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Representations of the Generalized Inverse

C. W. GROETSCH

Department of Mathematics, University of Cincinnati, Cincinnati, Ohio 45221

Submitted by Ky Fan

1. INTRODUCTION

In 1950 Bellman [3] proved that the formal Liouville–Neumann series solution of the Fredholm integral equation

$$u(s) = f(s) + \lambda \int_0^1 K(s, t)\, u(t)\, dt \tag{1.1}$$

is summable to the Fredholm solution of (1.1) for λ not a characteristic value by any summability method which sums the series $1 + z + z^2 + \cdots$ to to $1/(1 - z)$ for all $z \neq 1$. Bellman's theorem is the genesis of this note which is concerned with representing the generalized inverse of a linear operator in terms of summability methods. As special cases of our general result we shall obtain most of the representations of the generalized inverse which are known to the author as well as several new representations.

Let H_1 and H_2 be two Hilbert spaces over the same scalars and suppose T is a bounded linear operator on H_1 into H_2 (i.e., $T \in L(H_1, H_2)$) with closed range $R(T)$. The linear operator which assigns to each $b \in H_2$ the unique element $u \in H_1$ with minimal norm satisfying

$$\| Tu - b \| = \inf\{\| Tx - b \| : x \in H_1\},$$

is called the generalized inverse of T and we write $u = T^\dagger b$.

Note that *formally* we have

$$(I - (I - T^*T))^{-1}\, T^* = T^{-1}. \tag{1.2}$$

In this note we shall show that the generalized inverse $T^\dagger$ can indeed be represented by "summing" the formal series expansion of (1.2) by suitable summation methods and that some well-known representations of $T^\dagger$ arise in just this way.

2. Main Result and Specific Representations

If $T \in L(H_1, H_2)$ has closed range, then $R(T^*) = H$ is a Hilbert space and we will denote by A the operator in $L(H, H)$ obtained by restricting $I - T^*T$ to H, i.e., $A = (I - T^*T)|_H$. We note that since $T^*T|_H$ is positive definite [8, Lemma 1] and self-adjoint, the spectrum of A satisfies $\sigma(A) \subset (-\infty, 1)$.

THEOREM. *Suppose Ω is an open set with $\sigma(A) \subset \Omega \subseteq (-\infty, 1)$ and let $\{S_\beta(x)\}$ be a net of continuous real functions on Ω such that*

$$\lim_\beta S_\beta(x) = 1/(1-x)$$

uniformly on $\sigma(A)$, then the representation $T^\dagger = \lim_\beta S_\beta(A)\, T^*$ holds in the uniform topology for $L(H_2, H_1)$.

Proof. We will use the characterization of $T^\dagger$ given by Desoer and Whalen [5], namely $T^\dagger$ is the unique operator B in $L(H_2, H_1)$ satisfying

$$BTy = y \qquad \text{for all } y \in R(T^*),$$

and $By = 0$ for all $y \in N(T^*) = \{y\colon T^*y = 0\}$. An application of the spectral theorem for self-adjoint operators (see, e.g. [11, p. 345]) gives

$$\lim_\beta S_\beta(A)\, T^* = (I - A)^{-1}\, T^*.$$

We clearly have $(I - A)^{-1}\, T^*y = 0$ for $y \in N(T^*)$ and if $y \in R(T^*) = H$ then

$$(I - A)^{-1}\, T^*Ty = (T^*T\,|_H)^{-1}\, T^*Ty = y,$$

completing the proof.

We now give several specific representations of $T^\dagger$ by making various choices for the net $\{S_\beta(x)\}$. It is interesting to note that the two best known representations of $T^\dagger$ ((2.1) and (2.2) below) result immediately from the theorem by taking $\{S_\beta(x)\}$ to be very well-known classical summability transforms of the series $1 + x + x^2 + \cdots$.

COROLLARY. *If $T \in L(H_1, H_2)$ has closed range, then the following representations of $T^\dagger$ are valid in the uniform topology for $L(H_2, H_1)$.*

$$T^\dagger = \sum_{k=0}^{\infty} \alpha[I - \alpha T^*T]^k\, T^* \qquad (0 < \alpha < 2\,\|T\|^{-2}), \tag{2.1}$$

$$T^\dagger = \int_0^\infty e^{-T^*Ts} T^*\, ds, \tag{2.2}$$

$$T^{\dagger} = \sum_{k=0}^{\infty} \frac{1}{k+2} \left\{ \prod_{j=1}^{k} \left[I - \frac{1}{j+1} T^*T \right] \right\} T^*, \tag{2.3}$$

$$T^{\dagger} = \lim_{t \to 0^+} (T^*T + tI)^{-1} T^*, \tag{2.4}$$

$$T^{\dagger} = \lim_{t \to 0^+} \sum_{k=0}^{\infty} a_k(t) [I - T^*T]^k T^*, \tag{2.5}$$

where

$$a_0(t) \equiv 1 \qquad \textit{and} \qquad a_k(t) = e^{-tk\log k} \qquad (k \geqslant 1),$$

$$a_k(t) = 1/\Gamma(1 + tk) \qquad (k \geqslant 1),$$

or

$$a_k(t) = \Gamma(1 + (1 - t)\, k)/\Gamma(1 + k) \qquad (k \geqslant 1).$$

Proof. To prove (2.1) we note that the Euler–Knopp transform with parameter α sums the series $1 + x + x^2 + \cdots$ to $1/(1 - x)$ uniformly on compact subsets of $\Omega = \{x: |(1 - \alpha) + \alpha x| < 1\}$ [7, p. 178]. Also if $0 < \alpha < 2 \| T \|^{-2}$ then $\sigma(A) \subset \Omega$. Hence setting

$$S_n(x) = \sum_{k=0}^{n} \alpha[(1 - \alpha) + \alpha x]^k$$

in the theorem we obtain the stated result.

To prove (2.6) we set $\Omega = (-\infty, 1)$ and $S_t(x) = \int_0^t e^{(x-1)s}\, ds$ (the Borel integral transform of $1 + x + x^2 + \cdots$). Since $(-\infty, 1)$ is contained in the Borel polygon of $1/(1 - x)$ it follows that $S_t(x) \to 1/(1 - x)$ as $t \to \infty$ uniformly on compact subsets of $(-\infty, 1)$ [7, p. 189] and hence the result.

The Lototsky transform [1] (see also [6]) of $1 + x + x^2 + \cdots$ is

$$S_n(x) = \sum_{k=0}^{n} \frac{1}{k+2} \prod_{j=1}^{k} \left[\left(1 - \frac{1}{j+1} \right) + \frac{1}{j+1} x \right]$$

and has the property that $S_n(x) \to 1/(1 - x)$ uniformly on compact subsets of $(-\infty, 1)$ (see [1]). Hence we obtain (2.3).

The representation (2.4) follows by setting $\Omega = (-\infty, 1)$ and

$$S_t(x) = (1 + t - x)^{-1} \qquad (t > 0).$$

Finally, representation (2.5) follows by setting $\Omega = (-\infty, 1)$ and using for $S_t(x)$ the Lindelöf, Mittag–Leffler or LeRoy transform of $1 + x + x^2 + \cdots$, respectively (see [7, pp. 78–79]).

Remarks. Formula (2.1) is the representation given by Altman [2] for the nonsingular case and by Showalter [10] for the general case. Except for a change of variables (2.2) is the integral representation given in [10]. Formula (2.4) appears in the survey article of Nashed [9, p. 341] and was apparently first given for square matrices by den Broeder and Charnes (see [4]).

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