On the Continuation and Boundedness of Solutions of a Nonlinear Differential Equation

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Sufficient conditions are given so that all solutions of the nonlinear differential equation

\[ u'' + \phi(t, u, u')u' + p(t)f(u)g(u') = h(t, u, u') \]

are continuable to the right of an initial \( t \)-value \( t_0 > 0 \). These conditions are then extended so that all solutions \( u \) of the equation in question together with their derivative \( u' \) are bounded for \( t > t_0 \).

1. In this paper, we will examine some of the properties of solutions of the damped and forced nonlinear ordinary differential equation

\[ u'' + \phi(t, u, u')u' + p(t)f(u)g(u') = h(t, u, u'), \quad (1) \]

where \( \phi: I \times \mathbb{R}^2 \to \mathbb{R}, f: \mathbb{R} \to \mathbb{R}, g: \mathbb{R} \to \mathbb{R}_+, h: I \times \mathbb{R}^2 \to \mathbb{R} \), and \( p: I \to \mathbb{R}_+ \) are continuous and \( R = (-\infty, \infty), \mathbb{R}_+ = (0, \infty), \) and \( I = [0, \infty) \). In particular, we will give sufficient conditions for all solutions \( u \) of (1) to be continuable to the right of their initial \( t \)-value \( t_0 \in I \) and for all solutions \( u \) of (1) and their derivative \( u' \) to be bounded on \([t_0, \infty)\). Our continuation theorem will generalize an unpublished result of Gollwitzer [7] and will yield a special case of the continuation theorem of Izyumova and Kiguradze [13] when \( \phi = h = 0 \) and \( g = 1 \). As was observed by Gollwitzer, [7] and [8], the technique also provides an alternate proof of the well-known continuation theorem of Coffman and Ullrich [4]. The boundedness theorems extend some of the results found in the papers of Chang [3], DeKleine [6], Gollwitzer [8], Petty and Johnson [15], and the references given therein.

2. In addition to the assumptions given above, we assume in what follows that

(H1) there is a continuous function \( q: I \to I \) such that

\[ -q(t) \leq \phi(t, x, y) \quad \text{for all} \quad (t, x, y) \in I \times \mathbb{R}^2. \]
CONTINUATION AND BOUNDEDNESS

(H2) \( F(x) = \int_{0}^{x} f(s) \, ds \geq 0 \) for all \( x \in \mathbb{R} \).

(H3) \( G(y) = \int_{0}^{y} \left[ \frac{s}{g(s)} \right] \, ds \), \( \lim_{|y| \to \infty} G(y) = \infty \), and there is a positive constant \( M \) such that \( y^2|g(y)| \leq MG(y) \) for all \( |y| \geq 1 \).

(H4) there are continuous functions \( e_k : I \to I \), \( k = 1, 2 \), such that \( \left| h(t, x, y) \right| \leq e_1(t) + e_2(t) \left| y \right|, \) for all \((t, x, y) \in I \times \mathbb{R}^2 \).

In what follows, we write \( p \in CBV_{10c}(I) \) whenever \( p \) is continuous on \( I \) and of bounded variation on compact subsets of \( I \).

**Theorem 2.1.** Suppose that (H1)–(H4) hold. If \( p \in CBV_{10c}(I) \), then each solution \( u \) of (1) is continueable to the right of its initial \( t \)-value.

**Proof.** Let \( u \) be a solution of (1) with initial \( t \)-value at \( t_0 \in I \), and suppose that \( u \) cannot be continued past the (finite) point \( T \). It suffices (see [11, p. 14]) to show that \( u' \) remains bounded as \( t \) approaches \( T \) from the left.

Multiplying (1) by \( g^{-1}u' \), using (H1), and integrating on \([t_0, t] \subset [t_0, T)\), we get

\[
G(u'(t)) - G(u'(t_0)) = \int_{t_0}^{t} \{ q(s) [u'(s)]^2/g(u'(s)) \} \, ds + \int_{t_0}^{t} p(s) f(u(s)) \, ds
\]

\[
\leq \int_{t_0}^{t} \left[ | h(s, u(s), u'(s)) u'(s)|/g(u'(s)) \right] \, ds. \tag{2}
\]

By (H3), there are positive constants \( M, N_1, \) and \( N_2 \) such that

\[
y^2/g(y) \leq MG(y) + N_1 \quad \text{for all } y \in \mathbb{R}, \tag{3a}
\]

\[
| y |/g(y) \leq MG(y) + N_2 \quad \text{for all } y \in \mathbb{R}. \tag{3b}
\]

Hence, we obtain from (2),

\[
G(u'(t)) - G(u'(t_0)) = \int_{t_0}^{t} p(s) \, dF(u(s))
\]

\[
\leq K(T) + M \int_{t_0}^{t} \{ e_1(s) + e_2(s) + q(s) \} G(u'(s)) \, ds, \tag{4}
\]

where the first integral in (4) is a Riemann–Stieltjes integral and \( K \) is the real number defined by

\[
K(T) = N_1 \int_{t_0}^{T} \{ e_2(s) + q(s) \} \, ds + N_2 \int_{t_0}^{T} e_1(s) \, ds.
\]
Following Gollwitzer, [7] or [8], we integrate the first integral in (4) by parts to obtain, after simplification,

$$E(t) \leq E(t_0) + K(T) + M \int_{t_0}^{t} \{ e_1(s) + e_2(s) + q(s) \} E(s) \, ds$$

$$+ \int_{t_0}^{t} [p(s)]^{-1} E(s) \, dp(s),$$

(5)

where $E$ is the "energy" function defined by

$$E(t) = G(u'(t)) + p(t) F(u(t)).$$

(6)

Writing the Jordan decomposition of $p$ (see [12, Chap. 2]) as

$$p(t) = P^+ - P^-,$$

where $p^+$ and $p^-$ are the positive and negative variations of $p$, it follows from (5) that

$$E(t) \leq E(t_0) + K(T) + M \int_{t_0}^{t} \{ e_1(s) + e_2(s) + q(s) \} E(s) \, ds$$

$$+ \int_{t_0}^{t} [p(s)]^{-1} E(s) \, dp^+(s).$$

(7)

Let $Q: [t_0, T) \rightarrow I$ be defined by

$$Q(t) = M \int_{t_0}^{t} \{ e_1(s) + e_2(s) + q(s) \} \, ds + \int_{t_0}^{t} [p(s)]^{-1} \, dp^+(s).$$

(8)

Then $Q \in CBV([t_0, T))$ and is nondecreasing, and (7) can be written as

$$E(t) \leq E(t_0) + K(T) + \int_{t_0}^{t} E(s) \, dQ(s).$$

By a form of the Gronwall–Bellman inequality (see Gollwitzer [8], Györi [10], or Schmaedeke and Sell [16]), there is a constant $L$, depending on $Q$ but not on $E$, such that, for all $t \in [t_0, T)$,

$$E(t) \leq [E(t_0) + K(T)]L.$$

Therefore, $G \circ u'$ remains bounded as $t \to T$ from the left and (H3) applies to show that $u'$ remains bounded as $t \to T$ from the left. This completes the proof of the theorem.

**Remark 2.2.** Another continuation theorem, extending results of Coffman
continuation and boundedness

and Wong [5], has recently been proved by Teufel [17] using techniques different from those presented here.

3. In this section, we will prove a boundedness theorem for solutions $u$ of (1) and their derivative $u'$ by using a modification of the technique of the previous section. In addition to our previous assumptions, we suppose when necessary that

\[(H5) \lim_{|x| \to \infty} F(x) = \infty.\]

**Theorem 3.1.** Suppose that (H1)–(H4) hold and that $e_1$, $e_2$, and $q$ are integrable on $I$. Let $p(t) = a(t) b(t)$, where $a$ is nondecreasing and bounded above on $I$ and $b$ is nonincreasing on $I$. Then, for each solution $u$ of (1) with an initial $t$-value $t_0 \in I$, $u'$ is bounded on $[t_0, \infty)$. If, in addition, (H5) holds and $b$ is bounded below by $b_0 > 0$, then $u$ is bounded on $[t_0, \infty)$.

**Proof.** Let $u$ be a solution of (1) with initial $t$-value at $t_0 \in I$. Multiplying (1) by $(a(t) u'(t))^2$, integrating on $[t_0, t]$, and proceeding as in the proof of Theorem 2.1, we obtain

\[
\int_{t_0}^{t} [a(s)]^{-1} \frac{d}{ds} G(u'(s)) \, ds - \int_{t_0}^{t} [a(s) g(u'(s))]^{-1} q(s) [u'(s)]^2 \, ds
\]

\[+ \int_{t_0}^{t} b(s) \frac{d}{ds} F(u(s)) \, ds\]

\[+ \int_{t_0}^{t} h(s, u(s), u'(s)) u'(s) [a(s) g(u'(s))]^{-1} \, ds.\]

Applying the second mean value theorem for integrals (see [12, p. 68]) to the first and third integrals in (9), there are points $\alpha, \beta \in [t_0, t]$ such that

\[
\frac{G(u'(\alpha)) - G(u'(t_0))}{a(t_0)} + \frac{G(u'(t)) - G(u'(\alpha))}{a(t)} + b(t_0) [F(u(\beta)) - F(u(t_0))]
\]

\[+ b(t) [F(u(t)) - F(u(\beta))]\]

\[\leq M \int_{t_0}^{t} \{e_1(s) + e_2(s) + q(s)/a(s)\} G(u'(s)) \, ds\]

\[+ \int_{t_0}^{t} \{N_1 e_2(s) + N_2 e_1(s) + N_1 q(s)/a(s)\} \, ds.\]

Define a "modified energy" function $E$ on $[t_0, \infty)$ by

\[E(t) = [a(t)]^{-1} G(u'(t)) + b(t) F(u(t)).\]

(11)
Since \( a^{-1} \) and \( b \) are nonincreasing on \( I \), then we obtain from (10), after simplifying,
\[
E(t) \leq E(t_0) + m(t) + M \int_{t_0}^{t} \{ a(s) [e_1(s) + e_2(s)] + q(s) \} E(s) \, ds,
\]
(12)
where
\[
m(t) = \int_{t_0}^{t} \{ N_1 e_2(s) + N_2 e_1(s) + N_4 q(s)/a(s) \} \, ds.
\]
Furthermore,
\[
m(t) \leq \int_{t_0}^{\infty} \{ N_1 e_2(s) + N_2 e_1(s) + N_4 q(s)/a(t_0) \} \, ds = m_0 < \infty.
\]
Since \( a \) is bounded above, say by \( a_0 \), then (12) gives
\[
E(t) \leq E(t_0) + m_0 + M \int_{t_0}^{t} \{ q(s) + a_0 [e_1(s) + e_2(s)] \} E(s) \, ds.
\]
(13)
The Gronwall–Bellman inequality then applies to show that there is a constant \( L \), depending on \( e_1, e_2, \) and \( q \), but not \( E \), such that
\[
E(t) \leq (E(t_0) + m_0)L.
\]
In view of (11) and (H3) and (H5), the desired conclusion follows.

**Remark 3.2.** The use of the second mean value theorem for integrals in proving boundedness theorems dates back to a paper of Kamenev [14]. Its use was independently suggested by Gollwitzer and was explored by the author in [1].

**Remark 3.3.** The integrability of \( q \) is necessary for all solutions \( u \) of (1) to have a bounded derivative \( u' \), given that the remaining assumptions of Theorem 3.1 hold. This is easily seen from the fact that the equation
\[
 u'' + \{ [4(t+1)^{-\beta/3}] |uu'|-8(t+1)^{-1} u' + 6(t+1)^{1-3\beta} u^\alpha = 0,
\]
with \( \alpha \) the quotient of odd, positive integers, has \( u(t) = (t+1)^3 \) as a solution.

The following two theorems are corollaries to Theorem 3.1. In the first, we use the notation
\[
p'(t)_+ = \operatorname{Max}(p'(t), 0), \ p'(t)_- = \operatorname{Min}(-p'(t), 0)
\]
whenever \( p \in C'((R_+); thus, \ p'(t) = p'(t)_+ - p'(t)_- \). In the second, we use the notation for the Jordan decomposition of \( p \in CBV_{loc}(I) \) that was used in Section 2.
**Theorem 3.4.** Suppose that (H1)–(H4) hold. If $p \in C'(R_+)$ and $e_1, e_2, q,$ and $p' \ln p$ are integrable on $I$, then, for each solution $u$ of (1) with initial $t$-value at $t_0 \in I$, $u'$ is bounded on $[t_0, \infty)$. If, in addition, (H5) holds and $p' \ln p$ is integrable on $[t_0, \infty)$, then $u$ is bounded on $[t_0, \infty)$.

*Proof.* Let $a$ and $b$ be defined by

$$a(t) = p(t_0) \exp \left( \int_{t_0}^{t} \left[ \frac{p'(s)}{p(s)} \right] ds \right),$$

$$b(t) = \exp \left( - \int_{t_0}^{t} \left[ \frac{p'(s)}{p(s)} \right] ds \right),$$

and use Theorem 3.1.

**Theorem 3.5.** Suppose that (H1)–(H4) hold, and let $p \in CBV_{\text{loc}}(I)$. If $e_1, e_2,$ and $q$ are integrable on $I$, then, for each solution $u$ of (1) with initial $t$-value at $t_0 \in I$, $u'$ is bounded on $[t_0, \infty)$, provided that $\int_{t_0}^{\infty} [p(s)]^{-1} dp_{+}(s) < \infty$. If, in addition, (H5) holds and $\int_{t_0}^{\infty} [p(s)]^{-1} dp_{-}(s) < \infty$, then $u$ is bounded on $[t_0, \infty)$.

*Proof.* Using the fact that $\ln[p(t)/p(t_0)] = \int_{t_0}^{t} [p(s)]^{-1} dp(s)$ (see Gollwitzer [8]), let $a$ and $b$ be defined by

$$a(t) = p(t_0) \exp \left( \int_{t_0}^{t} [p(s)]^{-1} dp_{+}(s) \right),$$

$$b(t) = \exp \left( - \int_{t_0}^{t} [p(s)]^{-1} dp_{-}(s) \right),$$

and use Theorem 3.1.

**Remark 3.6.** In [8] and [9], Gollwitzer gives conditions on $p$ under which the assumed integral conditions in Theorem 3.5 are satisfied.

**Remark 3.7.** If $h = 0$, then an inspection of the proof of Theorem 3.1 shows that the assumption that $a$ is bounded above is not necessary to conclude that $u$ is bounded on $[t_0, \infty)$. Thus, for example, (11) implies that all solutions $u$ of (1) are bounded on $[t_0, \infty)$ when $h = 0$ provided that $q$ is integrable on $[t_0, \infty)$ and

$$\int_{t_0}^{s} [p(s)]^{-1} dp_{-}(s) < \infty.$$ 

However, it does not follow from (11) that $u'$ is bounded on $[t_0, \infty)$. 
4. We now consider the differential equation

\[ u'' + \phi(t, u, u')u' + p(t)f(u) = h(t, u, u') \]  

(1*)

which is a special case of (1) with \( g = 1 \). This equation was studied by the author in [1] and [2] under the assumptions that \( e_1 = 0 \) and \( \phi(t, x, y) = -q(t) + \phi_1(t, x, y) \) with \( q(t) \geq 0 \) for all \( t \in I \), \( \phi_1(t, x, y) \geq 0 \) for all \( (t, x, y) \in I \times R^2 \), and \( \int_0^\infty q(s) \, ds < \infty \). We will prove a boundedness theorem in this section which eliminates the last assumption mentioned above on \( q \). We first observe that if \( r \) is defined by \( r(t) = \exp(-\int_0^t q(s) \, ds) \), then (1*) can be written as

\[ (r(t)u')' + r(t)(q(t) + \phi(t, u, u'))u' + p(t)r(t)f(u) = r(t)h(t, u, u'). \]  

(14)

**Theorem 4.1.** Suppose that (H1)-(H5) hold and that \( e_1 \) and \( e_2 \) are integrable on \( I \). If \( p(t) \exp(-2\int q(x) \, dx) \) is nondecreasing on \( I \), then all solutions \( u \) of (1*) with initial \( t \)-value at \( t_0 \in I \) are bounded on \([t_0, \infty)\).

**Proof.** Let \( u \) be a solution of (1*) with initial \( t \)-value \( t_0 \in I \). Then, with \( r \) defined as above, \( u \) is a solution of (14). Multiplying (14) by \((rp)^{-1}u'\) and integrating on \([t_0, t]\) gives, after a simplification,

\[
\int_{t_0}^{t} \left( r(s)u'(s) \right) \left( r(s)u'(s) \right)' \, ds + \int_{t_0}^{t} f(u(s))u'(s) \, ds
\]

\[
\leq \int_{t_0}^{t} \left[ p(s) \right]^{-1} \left[ e_1(s) + e_2(s) \right] \left[ u'(s) \right] \left[ u'(s) \right] \, ds.
\]

(15)

Applying the second mean value theorem to the first integral in (15), using the fact that the function \( pr^2 \) is nondecreasing on \( I \), defining the function \( E \) by

\[
E(t) = \frac{[u'(t)]^2}{2p(t)} + F(u(t)),
\]

(16)

and using the inequality \( 2|y| \leq y^2 + 1 \), we obtain from (15),

\[
E(t) \leq E(t_0) + \int_{t_0}^{t} e_1(s) \, ds + \int_{t_0}^{t} (2e_2(s) + e_1(s)) \frac{[u'(s)]^2}{2p(s)} \, ds
\]

\[
\leq E(t_0) + \left\{ 2p(t_0) \left[ r(t_0) \right]^{2} \right\}^{-1} \int_{t_0}^{\infty} e_1(s) \, ds + \int_{t_0}^{t} \left\{ 2e_2(s) + e_1(s) \right\} E(s) \, ds.
\]

(17)

Therefore, an application of the Gronwall–Bellman inequality shows that \( E \) is bounded on \([t_0, \infty)\). Clearly, this implies that \( u \) is bounded on \([t_0, \infty)\).
Remark 4.2. It should be noted that $u'$ is not necessarily bounded on $[t_0, \infty)$; this follows from the fact that the equation
\[ u'' - u' + e^{2t}u = 0 \]
has the bounded solution $u(t) = \sin(e^t)$ with an unbounded derivative while $p \rho^2 = 1$.

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