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Homogeneous hyper-complex structures and the Joyce's construction

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ABSTRACT

We prove that any invariant hyper-complex structure on a homogeneous space $M = G/L$ where G is a compact Lie group is obtained via the Joyce's construction, provided that there exists a hyper-Hermitian naturally reductive invariant metric on M.

1. Introduction

A hyper-complex structure $\mathcal{Q}$ on a manifold M is a set of integrable complex structures on M of the form $\mathcal{Q} = \{aI + bJ + cK;\ a^2 + b^2 + c^2 = 1\}$, where I, J, $K = IJ$ are complex structures satisfying $IJK = -1$. A Riemannian metric g on M is called hyper-Hermitian if it is Hermitian w.r.t. every complex structure in $\mathcal{Q}$; it is easy to see that hyper-Hermitian metrics always exist. A hyper-Hermitian manifold $(M, g, \mathcal{Q})$ admits a HKT-structure, where HKT means hyper-Kähler with torsion, when there exists a metric connection ∇ that leaves every complex structure in $\mathcal{Q}$ parallel and whose torsion tensor T is totally skew. When a HKT-structure exists, the connection ∇ is unique and it is called the HKT-connection; actually it coincides with the Bismut connection of every complex structure in $\mathcal{Q}$ (see e.g. [7]). We refer the reader to [9,18,19] for equivalent definitions and basic properties of HKT-structures, which have also played an important role in theoretical physics (see e.g. [11,12]).

Hyper-Kähler structures are a special case of HKT-structures, namely when the HKT-connection coincides with the Levi-Civita connection of g, i.e. it is torsionfree. Actually, the hyper-Kähler condition is extremely stringent and examples are rare; for instance homogeneity forces the manifold to be flat (see e.g. [5]). On the contrary, there are plenty of examples of HKT-structures, even when M is supposed to be compact and homogeneous. In [17] the authors described and classified all the left invariant hyper-complex structures on compact Lie groups, for which there exists a bi-invariant, hyper-Hermitian Riemannian metric. Joyce [13] then described a way to construct hyper-complex structures on homogeneous spaces of compact Lie groups; this construction, which we recall in Section 2.3, has been then used and revisited by several authors, see e.g. [15,16]. Our first main result states that, if G is a compact Lie group, then every G-invariant hyper-complex structure on a homogeneous G-space is obtained via the Joyce construction, provided that there exists a naturally reductive G-invariant,

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hyper-Hermitian metric; this metric automatically endows the homogeneous space with an invariant HKT-structure. As a corollary of the proof of this first statement, we also get the fact that the semi-simplicity of G forces the group to be of a special kind, namely with every simple factor of type A_n. These results are summarised in the following

Theorem 1.1. *Let G be a compact connected Lie group acting transitively and almost effectively on some manifold M preserving a hyper-complex structure $\mathcal{Q}$. Suppose that there exists a naturally reductive G-invariant metric g on M which is hyper-Hermitian w.r.t. $\mathcal{Q}$. Then*

(1) *there exists a Cartan subalgebra of the complex reductive algebra $\mathfrak{g}^{\mathbb{C}}$ and a corresponding root system for the semi-simple part of $\mathfrak{g}^{\mathbb{C}}$ such that the hyper-complex structure $\mathcal{Q}$ coincides with the one given by the Joyce's construction*;
(2) *if G is semi-simple, then every simple factor of $\mathfrak{g}$ is of type A_n.*

The existence of a naturally reductive metric which is hyper-Hermitian is supposed and extensively used in [17] as well as in our arguments, while we are unaware of any result proving it. We refer the reader also to [15, Theorem 4], where the existence of a family of invariant HKT structures on compact homogeneous spaces is discussed. Our last result reduces the existence of a hyper-Hermitian naturally reductive metric on a homogeneous space to the case of a Lie group.

Proposition 1.2. *Let G be a compact Lie group and $M = G/L$ a G-homogeneous space endowed with an invariant hyper-complex structure $\mathcal{Q}$. Suppose L is not trivial and connected. Then*

(1) *the connected component Y of the fixed point set M^L of L containing the origin $[eL]$ is a positive dimensional hyper-complex submanifold. In particular $\chi(M) = 0$;*
(2) *if g is an invariant naturally reductive metric, it is hyper-Hermitian if and only if its restriction to Y is.*

Notation. Lie groups and their Lie algebras will be indicated by capital and gothic letters respectively. We will denote the Cartan–Killing form by $\mathcal{B}$.

2. Preliminaries

2.1. Invariant complex structures

In order to establish the notation we briefly recall the structure theory of compact homogeneous complex manifolds; the reader is referred e.g. to [2] for a more detailed exposition.

Let M be a compact complex manifold and let G be a compact connected Lie group acting almost effectively, transitively and holomorphically on M. We will write $M = G/L$ for some compact subgroup L. Up to a finite covering we can assume that L is connected. We will also denote by I the G-invariant complex structure on M.

The complexified group $G^{\mathbb{C}}$ acts holomorphically on G/L, so that $M = G^{\mathbb{C}}/Q$ for some connected complex subgroup $Q \subset G^{\mathbb{C}}$. It is well known that the *Tits fibration* ϕ provides a holomorphic fibering of the homogeneous space M over a compact rational homogeneous space $G^{\mathbb{C}}/P$, where the parabolic subgroup P is in general defined to be the normaliser $N_{G^{\mathbb{C}}}(Q^o)$ of Q^o (see [2]); since Q is connected the fibres of ϕ are complex tori. The flag manifold $G^{\mathbb{C}}/P$ can be written as G/C endowed with a G-invariant complex structure $\mathcal{I}$, where C is the centraliser of some torus in G. Accordingly the Lie algebra $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{t} \oplus \mathfrak{n}, \tag{2.1}$$

where $\mathfrak{c} = \mathfrak{l} \oplus \mathfrak{t}$ and $\mathfrak{n}$ is an $\mathrm{Ad}(C)$-invariant complement of $\mathfrak{c}$ in $\mathfrak{g}$. Since $\mathfrak{t}$ identifies with the tangent space to the fibre we have $[\mathfrak{t}, \mathfrak{t}] = 0$. Moreover the algebra $\mathfrak{c}$ is contained in the normaliser of $\mathfrak{l}$ in $\mathfrak{g}$ by construction, hence $[\mathfrak{l}, \mathfrak{t}] \subset \mathfrak{l} \cap \mathfrak{t} = 0$. We can choose a Cartan subalgebra $\mathfrak{h}$ of the form $\mathfrak{h} = \mathfrak{t}_{\mathfrak{l}}^{\mathbb{C}} \oplus \mathfrak{t}^{\mathbb{C}}$, where $\mathfrak{t}_{\mathfrak{l}}$ is a maximal abelian subalgebra of $\mathfrak{l}$. Denote by R the corresponding root system of $\mathfrak{g}^{\mathbb{C}}$, by $R_{\mathfrak{l}}$ the subsystem relative to $\mathfrak{l}$ so that $\mathfrak{l}^{\mathbb{C}} = \mathfrak{t}_{\mathfrak{l}} \oplus \bigoplus_{\alpha \in R_{\mathfrak{l}}} \mathfrak{g}_\alpha$, and by $R_{\mathfrak{n}}$ the symmetric subset of R such that $\mathfrak{n}^{\mathbb{C}} = \bigoplus_{\alpha \in R_{\mathfrak{n}}} \mathfrak{g}_\alpha$. The G-invariant complex structure $\mathcal{I}$ induces an endomorphism of $\mathfrak{n}^{\mathbb{C}}$ that is $\mathrm{Ad}(C)$-invariant and therefore the corresponding subspace $\mathfrak{n}^{1,0}$ is a sum of root spaces. The integrability of $\mathcal{I}$ is equivalent to the condition

$$\left[\mathfrak{n}^{1,0}, \mathfrak{n}^{1,0}\right]_{\mathfrak{n}^{\mathbb{C}}} \subseteq \mathfrak{n}^{1,0}$$

and one can prove (see e.g. [3]) that there is a suitable ordering of $R_{\mathfrak{n}} = R_{\mathfrak{n}}^+ \cup R_{\mathfrak{n}}^-$ such that

$$\mathfrak{n}^{1,0} = \bigoplus_{\alpha \in R_{\mathfrak{n}}^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^{0,1} = \bigoplus_{\alpha \in R_{\mathfrak{n}}^-} \mathfrak{g}_\alpha.$$

The G-invariant complex structure I on G/L induces an $\mathrm{Ad}(L)$-invariant endomorphism, still denoted by I, of $\mathfrak{m}^{\mathbb{C}}$, where $\mathfrak{m} := \mathfrak{t} + \mathfrak{n}$. It leaves both $\mathfrak{t}$ and $\mathfrak{n}$ invariant with $I|_{\mathfrak{n}} = \mathcal{I}$ and the integrability of I is equivalent to the vanishing of the Nijenhuis tensor N_I, namely for $X, Y \in \mathfrak{m}$

$$[IX, IY]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}} - I[IX, Y]_{\mathfrak{m}} - I[X, IY]_{\mathfrak{m}} = 0. \tag{2.2}$$

Eq. (2.2) is trivial for $X, Y \in \mathfrak{t}$ and with $X \in \mathfrak{t}$ and $Y \in \mathfrak{n}$ it reduces to the $\mathrm{ad}(\mathfrak{t})$-invariance of I. When $X, Y \in \mathfrak{n}$, then (2.2) is the integrability of $\mathcal{I}$ because $[\mathfrak{n}^{1,0}, \mathfrak{n}^{1,0}] \subseteq \mathfrak{n}^{1,0}$.

Vice versa, we start with a decomposition as in (2.1), where $\mathfrak{l} + \mathfrak{t} = \mathfrak{c}$ and $\mathfrak{c}$ is the centraliser of an abelian subalgebra. If we fix an $\mathrm{ad}(\mathfrak{c})$-invariant integrable complex structure $\mathcal{I}$ on $\mathfrak{n}$ and we extend it by choosing an arbitrary complex structure $I_{\mathfrak{t}}$ on $\mathfrak{t}$, then $I_{\mathfrak{t}} + \mathcal{I}$ will provide an integrable L-invariant complex structure on the homogeneous space G/L.

2.2. Hyper-complex and HKT structures

A *hyper-complex* structure on a manifold M is determined by a pair of anticommuting complex structures. Whenever such a pair (I, J) is given, one has a 2-sphere of complex structures on M given by $\{aI + bJ + cIJ\colon a, b, c \in \mathbb{R} \text{ and } a^2 + b^2 + c^2 = 1\}$.

A metric g on a hyper-complex manifold M is called *hyper-Hermitian* if it is Hermitian with respect to both the complex structures. A metric connection ∇ on a hyper-Hermitian manifold (M, g, I, J) is called *hyper-Kähler with torsion* (HKT) if $\nabla I = \nabla J = 0$ and the torsion T^{∇} is totally skew-symmetric, i.e. the tensor $\tau(X, Y, Z) = g(T^{\nabla}(X, Y), Z)$ is a 3-form.

Note that if such a connection exists, then it is unique since it is the *Bismut connection* of each complex structure (see [7]).

An important technical tool we use in Section 3 is the following well-known fact (see [20]).

Proposition 2.1. *Let (M, I) be a complex manifold. An almost complex structure J on M anticommuting with I is integrable if and only if the tensor N_{IJ} defined for $X, Y \in \Gamma(TM)$ as*

$$N_{IJ}(X, Y) = [IX, JY] + [JX, IY] - I[JX, Y] - J[IX, Y] - I[X, JY] - J[X, IY] \tag{2.3}$$

vanishes identically on M.

Using the notation of Section 2.1 we consider a homogeneous space $M = G/L$ where the compact group G preserves a hyper-complex structure generated by I, J and $K = IJ$.

Given a G-invariant decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{m}$ and the corresponding *canonical connection* D, it is known that any G-invariant tensor on the homogeneous space G/L is D-parallel (see e.g. [14]). Since the torsion of D is given by $T^D(X, Y) = -[X, Y]_{\mathfrak{m}}$, we see that D becomes the HKT-connection if there exists a G-invariant naturally reductive metric on M that is Hermitian with respect to I and J.

2.3. The Joyce construction

In [13] Joyce explains how to construct invariant hyper-complex structures on suitable compact homogeneous spaces. His construction can be described as follows. Given a compact Lie algebra $\mathfrak{g}$ we fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$ and denote by R the set of corresponding roots. We can find a sequence $\theta_1, \dots, \theta_k$ of roots such that if $\mathfrak{s}_i^{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C})$ is the subalgebra generated by the root spaces $\mathfrak{g}_{\theta_i}$, $\mathfrak{g}_{-\theta_i}$ and

$$\mathfrak{f}_i := \mathfrak{g} \cap \bigoplus_{\mathcal{B}(\alpha, \theta_i) \neq 0} \mathfrak{g}_\alpha, \qquad \mathfrak{b}_i := \mathfrak{g} \cap \bigcap_{j=1}^{i} \mathfrak{z}_{\mathfrak{g}^{\mathbb{C}}}\big(\mathfrak{s}_j^{\mathbb{C}}\big),$$

where $\mathfrak{z}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}_j^{\mathbb{C}})$ denotes the centraliser of $\mathfrak{s}_j^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}}$, then one has the decomposition

$$\mathfrak{g} = \mathfrak{b}_k \oplus \bigoplus_{i=1}^{k} \mathfrak{s}_i \oplus \bigoplus_{i=1}^{k} \mathfrak{f}_i. \tag{2.4}$$

The sequence $\{\theta_1, \dots, \theta_k\}$ can be obtained as follows. If we fix an ordering $R = R^+ \cup R^-$ we define θ_1 to be the highest root. Then θ_j can be inductively chosen to be a highest root of the root subsystem relative to $\mathfrak{z}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}_{j-1}^{\mathbb{C}})$. The roots $\{\theta_1, \dots, \theta_k\}$ form a maximal system of *strongly orthogonal* roots. In [1] the maximal strongly orthogonal subsets (MSOS) in irreducible root systems have been classified up to equivalence under the action of the Weyl group. For the reader's convenience in Appendix A we reproduce Table 1 with the indication of a natural MSOS for each simple Lie algebra, as given in [8].

The Lie algebra of the isotropy $\mathfrak{l} \subset \mathfrak{b}_k$ is chosen as follows: the semi-simple part of $\mathfrak{l}$ coincides with the semi-simple part of $\mathfrak{b}_k$ and the center $\mathfrak{z}_{\mathfrak{l}}$ of $\mathfrak{l}$ is a subset of the center $\mathfrak{z}'$ of $\mathfrak{b}_k$ such that $\dim \mathfrak{z}' - \dim \mathfrak{z}_{\mathfrak{l}} - k \equiv_4 0$. We denote by $\mathfrak{m}$ the $\mathcal{B}$-orthogonal complement of $\mathfrak{l}$ in $\mathfrak{g}$.

The invariant hyper-complex structure on G/L is obtained by the following $\mathrm{Ad}(L)$-invariant hyper-complex structure $\mathcal{Q}$ on $\mathfrak{m}$. The structure $\mathcal{Q}_{|\mathfrak{f}_i}$ coincides with $\mathrm{ad}(\mathfrak{s}_i)$. We select $\mathcal{B}$-orthogonal vectors $u_1, \dots, u_k$ in $\mathfrak{z}' \cap \mathfrak{m}$ and use the fact that $\mathfrak{q}_i = \mathfrak{s}_i \oplus \mathbb{R}u_i \cong \mathbb{H}$ to define $\mathcal{Q}_{|\mathfrak{q}_i}$. The complement of $\mathfrak{z}_{\mathfrak{l}} \oplus \sum_i \mathbb{R}u_i$ in $\mathfrak{z}'$ can be endowed with an arbitrary linear hyper-complex structure.

3. Proof of the main results

3.1. Proof of Theorem 1.1, part (1)

We write $M = G/L$ for some closed subgroup $L \subset G$. We will also suppose that L is connected, otherwise we pass to a finite covering. We will suppose that G/L admits a naturally reductive metric g with respect to the reductive decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{m}$, which is Hermitian w.r.t. every complex structure in $\mathcal{Q}$. We recall (see e.g. [14]) that the metric g induces a scalar product on $\mathfrak{m}$ such that, for every $X, Y, Z \in \mathfrak{m}$

$$g\big([X, Y]_{\mathfrak{m}}, Z\big) + g\big(Y, [X, Z]_{\mathfrak{m}}\big) = 0. \tag{3.1}$$

Using (3.1) and the Ad(L)-invariance of g, it is immediate to see that $g(\mathfrak{t}, \mathfrak{n}) = 0$.

We fix one complex structure $I \in \mathcal{Q}$ and apply the structure theory explained in Section 2.1, keeping the same notation. If now $J \in \mathcal{Q}$ is an integrable complex structure anticommuting with I, we think of J as an Ad(L)-invariant endomorphism of $\mathfrak{m}$ and we may formulate Proposition 2.1 as follows: for every $X, Y \in \mathfrak{m}$

$$[IX, JY]_{\mathfrak{m}} + [JX, IY]_{\mathfrak{m}} - I[JX, Y]_{\mathfrak{m}} - J[IX, Y]_{\mathfrak{m}} - I[X, JY]_{\mathfrak{m}} - J[X, IY]_{\mathfrak{m}} = 0. \tag{3.2}$$

If we now extend J to the complexification $\mathfrak{m}^{\mathbb{C}}$, we see that $J(\mathfrak{m}^{1,0}) = \mathfrak{m}^{0,1}$ and $J(\mathfrak{m}^{0,1}) = \mathfrak{m}^{1,0}$. If $X \in \mathfrak{m}^{1,0}$ and $Y \in \mathfrak{m}^{0,1}$, Eq. (3.2) reduces to

$$i[X, JY]_{\mathfrak{m}^{\mathbb{C}}} - i[JX, Y]_{\mathfrak{m}^{\mathbb{C}}} - I[JX, Y]_{\mathfrak{m}^{\mathbb{C}}} - I[X, JY]_{\mathfrak{m}^{\mathbb{C}}} = 0,$$

which is automatically satisfied since $[\mathfrak{m}^{1,0}, \mathfrak{m}^{1,0}]_{\mathfrak{m}^{\mathbb{C}}} \subseteq \mathfrak{m}^{1,0}$.

If $X, Y \in \mathfrak{m}^{1,0}$ we see that $N_{IJ}(X, Y) \in \mathfrak{m}^{0,1}$ and therefore Eq. (3.2) is equivalent to $g(N_{IJ}(X, Y), Z) = 0$ for every $Z \in \mathfrak{m}^{1,0}$. Using the fact that g is naturally reductive and Hermitian w.r.t. I and J, we have that Eq. (3.2) is equivalent to the following condition: for every $X, Y, Z \in \mathfrak{m}^{1,0}$ the cyclic sum $\mathfrak{S}_{(X,Y,Z)} g(JX, [Y, Z]_{\mathfrak{m}^{\mathbb{C}}}) = 0$, i.e.

$$d\omega_J|_{\wedge^{3,0}\mathfrak{m}} = 0, \tag{3.3}$$

where ω_J is the fundamental 2-form associated to J. We now consider the root system R associated with the choice of the maximal abelian subalgebra $(\mathfrak{t}_{\mathfrak{l}} + \mathfrak{t})^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ as described in Section 2.1. The root subsystem $R_{\mathfrak{n}}$ where $\mathfrak{m} = \mathfrak{t} \oplus \mathfrak{n}$ has an ordering $R_{\mathfrak{n}} = R^+_{\mathfrak{n}} \cup R^-_{\mathfrak{n}}$ induced by the complex structure I and we can select a root $\theta \in R^+_{\mathfrak{n}}$ which is maximal w.r.t. this ordering, namely for every $\alpha \in R^+_{\mathfrak{n}}$

$$\theta + \alpha \notin R^+_{\mathfrak{n}}.$$

Throughout the following we will denote by $\{H_\alpha, E_\alpha\}_{\alpha \in R}$ the standard Chevalley's basis of the semi-simple part of $\mathfrak{g}^{\mathbb{C}}$.

We here remark that, since the metric g is naturally reductive, for $\alpha, \beta \in R^+_{\mathfrak{n}}$ we have $g(E_\alpha, E_\beta) = 0$ whenever $\alpha \neq -\beta$ and $g(E_\alpha, E_{-\alpha}) \neq 0$. Moreover if $iH_\alpha \in \mathfrak{t}$ one can see that $g(E_\alpha, E_{-\alpha}) = -\frac{\|\alpha\|^2}{|\alpha|^2}$, where $\|\alpha\|^2 = g(iH_\alpha, iH_\alpha)$.

Lemma 3.1. *We have $JE_\theta \in \mathfrak{t}^{\mathbb{C}}$. In particular E_θ is centralised by $\mathfrak{l}$.*

Proof. Since $g(\mathfrak{t}, \mathfrak{n}) = 0$, we need to show that $g(JE_\theta, E_\alpha) = 0$ whenever $\alpha \in R^+_{\mathfrak{n}}$. To do this we can first take $X = E_\theta$, $Y = E_\alpha$ and $Z = H \in \mathfrak{t}^{\mathbb{C}} \cap \mathfrak{m}^{(1,0)}$ in formula (3.3) and obtain

$$(\alpha + \theta)(H)\, g(JE_\theta, E_\alpha) = 0.$$

Now, if $\alpha + \theta$ does not vanish on $\mathfrak{t}^{\mathbb{C}} \cap \mathfrak{m}^{(1,0)}$, then the claim follows. Otherwise $\alpha + \theta$ vanishes on the whole $\mathfrak{t}^{\mathbb{C}}$ since $\alpha + \theta \in i\mathfrak{t}^*$; in this case we can take $H' \in \mathfrak{t}_{\mathfrak{l}}$ such that $(\alpha + \theta)(H') \neq 0$. For such a H' we have $[H', JE_\theta] = \theta(H')JE_\theta$ and, contracting with E_α and using the fact that g is J-Hermitian, once again we get

$$(\alpha + \theta)\big(H'\big)\, g(JE_\theta, E_\alpha) = 0,$$

obtaining our first claim. The second assertion follows from the fact that $[\mathfrak{l}, \mathfrak{t}] = 0$ and the ad$(\mathfrak{l})$-invariance of J. □

Now we want to compute the $\mathfrak{t}^{\mathbb{C}}$-component of JE_α for $\alpha \in R^+_{\mathfrak{n}}$.

Lemma 3.2. *Given $\alpha \in R^+_{\mathfrak{n}}$ the following statements hold:*

(i) *If $\alpha|_{\mathfrak{t}_{\mathfrak{l}}} \equiv 0$ then $JE_\alpha = k_\alpha(H_\alpha + iIH_\alpha) \bmod \mathfrak{n}^{\mathbb{C}}$ for some $k_\alpha \in \mathbb{C}$. In particular $JE_\theta = k_\theta(H_\theta + iIH_\theta)$, where $|k_\theta|^2 = \frac{1}{2|\theta|^2}$.*

(ii) *If $\alpha|_{\mathfrak{t}_{\mathfrak{l}}} \not\equiv 0$ then $JE_\alpha \in \mathfrak{n}^{\mathbb{C}}$.*

Proof. In order to prove (i) we first note that iH_α lies in $\mathfrak{t}$. We now apply (3.3) taking $X = E_\alpha$, $Y = H_1$ and $Z = H_2$ where $H_1, H_2 \in \mathfrak{t}^{\mathbb{C}} \cap \mathfrak{m}^{1,0}$. Thus we obtain

$$g\big(JE_\alpha, \alpha(H_2)H_1 - \alpha(H_1)H_2\big) = 0.$$

The linear space $\mathrm{span}_{\mathbb{C}}\{\alpha(H_2)H_1 - \alpha(H_1)H_2\colon\ H_1, H_2 \in \mathfrak{t}^{\mathbb{C}} \cap \mathfrak{m}^{1,0}\}$ coincides with $\{v - iIv\colon\ v, Iv \in (\mathrm{Ker}\,\alpha) \cap \mathfrak{t}\}$. This means that the $\mathfrak{t}^{\mathbb{C}}$-component of JE_α is of the form $\gamma(w + iIw)$ with $\gamma \in \mathbb{C}$ and $w \in \mathfrak{t}$ is g-orthogonal to $\mathrm{Ker}\,\alpha$. Since $g(iH_\alpha, \mathrm{Ker}\,\alpha) = 0$ we can choose $w = iH_\alpha$ and the claim follows for a suitable $k_\alpha \in \mathbb{C}$. The last assertion follows from the following computation

$$\begin{aligned} g(E_\theta, E_{-\theta}) &= |k_\theta|^2 g(H_\theta - iIH_\theta, H_\theta + iIH_\theta) = 2|k_\theta|^2 g(H_\theta, H_\theta) \\ &= 2|k_\theta|^2 g\big([E_\theta, E_{-\theta}], H_\theta\big) = 2|k_\theta|^2 |\theta|^2 g(E_\theta, E_{-\theta}). \end{aligned}$$

As for (ii), we select $H \in \mathfrak{t}_{\mathfrak{l}}$ with $\alpha(H) \neq 0$ and use the $\mathrm{ad}(\mathfrak{l})$-invariance of J to compute

$$\alpha(H) g(JE_\alpha, \mathfrak{t}) = g\big([H, JE_\alpha], \mathfrak{t}\big) = g\big(JE_\alpha, [H, \mathfrak{t}]\big) = 0. \qquad \square$$

We note that k_θ is determined up to multiplication by a complex number of unit norm, since J can be chosen in the circle of complex structures in $\mathcal{Q}$ which are orthogonal to I.

Lemma 3.3.

(i) *If $\alpha, \beta \in R^+_{\mathfrak{n}}$ and $\alpha + \beta \notin R$, then $g(JE_\alpha, E_\beta) = 0$.*
(ii) *If $\alpha, \beta \in R^+_{\mathfrak{n}}$ and $\alpha + \beta = \gamma \in R^+$ with $\gamma_{|\mathfrak{t}_{\mathfrak{l}}} \not\equiv 0$, then $g(JE_\alpha, E_\beta) = 0$.*
(iii) *If $\alpha, \beta \in R^+_{\mathfrak{n}}$ and $\alpha + \beta = \gamma \in R^+$ with $\gamma_{|\mathfrak{t}_{\mathfrak{l}}} \equiv 0$, then $g(JE_\alpha, E_\beta) = 2k_\gamma \frac{\|\gamma\|^2}{|\gamma|^2} N_{\alpha,\beta}$.*

Proof. The first assertion can be easily proved with the same argument used in Lemma 3.1. In order to prove (ii), let $H \in \mathfrak{t}_{\mathfrak{l}}$ with $\gamma(H) \neq 0$ and use the $\mathrm{ad}(\mathfrak{l})$-invariance of J to compute

$$\begin{aligned} \alpha(H) g(JE_\alpha, E_\beta) &= g\big(J[H, E_\alpha], E_\beta\big) = g\big([H, JE_\alpha], E_\beta\big) = -g\big(JE_\alpha, [H, E_\beta]\big) \\ &= -g\big(JE_\alpha, \beta(H)E_\beta\big) = -\beta(H) g(JE_\alpha, E_\beta) \end{aligned}$$

so that the claim follows.

As for (iii), we select $H \in \mathfrak{t}$ with $\gamma(H) \neq 0$ and set $H' = H - iIH$. Using (3.3) we have

$$\gamma\big(H'\big) g(JE_\alpha, E_\beta) = g\big(JH', [E_\alpha, E_\beta]\big) = N_{\alpha,\beta} g\big(JH', E_\gamma\big) = -N_{\alpha,\beta} g\big(JE_\gamma, H'\big).$$

Applying part (i) of the previous Lemma we get

$$\gamma\big(H'\big) g(JE_\alpha, E_\beta) = -N_{\alpha,\beta} k_\gamma \big(g\big(H_\gamma, H'\big) + ig\big(IH_\gamma, H'\big)\big) = -2N_{\alpha,\beta} k_\gamma g\big(H_\gamma, H'\big) = 2\gamma\big(H'\big) N_{\alpha,\beta} k_\gamma \frac{\|\gamma\|^2}{|\gamma|^2}$$

and the claim follows. $\square$

We now consider the highest root θ and define $R(\theta) = \{\alpha \in R^+_{\mathfrak{n}};\ \theta - \alpha \in R\}$. Note that $\alpha \in R^+_{\mathfrak{n}}$ lies in $R(\theta)$ if and only if $\alpha \neq \theta$ and $\mathcal{B}(\alpha, \theta) \neq 0$. Moreover if $\alpha \in R(\theta)$, then $\theta - \alpha \in R_{\mathfrak{n}}$: indeed, if $\theta - \alpha = \beta \in R_{\mathfrak{l}}$, we have $\theta - \beta = \alpha \in R$, hence $[E_\theta, E_{-\beta}] \neq 0$, contradicting the fact that $[\mathfrak{l}, E_\theta] = 0$ (see Lemma 3.1).

Lemma 3.4. *If $\alpha \in R(\theta)$, then $JE_\alpha \in \mathfrak{n}^{\mathbb{C}}$.*

Proof. Suppose JE_α has a component along $\mathfrak{t}^{\mathbb{C}}$. Using Lemma 3.2, we compute

$$\begin{aligned} 0 = g(JE_\alpha, JE_{-\theta}) &= k_\alpha k_\theta g(H_\alpha + iIH_\alpha, H_\theta + iIH_\theta) = 2k_\alpha k_\theta g(H_\alpha, H_\theta) \\ &= 2k_\alpha k_\theta g\big([E_\alpha, E_{-\alpha}], H_\theta\big) = -2k_\alpha k_\theta \alpha(H_\theta) g(E_\alpha, E_{-\alpha}). \end{aligned}$$

Since $\alpha \in R(\theta)$ we have that $\alpha(H_\theta) \neq 0$. Therefore $k_\alpha = 0$ and the claim follows. $\square$

Lemma 3.5. *If $\alpha \in R(\theta)$, then $g(JE_\alpha, E_\beta) = 0$ for every $\beta \in R^+_{\mathfrak{n}}$ unless $\alpha + \beta = \theta$.*

Proof. By Lemma 3.3(i), it is enough to take $\beta \in R_{\mathfrak{n}}^{+}$ so that $\alpha+\beta=\gamma \in R$. Moreover by Lemma 3.3(iii), we may suppose that $\gamma|_{\mathfrak{t}_{\mathfrak{l}}} \equiv 0$, hence $\gamma|_{\mathfrak{t}} \not\equiv 0$. Choose $H \in \mathfrak{t}^{1,0}$ with $\gamma(H) \neq 0$. Now, if $\gamma \neq \theta$, by Lemma 3.4 we have that $g(J[E_\alpha, E_\beta], \mathfrak{t}) = 0$. Eq. (3.3) with $X = E_\alpha$, $Y = E_\beta$ and $Z = H$ implies $\gamma(H) g(JE_\alpha, E_\beta) = 0$ and the claim follows. □

The previous lemma says that for every $\alpha \in R_{\mathfrak{n}}^{+}$ one has $JE_\alpha = \lambda_\alpha E_{\alpha-\theta}$ for some $\lambda_\alpha \in \mathbb{C} \setminus \{0\}$. Using Lemma 3.3(iii) we have

$$\lambda_\alpha g(E_{\alpha-\theta}, E_{\theta-\alpha}) = g(JE_\alpha, E_{\theta-\alpha}) = 2k_\theta \frac{\|\theta\|^2}{|\theta|^2} N_{\alpha,\theta-\alpha}. \tag{3.4}$$

Using the fact that g is naturally reductive we have

$$g(E_{\alpha-\theta}, E_{\theta-\alpha}) = -\frac{N_{\alpha,\theta-\alpha}}{N_{\alpha,-\theta}} g(E_{-\theta}, E_\theta) = \frac{N_{\alpha,\theta-\alpha}}{N_{\alpha,-\theta}} \frac{\|\theta\|^2}{|\theta|^2},$$

which, combined with (3.4) gives

$$JE_\alpha = 2k_\theta N_{\alpha,-\theta} E_{\alpha-\theta}.$$

Let $\mathfrak{s}(\theta)^{\mathbb{C}}$ be the subalgebra of $\mathfrak{g}^{\mathbb{C}}$ generated by E_θ and $E_{-\theta}$, and define $\mathfrak{s}(\theta) = \mathfrak{s}(\theta)^{\mathbb{C}} \cap \mathfrak{g}$. Obviously $\mathfrak{s}(\theta) \cong \mathfrak{sp}(1)$. Set also

$$Z_\theta = I(iH_\theta) \in \mathfrak{t}, \qquad \mathfrak{u}(\theta) = \mathfrak{s}(\theta) \oplus \mathbb{R}\, Z_\theta.$$

Then $\mathcal{Q}$ leaves $\mathfrak{u}(\theta)$ invariant and $\mathcal{Q}_{|\mathfrak{u}(\theta)}$ is determined by the formula $JE_\theta = k_\theta(H_\theta + iIH_\theta)$. We also define $\mathfrak{f}_\theta = \mathfrak{g} \cap \bigoplus_{\alpha\in R(\theta)}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ and $\mathfrak{c}_\theta = \mathfrak{g} \cap \bigoplus_{\alpha \in C(\theta)}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$, where $C(\theta) = \{\alpha \in R_{\mathfrak{n}}^{+};\ (\theta,\alpha) = 0\} = R_{\mathfrak{n}}^{+} \setminus (R(\theta) \cup \{\theta\})$, so that

$$\mathfrak{n} \oplus \mathrm{span}_{\mathbb{R}}\{iH_\theta, Z_\theta\} = \mathfrak{u}(\theta) \oplus \mathfrak{f}_\theta \oplus \mathfrak{c}_\theta. \qquad \square$$

Proposition 3.6. *The hyper-complex structure $\mathcal{Q}$ leaves $\mathfrak{f}_\theta$ invariant and $\mathcal{Q}_{|\mathfrak{f}_\theta} = \mathrm{ad}(\mathfrak{s}(\theta))_{|\mathfrak{f}_\theta}$.*

Proof. We will show that there exist $\sigma_\theta, \tau_\theta \in \mathfrak{s}(\theta)$ such that for every $X \in \mathfrak{f}_\theta$ we have $JX = [\sigma_\theta, X]$ and $IX = [\tau_\theta, X]$. Let $\sigma_\theta = 2(\bar{k}_\theta E_\theta - k_\theta E_{-\theta})$ and $\tau_\theta = \frac{2iH_\theta}{|\theta|^2}$. The claim is a consequence of the following direct computations

$$[\sigma_\theta, E_\alpha] = -2k_\theta[E_{-\theta}, E_\alpha] = -2k_\theta N_{-\theta,\alpha} E_{\alpha-\theta} = JE_\alpha,$$
$$[\tau_\theta, E_\alpha] = \frac{2i}{|\theta|^2}[H_\theta, E_\alpha] = 2\frac{\mathcal{B}(\alpha,\theta)}{|\theta|^2} iE_\alpha = iE_\alpha,$$

where in the last equation we have used the fact that $2\frac{\mathcal{B}(\alpha,\theta)}{(\theta,\theta)} = 1$ since the θ-string of α is formed only by $\alpha - \theta$ and α (see e.g. [10]). □

We now set $\theta_1 := \theta$, $k_1 := k_\theta$ and define inductively the roots θ_j as follows:

(1) θ_{j+1} is maximal in $C(\theta_j)$, i.e. $\theta_{j+1} + \alpha \notin R$ for every $\alpha \in C(\theta_j)$;
(2) $C(\theta_{j+1}) := \{\alpha \in C(\theta_j):\ \theta_{j+1} - \alpha \notin R\}$.

We then set $R(\theta_{j+1}) = \{\alpha \in C(\theta_j);\ \theta_{j+1} - \alpha \in R\}$ and $\mathfrak{f}_{j+1} = \mathfrak{g} \cap \bigoplus_{\alpha\in R(\theta_{j+1})}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. Moreover we define $\mathfrak{s}_{j+1} \cong \mathfrak{sp}(1)$ as the real subalgebra generated by $E_{\theta_{j+1}}, E_{-\theta_{j+1}}$ (note that $\mathfrak{s}_1 = \mathfrak{s}(\theta)$) and $\mathfrak{u}_{j+1} = \mathfrak{s}_{j+1} \oplus \mathbb{R}Z_{j+1}$ where $Z_{j+1} = iIH_{\theta_{j+1}} \in \mathfrak{t}$.

Now we have

Proposition 3.7. *There exists a set of roots $\theta_1, \dots, \theta_\ell$ such that for $j = 1, \dots, \ell$ we have:*

(i) *the subset $C(\theta_\ell)$ is empty;*
(ii) *the hyper-complex structure $\mathcal{Q}$ leaves $\mathfrak{f}_j$ and $\mathfrak{u}_j$ invariant. In particular $\mathcal{Q}_{|\mathfrak{f}_j} = \mathrm{ad}(\mathfrak{s}_j)_{|\mathfrak{f}_j}$ and we have $JE_{\theta_j} = k_j(H_{\theta_j} + iIH_{\theta_j})$ for a suitable $k_j \in \mathbb{C}$ (hence $\mathfrak{l}$ centralises $\mathfrak{s}_j$);*
(iii) *there is a g-orthogonal decomposition $\mathfrak{g} = \mathfrak{l} \oplus \tilde{\mathfrak{t}} \oplus \bigoplus_{j=1}^{\ell} \mathfrak{u}_j \oplus \bigoplus_{j=1}^{\ell} \mathfrak{f}_j$, where $\tilde{\mathfrak{t}}$ lies in $\mathfrak{t}$ and is $\mathcal{Q}$-invariant. Moreover $[\mathfrak{l}, \mathfrak{u}_j] = 0$, $[\mathfrak{u}_j, \mathfrak{u}_k] = 0$ for $j \neq k$ and $[\mathfrak{u}_j, \mathfrak{f}_j] \subseteq \mathfrak{f}_j$;*
(iv) *the root θ_1 can be chosen as the highest root $\tilde{\theta}$ of the whole root system R of $\mathfrak{g}$ with respect to an ordering such that $R^+ \supseteq R_{\mathfrak{n}}^{+}$.*

Proof. The first three statements can be proved by induction using exactly the same arguments as in the previous lemmas and in Proposition 3.6. The only new statement to prove is (iv). To do this it is enough to show that the highest root space $\mathfrak{g}_{\tilde{\theta}}$ does not belong to $\mathfrak{l}^{\mathbb{C}}$. Suppose now by contradiction that $E_{\tilde{\theta}} \in \mathfrak{l}^{\mathbb{C}}$. Given $\alpha \in R_{\mathfrak{n}}^{-}$, we have $JE_\alpha = H + \sum_{\beta\in R_{\mathfrak{n}}^{+}} c_\beta E_\beta$ for some $H \in \mathfrak{t}^{\mathbb{C}}$ and $c_\beta \in \mathbb{C}$ and therefore $[E_{\tilde{\theta}}, E_\alpha] = -J[E_{\tilde{\theta}}, JE_\alpha] = 0$ because $\tilde{\theta} + R_{\mathfrak{n}}^{+} \not\subset R$ and $[\mathfrak{l}, \mathfrak{t}] = 0$. Hence $[E_{\tilde{\theta}}, \mathfrak{n}] = 0$ and therefore $[E_{\tilde{\theta}}, \mathfrak{m}^{\mathbb{C}}] = 0$. But this cannot happen otherwise $\exp_G(E_{\tilde{\theta}} - E_{-\tilde{\theta}})$ would act trivially on M, contradicting the (almost) effectiveness of the G-action. □

Note that the decomposition obtained above matches with decomposition (2.4) if we take $\mathfrak{b}_k = \mathfrak{l} \oplus \tilde{\mathfrak{t}} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\bigoplus_{j=1}^{\ell} \mathfrak{u}_j$. We also note that we have the following necessary condition: if $\mathfrak{z}_\ell$ is the center of the centraliser in $\mathfrak{g}$ of $\{s(\theta_1), \dots, s(\theta_\ell)\}$, then

$$\dim \mathfrak{z}_\ell \geqslant \ell. \tag{3.5}$$

3.2. Proof of Theorem 1.1, part (2)

We first prove the claim in the case in which G is simple, using Proposition 3.7 and condition (3.5).

Since the root θ_1 can be chosen as the highest root $\tilde{\theta}$ of the whole root system, we can start from the "Wolf decomposition" of $\mathfrak{g}$ with respect $\theta_1 = \tilde{\theta}$:

$$\mathfrak{g} = \mathfrak{s}(\theta_1) \oplus \mathfrak{z}_{\mathfrak{g}}\big(\mathfrak{s}(\theta_1)\big) \oplus \mathfrak{m}_1,$$

where $\mathfrak{m}_1$ is identified with the tangent space of the corresponding Wolf space.

By a case-by-case inspection for simple groups it is not difficult to see that for every set of strongly orthogonal roots $\theta_1 = \tilde{\theta}, \dots, \theta_\ell$ of $\mathfrak{g}$, we have $\dim \mathfrak{z}_\ell < \ell$ unless $\mathfrak{g}$ is of type A_n. If $\mathfrak{g} = \mathfrak{su}(n)$ we have indeed $\dim \mathfrak{z}_\ell = \ell$ for every choice of $\theta_1 = \tilde{\theta}, \dots, \theta_\ell$ (see also [16, Proposition 1]).

Suppose now that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ where the $\mathfrak{g}_j$'s are simple Lie algebras. The set of roots $\Theta = \{\theta_1, \dots, \theta_\ell\}$ is the disjoint union of the subsets Θ_j of all roots in Θ belonging to $\mathfrak{g}_j$. Now $\mathfrak{z}_\ell$ splits as a direct sum of the centres $\mathfrak{z}_j$ of the centralisers in $\mathfrak{g}_j$ of the subalgebras generated by the roots in Θ_j. Then $\dim \mathfrak{z}_\ell = \sum_{j=1}^{r} \dim \mathfrak{z}_j < \sum_{j=1}^{r} \sharp\Theta_j = \ell$ if at least one factor of $\mathfrak{g}$ is not of type A_n by the previous discussion.

3.3. Proof of Proposition 1.2

(1) Suppose that Y is reduced to a point. For any $I \in \mathcal{Q}$ the corresponding Tits fibration π has a typical fiber that is pointwisely fixed by the isotropy L, hence trivial. This means that M is a flag manifold with an invariant hyper-complex structure. If we decompose $\mathfrak{g} = \mathfrak{l} + \mathfrak{m}$ with $\mathfrak{m}$ an $\mathrm{ad}(\mathfrak{l})$-invariant subspace, it is known that the $\mathrm{ad}(\mathfrak{l})$-irreducible submodules $\mathfrak{m}_j$ $(j = 1, \dots, k)$ of $\mathfrak{m}$ are mutually inequivalent (see e.g. [6]) and therefore $\mathcal{Q}$-invariant. Now $\mathfrak{l}$ has a non trivial center $\mathfrak{c}$ and there is a submodule, say $\mathfrak{m}_1$, such that $\mathrm{ad}(\mathfrak{c})|_{\mathfrak{m}_1}$ is not trivial. Then using the irreducibility of $\mathfrak{m}_1$, we see that $\mathcal{Q}|_{\mathfrak{m}_1}$ belongs to $\mathrm{ad}(\mathfrak{c})|_{\mathfrak{m}_1}$, contradicting the fact that the $\mathcal{Q}|_{\mathfrak{m}_1}$ contains anti-commuting elements. Therefore Y has positive dimension and is $\mathcal{Q}$-invariant. Since L is not trivial, we see that Y is also a proper submanifold.

(2) Suppose now that the restriction of g to Y is hyper-Hermitian and consider the decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{t} + \mathfrak{n}$ as in Section 2.1, relative to some $I \in \mathcal{Q}$. Note that $[\mathfrak{l}, \mathfrak{t}] = 0$ means that $\mathfrak{t}$ projects to a subspace of $T_{[eL]}Y$ and therefore $g|_{\mathfrak{t}\times\mathfrak{t}}$ is I-Hermitian. Now $\mathfrak{n}^{\mathbb{C}}$ is a sum of root spaces w.r.t. the Cartan subalgebra $(\mathfrak{t}_{\mathfrak{l}} + \mathfrak{t})^{\mathbb{C}}$ and a simple computation using the natural reductiveness and the $\mathrm{ad}(\mathfrak{l})$-invariance of g shows that $g(E_\alpha, E_\beta) = 0$ for every roots α, β with $\alpha + \beta \neq 0$. Our claim now follows form the fact that $g(IE_\alpha, IE_{-\alpha}) = g(E_\alpha, E_{-\alpha})$ for every root α.

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Appendix A

In this appendix we list a natural choice $\{\theta_1, \dots, \theta_k\}$ of a MSOS for each simple Lie algebra as given in [8]. As for notations we follow [4]. The standard linear forms on the Cartan subalgebra are here denoted by ε_i, while for the exceptional Lie algebra $\mathbf{G}_2$ the standard simple roots are denoted by α_1, α_2.

Table 1
Maximal strongly orthogonal subsets.

$\mathbf{A}_n$	$\theta_1 = \varepsilon_1 - \varepsilon_{n+1}, \theta_2 = \varepsilon_2 - \varepsilon_n, \dots, \theta_n = \varepsilon_{\frac{n}{2}} - \varepsilon_{\frac{n}{2}+2}$ (n even)
	$\theta_1 = \varepsilon_1 - \varepsilon_{n+1}, \theta_2 = \varepsilon_2 - \varepsilon_n, \dots, \theta_n = \varepsilon_{\frac{n+1}{2}} - \varepsilon_{\frac{n+1}{2}+2}$ (n odd)
$\mathbf{B}_n$	$\theta_1 = \varepsilon_1 + \varepsilon_2, \theta_2 = \varepsilon_1 - \varepsilon_2, \dots, \theta_{n-1} = \varepsilon_{n-1} + \varepsilon_n, \theta_n = \varepsilon_{n-1} - \varepsilon_n$ (n even)
	$\theta_1 = \varepsilon_1 + \varepsilon_2, \theta_2 = \varepsilon_1 - \varepsilon_2, \dots, \theta_{n-2} = \varepsilon_{n-2} + \varepsilon_{n-1}, \theta_{n-1} = \varepsilon_{n-2} - \varepsilon_{n-1}, \theta_n = \varepsilon_n$ (n odd)
$\mathbf{C}_n$	$\theta_1 = 2\varepsilon_1, \theta_2 = 2\varepsilon_2, \dots, \theta_n = 2\varepsilon_n$
$\mathbf{D}_n$	$\theta_1 = \varepsilon_1 + \varepsilon_2, \theta_2 = \varepsilon_1 - \varepsilon_2, \dots, \theta_{n-1} = \varepsilon_{n-1} + \varepsilon_n, \theta_n = \varepsilon_{n-1} - \varepsilon_n$ (n even)
	$\theta_1 = \varepsilon_1 + \varepsilon_2, \theta_2 = \varepsilon_1 - \varepsilon_2, \dots, \theta_{n-2} = \varepsilon_{n-2} + \varepsilon_{n-1}, \theta_{n-1} = \varepsilon_{n-2} - \varepsilon_{n-1}$ (n odd)
$\mathbf{G}_2$	$\theta_1 = 3\alpha_1 + 2\alpha_2$, $\theta_2 = \alpha_1$
$\mathbf{F}_4$	$\theta_1 = \varepsilon_1 + \varepsilon_2$, $\theta_2 = \varepsilon_1 - \varepsilon_2$, $\theta_3 = \varepsilon_3 + \varepsilon_4$, $\theta_4 = \varepsilon_3 - \varepsilon_4$
$\mathbf{E}_6$	$\theta_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8)$, $\theta_2 = \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8)$, $\theta_3 = \varepsilon_4 - \varepsilon_1$, $\theta_4 = \varepsilon_3 - \varepsilon_2$
$\mathbf{E}_7$	$\theta_1 = \varepsilon_8 - \varepsilon_7$, $\theta_2 = \varepsilon_6 + \varepsilon_5$, $\theta_3 = \varepsilon_6 - \varepsilon_5$, $\theta_4 = \varepsilon_4 + \varepsilon_3$, $\theta_5 = \varepsilon_4 - \varepsilon_3$, $\theta_6 = \varepsilon_2 + \varepsilon_1$, $\theta_7 = \varepsilon_2 - \varepsilon_1$
$\mathbf{E}_8$	$\theta_1 = \varepsilon_7 + \varepsilon_8$, $\theta_2 = \varepsilon_8 - \varepsilon_7$, $\theta_3 = \varepsilon_6 + \varepsilon_5$, $\theta_4 = \varepsilon_6 - \varepsilon_5$, $\theta_5 = \varepsilon_4 + \varepsilon_3$, $\theta_6 = \varepsilon_4 - \varepsilon_3$, $\theta_7 = \varepsilon_2 + \varepsilon_1$, $\theta_8 = \varepsilon_2 - \varepsilon_1$

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