2) $\Rightarrow$ (1) Use the premised $\{U_n\colon n \in \mathbb{N}\} \in \mathcal{O}_\mathcal{I}$: Assume that $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O}_\mathcal{I}$. For each n, let

$$\mathcal{V}_n = \{U_n \cap U\colon U \in \mathcal{U}_n\}.$$

Now, $\bigcup_n \mathcal{V}_n = \mathcal{U} \in \mathcal{U}_\mathcal{I}$, and for each n, $\mathcal{V}_n \notin \mathcal{U}_\mathcal{I}$. By thinning out the sets $\mathcal{V}_n$ if necessary, we may assume that they are disjoint. By (2), there is $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \in \mathcal{O}_\mathcal{I}$ and $|\mathcal{V} \cap \mathcal{V}_n| \leqslant 1$ for all n. $\square$

For T_1 topological spaces, the assumption that X has a countable open $\mathcal{I}$-cover can be simplified.

Lemma 2.9. *Let $\mathcal{I}$ be an ideal on a T_1 space X. There is a countable $\mathcal{I}$-cover of X if, and only if, there is a countable $D \subseteq X$ such that $D \notin \mathcal{I}$.*

Proof. ($\Rightarrow$) Let $\mathcal{U}$ be a countable $\mathcal{I}$-cover of X. For each $U \in \mathcal{U}$, pick $x_U \in X \setminus U$. Take $D = \{x_U\colon U \in \mathcal{U}\}$.

($\Leftarrow$) $\mathcal{U} = \{X \setminus \{x\}\colon x \in D\}$ is a countable $\mathcal{I}$-cover of X. □

In particular, Scheepers's Theorem 2.1 is true *for all T_1 spaces*: It is trivially true for finite spaces, and in the remaining case there is a countably infinite subset.

In the case of k-covers, it suffices to assume that X has a countable subset with noncompact closure.

3. Back to van der Waerden's Theorem

We reconsider van der Waerden's superfilter AP of all sets containing arbitrarily long arithmetic progressions.

Example 3.1. Furstenberg and Weiss (unpublished) proved that $\mathrm{AP} \nrightarrow (\mathrm{AP})^2_2$. Using Lemma 1.13, we can reproduce their observation by showing that AP is not even *weakly Ramsey*: Let $A_1 = \{1\}$, and for each $n > 1$, let $m_n = 2\max A_{n-1}$, and $A_n = \{m_n + 1, m_n + 2, \ldots, m_n + n\}$. For each n, $A_n \notin \mathrm{AP}$, and $\bigcup_n A_n \in \mathrm{AP}$. But there is no arithmetic progression of length 3 with at most one element in each A_n.

Example 3.1 motivates us to look for a property which is weaker than being Ramsey but still implies Ramsey's Theorem, and which is satisfied by AP. A natural candidate is available in the literature.

Definition 3.2 *(Baumgartner and Taylor [2]).* $\mathcal{S} \to \lceil \mathcal{S} \rceil^n_k$ is the statement: For each $A \in \mathcal{S}$ and each coloring $c : [A]^n \to \{1, 2, \ldots, k\}$, there is $M \subseteq A$ such that $M \in \mathcal{S}$, and a partition of M into finite pieces, such that c is constant on elements of $[M]^n$ containing at most one element from each piece.

Any provable assertion of the form $\mathcal{S} \to \lceil \mathcal{S} \rceil^n_k$ with $\emptyset \neq \mathcal{S} \subseteq [\mathbb{N}]^\infty$ and $n, k \geqslant 2$ is an improvement of Ramsey's Theorem: Given a coloring of $\mathbb{N}$, take $M \in \mathcal{S}$ and a partition of M into finite sets as promised by $\mathcal{S} \to \lceil \mathcal{S} \rceil^n_k$. Then any choice of one element from each piece gives an infinite monochromatic set. $\mathcal{S} \to \lceil \mathcal{S} \rceil^n_k$ also implies that $\mathcal{S}$ is a superfilter.

Lemma 3.3. *For each upwards-closed $\emptyset \neq \mathcal{S} \subseteq [\mathbb{N}]^\infty$:*

(1) *If $\mathcal{S} \to \lceil \mathcal{S} \rceil^n_k$, $l \leqslant n$, and $m \leqslant k$, then $\mathcal{S} \to \lceil \mathcal{S} \rceil^l_m$.*
(2) *For each k, $\mathcal{S} \to \lceil \mathcal{S} \rceil^1_k$ is equivalent to $\mathcal{S} \to (\mathcal{S})^1_k$.*

Proof. (1) Given $c : [A]^l \to \{1, \ldots, m\}$, define $f : [A]^n \to \{1, \ldots, k\}$ by letting $f(F)$ be the c-color of the l smallest elements of F. Use $\mathcal{S} \to \lceil \mathcal{S} \rceil^n_k$ to obtain $M \subseteq A$ such that $M \in \mathcal{S}$, and a partition of M into finite sets, such that sets with elements coming from distinct pieces of M all have the same f-color i.

For each $F \in [A]^l$ with elements coming from distinct pieces of M, take arbitrary $n - l$ elements from other pieces of M, which are greater than all elements of F (this can be done since M is infinite, and the pieces are finite). Add these elements to F, to obtain F'. Then $c(F) = f(F') = i$.

(2) Immediate from the definition. □

Definition 3.4. A superfilter $\mathcal{S}$ is a *P-point* if for all members $A_1 \supseteq A_2 \supseteq \cdots$ of $\mathcal{S}$, there is $A \in \mathcal{S}$ such that $A \setminus A_n$ is finite for all n.

Definition 3.5 *(Scheepers [23]).* $\mathsf{S}_{\mathrm{fin}}(\mathcal{S}, \mathcal{S})$ is the statement: Whenever $S_1, S_2, \ldots \in \mathcal{S}$, there are finite $F_n \subseteq S_n$, $n \in \mathbb{N}$, such that $\bigcup_n F_n \in \mathcal{S}$.

Theorem 3.6. *The following are equivalent for superfilters $\mathcal{S}$:*

(1) *$\mathcal{S}$ is a P-point.*
(2) $\mathsf{S}_{\mathrm{fin}}(\mathcal{S}, \mathcal{S})$.
(3) *For all disjoint $A_1, A_2, \ldots$ with $\bigcup_{n \geqslant m} A_n \in \mathcal{S}$ for all m, there is $A \subseteq \bigcup_n A_n$ such that $A \in \mathcal{S}$ and $A \cap A_n$ is finite for all n.*
(4) *For each partition $\mathbb{N} = \bigcup_n A_n$ with $\bigcup_{n \geqslant m} A_n \in \mathcal{S}$ for all m, there is $A \in \mathcal{S}$ such that $A \cap A_n$ is finite for all n.*
(5) *$\mathcal{S} \to \lceil \mathcal{S} \rceil^2_2$ and $\mathcal{S}$ is shrinkable.*
(6) *$\mathcal{S} \to \lceil \mathcal{S} \rceil^n_k$ for all n, k, and $\mathcal{S}$ is shrinkable.*

Proof. (1) $\Rightarrow$ (2) Assume that $S_1, S_2, \ldots \in \mathcal{S}$. For each n, let $A_n = \bigcup_{m\geqslant n} S_m$. By (1), there is $A \in \mathcal{S}$ such that $A \setminus A_n$ is finite for all n. For each n, let $F_n = (A \cap S_n) \setminus A_{n+1}$. Let $B = A \cap \bigcap_n A_n$. For each n, add at most finitely many elements of B to F_n, in a way that F_n remains finite, $F_n \subseteq S_n$, and $\bigcup_n F_n \supseteq B$. Then $A \setminus \bigcup_n F_n$ is finite, and thus $\bigcup_n F_n \in \mathcal{S}$.

(2) $\Rightarrow$ (3) Apply $\mathsf{S}_{\mathrm{fin}}(\mathcal{S}, \mathcal{S})$ to the sequence $\bigcup_{n\geqslant m} A_n$, $m \in \mathbb{N}$.

(3) $\Rightarrow$ (4) Trivial.

(4) $\Rightarrow$ (1) Assume that $B_1 \supseteq B_2 \supseteq \cdots$ are members of $\mathcal{S}$. We may assume that $B_1 = \mathbb{N}$. Let $A_0 = \bigcap_n B_n$. If $A_0 \in \mathcal{S}$ we are done, so assume that $A_0 \notin \mathcal{S}$.

For each n, let $A_n = B_n \setminus B_{n+1}$. $\mathbb{N} = A_0 \cup \bigcup_n A_n$ is a partition of $\mathbb{N}$ as required in (3): $\bigcup_n A_n \in \mathcal{S}$ as $A_0 \notin \mathcal{S}$. For each n, $\bigcup_{m\geqslant n} A_m = B_n \setminus A_0 \in \mathcal{S}$, since $B_n \in \mathcal{S}$. Take $A \in \mathcal{S}$ such that $A \cap A_n$ is finite for all n. Then $A \setminus B_n$ is finite for all n.

(5) $\Rightarrow$ (3) Consider disjoint $A_1, A_2, \ldots$ with $\bigcup_{n\geqslant m} A_n \in \mathcal{S}$ for all m. As $\mathcal{S}$ is shrinkable, we may assume that $A_n \notin \mathcal{S}$ for all n. Let $D = \bigcup_n A_n$, and define a coloring $c : [D]^2 \to \{1, 2\}$ by

$$c(m,k) = \begin{cases} 1 & (\exists n)\ m, k \in A_n, \\ 2 & \text{otherwise.} \end{cases}$$

By $\mathcal{S} \to \lceil \mathcal{S} \rceil^2_2$, there is a partition $M = \bigcup_n F_n \subseteq D$ into finite sets, such that $M \in \mathcal{S}$ and c is constant on pairs of elements coming from different F_n's.

Assume that these pairs have color 1. Fix $k \in F_1$, and n such that $k \in A_n$. For each $m \neq 1$ and each $i \in F_m$, $c(k, i) = 1$ and thus $i \in A_n$, too. But then each $l \in F_1$ has $c(i, l) = 1$, and thus $l \in A_n$, too. Thus, $M \subseteq A_n$. As $M \in \mathcal{S}$, we have that $A_n \in \mathcal{S}$; a contradiction. Thus, all pairs coming from different F_n's, must come from different A_n's. Take $A = \bigcup_n F_n$.

(1), (3) $\Rightarrow$ (6) Clearly, (3) implies that $\mathcal{S}$ is shrinkable. We prove that $\mathcal{S} \to \lceil \mathcal{S} \rceil^d_k$ for all d, k, by induction on d.

Let $\mathcal{S}$ be a P-point superfilter, $A \in \mathcal{S}$, and $c : [A]^d \to \{1, \ldots, k\}$. The case $d = 1$ follows from $\mathcal{S}$ being a superfilter.

Induction step: Enumerate $A = \{a_n \colon n \in \mathbb{N}\}$. Choose $A_{a_1} \subseteq A \setminus \{a_1\}$ and a color i_{a_1} such that $A_{a_1} \in \mathcal{S}$, and a partition of A_{a_1} into finite sets, such that for each $F \in [A_{a_1}]^{d-1}$ with at most one element in each piece, $c(\{a_1\} \cup F) = i_{a_1}$. In a similar manner, choose inductively for each $n > 1$, $A_{a_n} \subseteq A_{a_{n-1}} \setminus \{a_n\}$ and a color i_{a_n} such that $A_{a_n} \in \mathcal{S}$, and a partition of A_{a_n} into finite sets, such that for each $F \in [A_{a_n}]^{d-1}$ with at most one element in each piece, $c(\{a_n\} \cup F) = i_{a_n}$.

As $\mathcal{S}$ is a P-point, there is $B \in \mathcal{S}$ such that $B \setminus A_{a_n}$ is finite for all n. Fix a color i such that $C = \{n \in B \colon i_n = i\} \in \mathcal{S}$.

Let $c_1 = \min C$. Inductively, for each $n > 1$ choose $c_n \in C$ such that:

(1) $c_n > c_{n-1}$;
(2) For each piece from the partitions of $A_{a_1}, \ldots, A_{a_n}$ which intersects $[1, c_{n-1})$, c_n is greater than all elements of that piece; and
(3) $C \setminus [1, c_n) \subseteq A_{c_{n-1}}$.

As

$$C = \left(C \cap \bigcup_{n\in\mathbb{N}} [c_{2n}, c_{2n+1})\right) \cup \left(C \cap \bigcup_{n\in\mathbb{N}} [c_{2n-1}, c_{2n})\right),$$

there is $l \in \{0, 1\}$ such that $M = C \cap \bigcup_n [c_{2n-l}, c_{2n+1-l}) \in \mathcal{S}$.

Let $m_1 < m_2 < \cdots < m_d$ be members of M coming from distinct intervals $[c_{2n-l}, c_{2n+1-l})$. Let n be minimal with $m_1 < c_n$. Then

$$m_2, \ldots, m_d \in C \setminus [1, c_{n+1}) \subseteq A_{c_n} \subseteq A_{m_1},$$

and $m_2, \ldots, m_d$ come from distinct pieces of the partition of A_{m_1}. Thus, $c(\{m_1, \ldots, m_d\}) = c(\{m_1\} \cup \{m_2, \ldots, m_d\}) = i_{m_1} = i$.

(6) $\Rightarrow$ (5) Trivial. □

The equivalence of (1) and (3) in the following corollary can be shown, using a well-known argument, to be the same as the equivalence of (i) and (iii) in Theorem 2.3 of Baumgartner and Taylor [2].

Corollary 3.7. *For ultrafilters $\mathcal{U}$, the following are equivalent*:

(1) *$\mathcal{U}$ is a P-point.*
(2) $\mathsf{S}_{\mathrm{fin}}(\mathcal{U}, \mathcal{U})$.
(3) $\mathcal{U} \to \lceil \mathcal{U} \rceil^2_2$.
(4) $\mathcal{U} \to \lceil \mathcal{U} \rceil^n_k$ *for all n, k.*

Proof. Recall that ultrafilters are shrinkable. □

Definition 3.8. A family $\mathcal{F}$ of subsets of $\mathbb{N}$ *generates* an upwards-closed family $\mathcal{S}$ if $\mathcal{F} \subseteq \mathcal{S}$ and each element of $\mathcal{S}$ contains an element of $\mathcal{F}$. An upwards-closed family $\mathcal{S} \subseteq [\mathbb{N}]^\infty$ is *compactly generated* if there are upwards-closed families $\mathcal{F}_1, \mathcal{F}_2, \ldots \subseteq P(\mathbb{N})$, each generated by finite subsets of $\mathbb{N}$, such that $\mathcal{S} = \bigcap_n \mathcal{F}_n$.

Example 3.9. $[\mathbb{N}]^\infty$ is compactly generated: Take $\mathcal{F}_n = [\mathbb{N}]^{\geqslant n}$, $n \in \mathbb{N}$.

AP is compactly generated: Let $\mathcal{F}_n$ be the family of all sets containing arithmetic progressions of length n.

Similarly, the *Folkman–Rado–Sanders* superfilter [22] of sets containing arbitrarily large finite subsets together with all of their subset sums is compactly generated.

Schur's Theorem [26] states that if the natural numbers are colored in finitely many colors, then there is a monochromatic solution to the equation $x + y = z$. Rado's Theorem [20] extends Schur's Theorem to arbitrary regular homogeneous systems of equations. A homogeneous system of equations $Ax = 0$ with integer coefficients is *regular* if the columns of A can be partitioned into sets $P_1, \ldots, P_k$ such that $\sum_{v \in P_1} v = 0$, and for each $i > 1$, each element of P_i is a linear combination of elements of $P_1 \cup \cdots \cup P_{i-1}$.

The family of all sets containing a solution to a regular homogeneous system of equations is not a superfilter. This problem can be solved by using the following operation on upwards-closed families (see Proposition 3.12 below).

Definition 3.10. For an upwards-closed family $\mathcal{F}$ of subsets of $\mathbb{N}$ and $k \in \mathbb{N}$, $\mathrm{Par}_k(\mathcal{F})$ is the family of all $A \subseteq \mathbb{N}$ such that for each partition of A into k pieces, one of the pieces belongs to $\mathcal{F}$. $\mathrm{Par}(\mathcal{F}) = \bigcap_k \mathrm{Par}_k(\mathcal{F})$.

For upwards-closed families $\mathcal{F}$, $\mathrm{Par}(\mathcal{F}) \subseteq \mathcal{F}$, and $\mathcal{F}$ is a superfilter if, and only if, $\mathrm{Par}(\mathcal{F}) = \mathcal{F}$.

Lemma 3.11. *Assume that $\mathcal{F} \subseteq P(\mathbb{N})$ is upwards-closed and generated by finite subsets of $\mathbb{N}$. Then the same is true for $\mathrm{Par}_k(\mathcal{F})$, for all k.*

Proof. This is a reformulation of the compactness theorem for partitions, see Theorem 2.5 in [19]. □

Note that $\mathbb{N} \in \mathrm{Par}(\mathcal{F})$ if, and only if, $\mathrm{Par}(\mathcal{F})$ is nonempty.

Proposition 3.12. *Let $\mathcal{F}$ be an upwards-closed family of subsets of $\mathbb{N}$. Assume that $\mathcal{F}$ does not contain any singleton, and $\mathbb{N} \in \mathrm{Par}(\mathcal{F})$. Then*:

(1) $\mathrm{Par}(\mathcal{F})$ *is the maximal superfilter contained in $\mathcal{F}$.*
(2) *If $\mathcal{F}$ is compactly generated, then so is* $\mathrm{Par}(\mathcal{F})$.

Proof. (1) It is easy to see that $\mathrm{Par}(\mathcal{F})$ is closed upwards.

Assume that $A \cup B \in \mathrm{Par}(\mathcal{F})$, and $A \cap B = \emptyset$. If $A, B \notin \mathrm{Par}(\mathcal{F})$, then there are a partition of A into n pieces and a partition of B into m pieces, such that none of the pieces belong to $\mathcal{F}$. But this yields a partition of $A \cup B$ into $n + m$ pieces, none of which from $\mathcal{F}$, that is, $A \cup B \notin \mathrm{Par}_{n+m}(\mathcal{F})$. A contradiction.

As $\mathrm{Par}(\mathcal{F}) \subseteq \mathcal{F}$, there are no singletons in $\mathrm{Par}(\mathcal{F})$, and consequently no finite sets.

If $\mathcal{S}$ is any superfilter contained in $\mathcal{F}$, then $\mathcal{S} = \mathrm{Par}(\mathcal{S}) \subseteq \mathrm{Par}(\mathcal{F})$.

(2) Assume that $\mathcal{F} = \bigcap_n \mathcal{F}_n$, with each $\mathcal{F}_n$ upwards-closed and generated by finite subsets. Replacing each $\mathcal{F}_n$ by $\bigcap_{m \leqslant n} \mathcal{F}_m$, we may assume that $\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \cdots$. It follows that for each k, $\mathrm{Par}_k(\bigcap_n \mathcal{F}_n) = \bigcap_n \mathrm{Par}_k(\mathcal{F}_n)$, and thus

$$\mathrm{Par}(\mathcal{F}) = \bigcap_{k \in \mathbb{N}} \mathrm{Par}_k(\mathcal{F}) = \bigcap_{k \in \mathbb{N}} \mathrm{Par}_k\left(\bigcap_{n \in \mathbb{N}} \mathcal{F}_n\right) = \bigcap_{k,n \in \mathbb{N}} \mathrm{Par}_k(\mathcal{F}_n).$$

By Lemma 3.11, each $\mathrm{Par}_k(\mathcal{F}_n)$ is upwards-closed and generated by finite sets. □

Example 3.13. Let $\mathcal{F}$ be the family of all subsets of $\mathbb{N}$ containing a solution to the equation $x + y = z$. Let $\mathrm{Par}(x + y = z) = \mathrm{Par}(\mathcal{F})$. Schur's Theorem tells that $\mathbb{N} \in \mathrm{Par}(x + y = z)$. By Proposition 3.12, $\mathrm{Par}(x + y = z)$ is a compactly-generated superfilter. We can define similarly $\mathrm{Par}(Ax = 0)$ for an arbitrary regular system of homogeneous equations, and by Rado's Theorem have that $\mathrm{Par}(Ax = 0)$ is a compactly-generated superfilter.

We now state the main application of Theorem 3.6.

Theorem π. *Assume that $\mathcal{S}$ is a compactly-generated superfilter. Then $\mathcal{S} \to \lceil \mathcal{S} \rceil^n_k$ for all n, k.*

Proof. By Theorem 3.6, it suffices to show that $\mathsf{S}_{\mathrm{fin}}(\mathcal{S}, \mathcal{S})$ holds. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots \subseteq P(\mathbb{N})$ be upwards-closed and generated by finite sets, such that $\mathcal{S} = \bigcap_n \mathcal{F}_n$. Assume that $A_1, A_2, \ldots \in \mathcal{S}$. For each n, pick a finite $F_n \in \mathcal{F}_n$ such that $F_n \subseteq A_n$. Then $\bigcup_n F_n \in \mathcal{S}$. □

Theorem π is a simultaneous improvement of the theorems of Ramsey, van der Waerden, Schur, Rado, Folkman–Rado–Sanders, and many more. In particular, we have the following.

Corollary 3.15. $\mathrm{AP} \to \lceil \mathrm{AP} \rceil^n_k$, *for all n, k.*

Theorem π can be restated as follows.

Corollary 3.16. *Assume that $\mathcal{S}$ is a superfilter compactly generated by $\mathcal{F}_1, \mathcal{F}_2, \ldots$. Then for all r, k, $A \in \mathcal{S}$, $c : [A]^r \to \{1, \ldots, k\}$, and $m_1 < m_2 < \cdots$, there are disjoint $F_n \in \mathcal{F}_{m_n}$, $n \in \mathbb{N}$, such that $\bigcup_n F_n \in \mathcal{S}$, and c is constant on sets with at most one element from each F_n.*

Proof. We may assume that $\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \cdots$. Assume that $A \in \mathcal{S}$ and $c : [A]^n \to \{1, 2, \ldots, k\}$. Using Theorem π, take $M \subseteq A$ such that $M \in \mathcal{S}$, and a partition of M into finite pieces, such that c is constant on sets containing at most one element from each piece. M contains some finite element of $\mathcal{F}_{m_1}$. Let F_1 be the union of as many pieces of M as required so that F_1 contains this element of $\mathcal{F}_{m_1}$. $M \setminus F_1 \in \mathcal{S}$, and is partitioned by the remaining pieces, thus we can take a union of finitely many of the remaining pieces, F_2, containing some element of $\mathcal{F}_{m_2}$, etc.

$\bigcup_n F_n$ contains an element of each $\mathcal{F}_n$, and thus belongs to $\mathcal{S}$. □

Example 3.17. Consider Corollary 3.16 with $\mathcal{S} = \mathrm{AP}$. Fix an arbitrarily quickly increasing sequence m_n, and assume that we color an arbitrarily sparse $A \in \mathrm{AP}$. Then each F_n contains, and thus may be assumed to be, an arithmetic progression of length m_n. The special case $A = \mathbb{N}$ is the main corollary in Bergelson and Hindman's 1989 paper [4].

Bergelson and Hindman's proof in [4] shows that it suffices to assume that the colored set A is an element of a combinatorially large ultrafilter (see [4]). Elements of AP need not lie in a combinatorially large ultrafilter, and we do not know a simple way to deduce Corollary 3.15 (or 3.16) from Bergelson and Hindman's Corollary, and not even from their much stronger Theorem 2.5 of [4].

4. An additional application to topological selection principles

Using Theorem 3.6 and arguments similar to those in the proof of Theorem 2.6, we also obtain the following Theorem 4.1. In the case that $\mathcal{I}$ is the ideal of finite sets ($\mathcal{O}_\mathcal{I} = \Omega$), the equivalence of (2) and (4) was proved by Just, Miller, Scheepers, and Szeptycki in [13]. In the case that $\mathcal{I}$ is the ideal of subsets of compact sets, the equivalence of (2) and (4) was proved by Di Maio, Kočinac, and Meccariello in [6].

Theorem 4.1. *Let $\mathcal{I}$ be an ideal on X. If X is $\mathcal{O}_\mathcal{I}$-Lindelöf, then the following are equivalent*:

(1) *For all $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots$ from $\mathcal{O}_\mathcal{I}$, there is $\mathcal{U} \in \mathcal{O}_\mathcal{I}$ such that $\mathcal{U} \setminus \mathcal{U}_n$ is finite for all n.*
(2) $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_\mathcal{I}, \mathcal{O}_\mathcal{I})$.
(3) *For all disjoint $\mathcal{U}_1, \mathcal{U}_2, \ldots$ with $\bigcup_{n \geqslant m} \mathcal{U}_n \in \mathcal{O}_\mathcal{I}$ for all m, there is $\mathcal{U} \subseteq \bigcup_n \mathcal{U}_n$ such that $\mathcal{U} \in \mathcal{O}_\mathcal{I}$ and $\mathcal{U} \cap \mathcal{U}_n$ is finite for all n.*
(4) $\mathcal{O}_\mathcal{I} \to \lceil \mathcal{O}_\mathcal{I} \rceil^2_2$.
(5) $\mathcal{O}_\mathcal{I} \to \lceil \mathcal{O}_\mathcal{I} \rceil^n_k$ *for all n, k.*

Here too, by using direct arguments as in the proof of Theorem 2.8, "X is $\mathcal{O}_\mathcal{I}$-Lindelöf" can be weakened to "X has a countable open $\mathcal{I}$-cover", or equivalently for T_1 spaces, to "there is a countable $D \subseteq X$ such that $D \notin \mathcal{I}$".

5. Final comments

Mathias defines in [18] *happy families*, certain types of superfilters which were later named *selective* by Farah [8]. Farah points out in [8] that every selective superfilter is Ramsey. It is immediate that every selective superfilter is strongly Ramsey, and arguments similar to those in the proof of Lemma 1.12 show that every strongly Ramsey superfilter is selective. Given Farah's observation, one can obtain a simpler proof of Lemma 1.12.

Recław has informed us of his independent work with Filipów, Mrożek, and Szuca [9], which contains related results, mainly of a descriptive set theoretic flavor.

In the topological results, considering from the start only countable covers removes any restriction from the considered topological spaces. For example, our results immediately apply to the corresponding families of countable *Borel* covers, since the Borel sets form a base for a topology on X. A general study of countable Borel covers in the context of selection principles is available in [25].

Theorem π and its Corollary 3.16 should be viewed as a simple way to lift one-dimensional Ramsey-theoretic results to higher dimensions. It does not generalize the Bergelson–Hindman Theorem from [4], but it extends it to cover additional classes of superfilters, and assumes less on the colored set.

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to these results, when presented in the conference *Ultramath 2008* (Pisa, Italy, June 2008). There is no doubt that these results owe much to the pioneering works of Scheepers and his followers, some of which are mentioned in the bibliography, and in the references therein.

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