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Dual mixed volumes and the slicing problem

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Abstract

We develop a technique using dual mixed-volumes to study the isotropic constants of some classes of spaces. In particular, we recover, strengthen and generalize results of Ball and Junge concerning the isotropic constants of subspaces and quotients of L_p and related spaces. An extension of these results to negative values of p is also obtained, using generalized intersection-bodies. In particular, we show that the isotropic constant of a convex body which is contained in an intersection-body is bounded (up to a constant) by the ratio between the latter's mean-radius and the former's volume-radius. We also show how type or cotype 2 may be used to easily prove inequalities on any isotropic measure.

Keywords: Slicing problem; Isotropic constant; Dual mixed volumes; Generalized intersection bodies; Type and cotype 2

1. Introduction

The main purpose of this note is to provide new types of bounds on a convex body's isotropic constant, by means of dual mixed-volumes with different families of bodies.

A centrally symmetric convex body K in $\mathbb{R}^n$ is said to be in isotropic position if $\int_K \langle x,\theta\rangle^2\,dx$ is constant for all $\theta \in S^{n-1}$, the Euclidean unit sphere. If in addition K is of volume 1, then its isotropic constant is defined to be the L_K satisfying $\int_K \langle x,\theta\rangle^2\,dx = L_K^2$ for all $\theta \in S^{n-1}$. It is easy to see that every body may be brought to isotropic position using an affine transformation, and that the isotropic position is unique modulo rotations and homothety [35]. Hence, for a gen-

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eral centrally symmetric convex body K we shall denote by L_K the isotropic constant of K in its isotropic position of volume 1.

A famous problem, commonly known as the slicing problem, asks whether L_K is bounded from above by a universal constant independent of n, for all centrally symmetric convex bodies K in $\mathbb{R}^n$. This was first posed in an equivalent form by J. Bourgain, who asked whether every centrally symmetric convex body of volume 1, has an $(n-1)$-dimensional section whose volume is bounded from below by some universal constant. This is known to be true for several families of bodies, such as sections of L_1, projection bodies and 1-unconditional bodies (see [4,35] or below). The best general bound is due to Bourgain, who showed in [8] that $L_K \leqslant Cn^{1/4} \times \log(1+n)$. Recently, the general problem has been reduced to the case that K has finite volume-ratio [13].

The main idea of this note is to compare a general convex body K (or its polar) with a less general body L chosen from a specific family, and thus gain some knowledge on its isotropic constant. We shall consider two main families: unit-balls of n-dimensional subspaces of L_p, denoted SL_p^n, and k-Busemann–Petty bodies, denoted BP_k^n, which are a generalization of intersection bodies (the class BP_1^n) introduced by Zhang in [41] (there they are referred to as "generalized $(n-k)$-intersection bodies," see Section 2 for definitions). The body L may not be necessarily convex, but we will assume that it is a centrally symmetric star-body, defined by a continuous radial function $\rho_L(\theta) = \max\{r \geqslant 0 \mid r\theta \in L\}$ for $\theta \in S^{n-1}$. Our main tool for comparing two star-bodies will be the dual mixed-volume of order p, defined in Section 2, which was first introduced by Lutwak in [30].

We will require a few more notations. Let $|x|$ denote the standard Euclidean norm of $x \in \mathbb{R}^n$, let D_n denote the Euclidean unit ball and let σ denote the Haar probability measure on S^{n-1}. Let $SL(n)$ denote the group of volume preserving linear transformations in $\mathbb{R}^n$, and let $\mathrm{Vol}(B)$ denote the Lebesgue measure of the set $B \subset \mathbb{R}^n$ in its affine hull. Let K° denote the polar body to a convex body K.

An equivalent characterization of the isotropic position [35] states that it is the position which minimizes the expression $\int_K |x|^2\,dx$, in which case the latter is equal to nL_K^2 if $\mathrm{Vol}(K) = 1$. By comparing with the value of this expression in a position for which the circumradius $a(K)$ of K is minimal, we immediately get the bound $L_K \leqslant a(K)/\sqrt{n}$. Equivalently, making this invariant to change of position or normalization, we get the following well-known elementary bound on L_K in terms of the outer volume-ratio of K:

$$L_K \leqslant C \inf\left\{ \left(\frac{\mathrm{Vol}(\mathcal{E})}{\mathrm{Vol}(K)} \right)^{1/n} \,\middle|\, K \subset \mathcal{E},\ \mathcal{E} \in SL_2^n \right\},$$

where SL_2^n is just the class of all ellipsoids in $\mathbb{R}^n$. This was generalized in [4] by K. Ball as follows:

Theorem *(Ball).*

$$L_K \leqslant C \inf\left\{ \left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(K)} \right)^{1/n} \,\middle|\, K \subset L,\ L \in SL_1^n \right\}. \tag{1.1}$$

In fact, Ball showed that the expression on the right is equivalent (up to universal constants) to the so-called weak right-hand Gordon–Lewis constant $wrgl_2(X_K^*)$ of the Banach space X_K^* whose unit ball is the polar of K. Ball showed that $wrgl_2(X^*)$ is majorized (up to a constant) by $gl_2(X)$, the Gordon–Lewis constant of X, and hence L_K is bounded for spaces X_K with uniformly bounded gl_2 constants. These include subspaces of L_p for $1 \leqslant p \leqslant 2$, quotients of L_q for $2 \leqslant q \leqslant \infty$, and spaces with a 1-unconditional basis (the latter were first shown to have a bounded isotropic constant by Bourgain). A complementary result was obtained in [22] by Junge, who showed the following (this is not explicit in his formulation but follows from the proof):

Theorem *(Junge).*

$$L_K \leqslant C \inf\left\{ \sqrt{p}q \left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(K)}\right)^{1/n} \;\middle|\; \begin{array}{l} K \subset L,\ L \in SQL_p^n, \\ 1 < p < \infty,\ 1/p + 1/q = 1 \end{array} \right\}, \tag{1.2}$$

where SQL_p^n is the class of all unit-balls of n-dimensional subspaces of quotients of L_p, and $q = p^*$ is the conjugate exponent to p.

In fact, Junge showed that L_p may be replaced by any Banach space X with bounded $gl_2(X)$ such that X has finite type, in which case $\sqrt{p}q$ above should be replaced by some constant depending on X.

As evident from their more general formulations, the results of Ball and Junge described above make heavy use of non-trivial functional analysis and operator theory, and as a result the geometric intuition behind the slicing problem is substantially lost. Of course, this is to be expected if the conditions on the space X_K are formulated using operator theory notions, such as (variants of) the Gordon–Lewis property. But for classical spaces such as subspaces or quotients of L_p, one may hope to simplify the approach, derive better bounds on L_K, and unify Ball and Junge's results into a single framework. Using an elementary argument, geometric in nature, we show the following generalizations of (1.1) and partial strengthening of (1.2) (the term "partial" refers to the fact that we restrict L to the class SL_p^n or QL_q^n defined below), for a convex isotropic body K with $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$:

Theorem 1.

$$L_K \leqslant C \inf\left\{ \frac{\sqrt{p_0}}{M_p(L)} \;\middle|\; K \subset L,\ L \in SL_p^n,\ p \geqslant 0 \right\},$$

where $p_0 = \max(1, \min(p, n))$, $M_p(L) = (\int_{S^{n-1}} \|x\|_L^p \, d\sigma(x))^{1/p}$ for $p > 0$, and by passing to the limit, $M_0(L) = \exp(\int_{S^{n-1}} \log \|x\|_L \, d\sigma(x))$.

Theorem 1′.

$$L_K \leqslant C \frac{T_2(X_K)}{M_2(K)},$$

where $T_2(X_K)$ is the (Gaussian) type-2 constant of X_K.

Theorem 2.

$$L_K \leqslant C \inf\left\{ \mathcal{L}_k \widetilde{M}_k(L) \;\middle|\; K \subset L,\ L \in BP_k^n,\ k = 1, \dots, n-1 \right\},$$

where $\mathcal{L}_k$ denotes the maximal isotropic constant of centrally symmetric convex bodies in $\mathbb{R}^k$ and $\widetilde{M}_k(L) = (\int_{S^{n-1}} \rho_L(x)^k \, d\sigma(x))^{1/k}$. We emphasize again that BP_1^n is exactly the class of intersection bodies.

Indeed, these are all generalizations of (1.1) and (1.2), since by passing to polar coordinates and applying Jensen's inequality (for $p, k > 0$):

$$\frac{1}{M_p(L)} \leqslant \widetilde{M}_k(L) \leqslant \left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(D_n)}\right)^{1/n}, \tag{1.3}$$

and since $T_2(X_K) \leqslant C\sqrt{p}$ by Kahane's inequality when $K \in SL_p^n$ for $p \geqslant 2$. This also applies to Theorem 2, since any $K \in SL_p^n$ for $0 < p \leqslant 2$ (and in particular $p = 1$) is an intersection body (see [24]), and hence a k-Busemann–Petty body for all $k \geqslant 1$ [20,34].

We also have the following dual counterparts to Theorems 1 and 2, for a convex isotropic body K with $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$:

Theorem 3.

$$L_K \leqslant C \inf\left\{ \sqrt{p_0} M_p^*\big(T(L)\big) \;\middle|\; \begin{array}{l} K \subset L,\ L \in QL_q^n,\ T \in SL(n), \\ 1 \leqslant q \leqslant \infty,\ 1/p + 1/q = 1 \end{array} \right\},$$

where QL_q^n is the class of all unit-balls of n-dimensional quotients of L_q, p_0 is defined as above for $p = q^*$, and $M_p^*(G) = M_p(G^\circ)$.

This is indeed a (partial) strengthening of (1.2), since by Lemma 4.8 (see also the Mean Norm Corollary below), there exists a position $T \in SL(n)$ of $L \in QL_q^n$ such that:

$$M_p^*\big(T(L)\big) \leqslant C\sqrt{p_0}\left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(D_n)}\right)^{1/n}.$$

It is also interesting to note that the proof of Theorem 3, although derived independently, closely resembles Bourgain's proof that $L_K \leqslant Cn^{1/4}\log(1+n)$.

Theorem 4.

$$L_K \leqslant C \inf\left\{ \frac{\mathcal{L}_{2k}^2}{\widetilde{M}_k(T(L))} \;\middle|\; \begin{array}{l} L \subset K^\circ,\ L \in BP_k^n, \\ T \in SL(n),\ k = 1, \dots, \lfloor n/3 \rfloor \end{array} \right\}.$$

Using an analogue of Lemma 4.8 (stated in the Mean Radius Corollary below), we may deduce the following bound on L_K for polars of bodies in CBP_k^n, the class of *convex* k-Busemann–Petty bodies:

$$L_K \leqslant C \inf\left\{ \mathcal{L}_{2k}^2 \mathcal{L}_k \left(\frac{\mathrm{Vol}(L^\circ)}{\mathrm{Vol}(K)}\right)^{1/n} \;\middle|\; \begin{array}{l} K \subset L^\circ,\ L \in CBP_k^n, \\ k = 1, \dots, \lfloor n/3 \rfloor \end{array} \right\}.$$

Since Jensen's inequality (1.3) is usually strict, it is not hard to construct examples for which Theorem 1 asymptotically out-performs Junge's bound. Indeed, for $K = [-1, 1]^n$, it is well known (see Section 6) that K is isomorphic to a body $L \in SL_p^n$, for $p = \log n$. Junge's bound therefore implies $L_K \leqslant C\sqrt{\log n}$, while Theorem 1 gives $L_K \leqslant C$, since $M_p(L) \simeq M_p(K) \simeq \sqrt{\log n}(\mathrm{Vol}(K)/\mathrm{Vol}(D_n))^{1/n}$.

As mentioned above, Theorems 1 and 3 imply, in particular, that $L_K \leqslant C\sqrt{p}$ for $K \in SL_p^n$ and $p \geqslant 1$, and $L_K \leqslant q^*$ for $K \in QL_q^n$ and $q > 1$. We note that this is not contained in Junge's result (1.2). The strength of (1.2) is that it applies simultaneously to all subspaces of quotients of L_p, which our method does not handle. Ironically, this is also its drawback, if one is interested in proper subspaces or quotients only: it gives the same bound on L_K in either case. Therefore, one cannot hope to have a good bound for SL_p^n with $1 \leqslant p < 2$ (QL_q^n with $q > 2$) without solving the Slicing Problem, because this would imply the same bound for QL_p^n (SL_q^n) in that range, which already contain all convex bodies. To fill the bound for SL_p^n with $1 \leqslant p < 2$ (QL_q^n with $q > 2$), one needs to use Ball's result in its general form (or simply use (1.1) combined with the fact that $SL_p^n \subset SL_1^n$ for $1 \leqslant p \leqslant 2$; by duality $QL_q^n \subset QL_\infty^n$ for $q \geqslant 2$, implying that the bodies in QL_q^n have finite outer volume-ratio as projection bodies). We therefore see that Theorems 1 and 3 combine the ranges $1 \leqslant p < 2$ and $p \geqslant 2$ into a single framework.

Evidently, Theorem 1′ has a somewhat different flavor, and indeed its proof is totally different from the proofs of the other theorems. The proof is based on a simple yet effective framework for combining isotropic measures with type and cotype 2, which is introduced in Section 3 (this section may be read independently from the rest of this note). This framework also enables us to easily recover several known lemmas on John's maximal volume ellipsoid position (originally proved using Operator Theory techniques), which we use in the proof of Lemma 4.8 (mentioned above). We remark that Theorem 1′ also follows from the work in [12] but in a more complicated manner.

The other theorems are all proved using another technique, involving dual mixed-volumes. Theorems 1 and 3 are proved in Section 4, and Theorems 2 and 4 are proved in Section 5. In Section 6, we give several corollaries of our main theorems, some of which are mentioned below.

Using the known fact that L_K is always bounded from below, Theorems 1′, 1 and 2, immediately yield the following useful corollary, for an isotropic convex body K with $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$:

Mean Norm/Radius Corollary.

(1) $M_2(K) \leqslant C T_2(X_K)$.
(2) *If* $K \in SL_p^n$ $(p > 0)$, *then* $M_p(K) \leqslant C\sqrt{p_0}$.
(3) *If* $K \in BP_k^n$ $(k = 1, \dots, n-1)$, *then* $\widetilde{M}_k(K) \geqslant C/\mathcal{L}_k$.

Jensen's inequality in (1.3) shows that these bounds are tight (to within a constant) for $p, k, T_2(X_K) \leqslant C$. One should also keep in mind that if K° is in isotropic position, this corollary is applicable to K°, providing different inequalities.

In addition, although this is a direct consequence of the extended formulation of Junge's Theorem (and also of Theorems 1 and 3), the following corollary about a centrally symmetric convex polytope P is worth explicit stating:

Polytope Corollary.

(1) *If* P *has* $2m$ *facets then* $L_P \leqslant C\sqrt{\log(1+m)}$.
(2) *If* P *has* $2m$ *vertices then* $L_P \leqslant C\log(1+m)$.

In particular, this implies that Gluskin's probabilistic construction in [17] of two convex bodies K_1 and K_2 with Banach–Mazur distance of order n, satisfies $L_{K_1}, L_{K_2} \leqslant C\log(1+n)$.

Theorem 2 should be understood as a partial complimentary result to Theorem 1. The reason for this may be better explained, if we first consider a second generalization of intersection bodies, introduced by Koldobsky in [25]. We shall call these bodies *k-intersection bodies* and denote this class of bodies by $\mathcal{I}_k^n$. It was shown in [25] that $BP_k^n \subset \mathcal{I}_k^n$, and the question of whether $BP_k^n = \mathcal{I}_k^n$ remains open (see [34] for an account of recent progress in this direction). The class $\mathcal{I}_k^n$ satisfies a certain characterization of being embedded in L_p, which has been continued analytically to the negative value $p = -k$, so in some sense $\mathcal{I}_k^n = SL_{-k}^n$. Therefore, in some sense, $BP_k^n \subset SL_{-k}^n$, hence our initial remark.

The class of star-bodies BP_k^n seems at first glance a non-natural object to work with when studying convex bodies. Nevertheless, we describe in Section 6 several potential ways in which this object may be harnessed to our advantage.

2. Definitions and notations

A convex body K will always refer to a compact, convex set in $\mathbb{R}^n$ with non-empty interior. We will always assume that the bodies in question are centrally symmetric, i.e. $K = -K$. The equivalence between convex bodies and norms in $\mathbb{R}^n$ is well known, with the correspondence $\|x\|_K = \min\{t > 0 \mid x/t \in K\}$. The associated normed space $(\mathbb{R}^n, \|\cdot\|_K)$ will be denoted by X_K. The dual norm is defined as $\|x\|_K^* = \sup_{y \in K} |\langle x, y\rangle|$, and its associated unit-ball is called the polar body to K, and denoted K°. The dual normed space $(\mathbb{R}^n, \|\cdot\|_K^*)$ is denoted by X_K^* $(= X_{K^\circ})$. We will say that a convex-body K is 1-unconditional, or simply unconditional, with respect to the given Euclidean structure (which we always assume to be fixed), if $(x_1, \dots, x_n) \in K$ implies $(\pm x_1, \dots, \pm x_n) \in K$ for all possible sign assignments.

We will also work with general star-bodies L, which are star-shaped bodies, meaning that $tL \subset L$ for all $t \in [0, 1]$, with the additional requirement that their radial function ρ_L is a continuous function on S^{n-1}. The radius of L in direction $\theta \in S^{n-1}$ is defined as $\rho_L(\theta) = \max\{r \geqslant 0 \mid r\theta \in L\}$. For a general star-body L, we define its Minkowski functional $\|x\|_L$ in the same manner as for a convex body (so $\|x\|_L$ is no longer necessarily a norm). Obviously, $\rho_L(\theta) = 1/\|\theta\|_L$ for all $\theta \in S^{n-1}$.

By identifying between a star-body and its radial function, a natural metric arises on the space of star-bodies. The radial metric, denoted by d_r, is defined as:

$$d_r(L_1, L_2) = \sup_{\theta \in S^{n-1}} \left|\rho_{L_1}(\theta) - \rho_{L_2}(\theta)\right|.$$

As mentioned in the Introduction, our main tool for comparing two star-bodies L_1 and L_2 will be the dual mixed-volume of order $p \in \mathbb{R}$, introduced by Lutwak in [30] (see also [32]), and defined as:

$$\widetilde{V}_p(L_1, L_2) = \frac{1}{n} \int\limits_{S^{n-1}} \rho_{L_1}(x)^p \rho_{L_2}(x)^{n-p}\, dx$$

(note that the integration is w.r.t. the Lebesgue measure on S^{n-1}). By polar integration, it is obvious that $\widetilde{V}_p(L, L) = \mathrm{Vol}(L)$ for all p. We will also use the following useful property of dual mixed-volumes (see [32]):

$$\widetilde{V}_p\big(T(L_1), T(L_2)\big) = \widetilde{V}_p(L_1, L_2), \tag{2.1}$$

for any $T \in SL(n)$ and $p \in \mathbb{R}$. We also constantly use the well-known formula for the volume of the Euclidean unit ball D_n:

$$\mathrm{Vol}(D_n) = \frac{\pi^{n/2}}{\Gamma(n/2+1)}. \tag{2.2}$$

Several useful notations for a star-body L will be used. For $p > 0$, the pth mean-norm, denoted by $M_p(L)$, is defined as:

$$M_p(L) = \left(\int_{S^{n-1}} \|x\|_L^p \, d\sigma(x) \right)^{1/p}.$$

Passing to the limit as $p \to 0$, we define $M_0(L) = \exp(\int_{S^{n-1}} \log \|x\|_L \, d\sigma(x))$. We will define the mean-norm as $M(L) = M_1(L)$. The pth mean-width, denoted $M_p^*(L)$, is defined as $M_p^*(L) = M_p(L^\circ)$, and as usual, the mean-width is defined as $M^*(L) = M_1^*(L)$. The pth mean-radius, denoted by $\widetilde{M}_p(L)$, is defined as:

$$\widetilde{M}_p(L) = \left(\int_{S^{n-1}} \rho_L(x)^p \, d\sigma(x) \right)^{1/p}.$$

We will define the mean-radius as $\widetilde{M}(L) = \widetilde{M}_1(L)$. The minimal $a, b > 0$ for which $1/a|x| \leqslant \|x\|_L \leqslant b|x|$, will be denoted by $a(L)$ and $b(L)$, respectively. Geometrically, $a(L)$ and $1/b(L)$ are the radii of the circumscribing and inscribed Euclidean balls of L, respectively. The expression $(\mathrm{Vol}(L)/\mathrm{Vol}(D_n))^{1/n}$ will be referred to as the volume-radius of L. The infimum of $(\mathrm{Vol}(L)/\mathrm{Vol}(\mathcal{E}))^{1/n}$ over all ellipsoids $\mathcal{E}$ contained in L is called the volume-ratio of L. Similarly, the infimum of $(\mathrm{Vol}(\mathcal{E})/\mathrm{Vol}(L))^{1/n}$ over all ellipsoids $\mathcal{E}$ containing L is called the outer volume-ratio of L. A position of a body L is a volume preserving linear image of L, i.e. $T(L)$ for $T \in SL(n)$.

Going back to convex bodies and normed spaces, we now define the (Gaussian) type- and cotype-2 constants of a normed space $X = (\mathbb{R}^n, \|\cdot\|)$. The (Gaussian) type-2 constant of X, denoted $T_2(X)$, is the minimal $T > 0$ for which:

$$\left(\int_\Omega \left\| \sum_{i=1}^m g_i(\omega) x_i \right\|^2 d\omega \right)^{1/2} \leqslant T \left(\sum_{i=1}^m \|x_i\|^2 \right)^{1/2}$$

for any $m \geqslant 1$ and any $x_1, \dots, x_m \in X$, where $g_1, \dots, g_m$ are independent real-valued standard Gaussian r.v.'s on a common probability space $(\Omega, d\omega)$. Similarly, the (Gaussian) cotype-2 constant of X, denoted $C_2(X)$, is the minimal $C > 0$ for which:

$$\left(\int_\Omega \left\| \sum_{i=1}^m g_i(\omega) x_i \right\|^2 d\omega \right)^{1/2} \geqslant 1/C \left(\sum_{i=1}^m \|x_i\|^2 \right)^{1/2}$$

for any $m \geqslant 1$ and $x_1, \dots, x_m \in X$. We will not distinguish between the Gaussian and the Rademacher type- (cotype-) 2 constants, since it is well known that the former constant is always majorated by the latter one (e.g., [37]), and all our results will involve upper bounds in terms of the Gaussian type (cotype) 2.

We will often identify between a normed space and its unit-ball. In particular, for the infinite dimensional Banach space $L_p = L_p([0,1], dx)$, whenever the expression "sections of L_p" is used, we will mean sections of its unit-ball. And when the expression "quotients of L_p" is used, we might refer to the unit-balls of these quotient spaces.

Throughout the paper, all constants used will be universal, independent of all other parameters, and in particular, independent of n. We reserve C, C', C_1, C_2 to denote these constants, which may take different values on separate instances. We will write $A \simeq B$ to signify that $C_1 A \leqslant B \leqslant C_2 A$ with universal constants $C_1, C_2 > 0$.

For the results of Sections 5 and 6, we shall need to define the class of k-Busemann–Petty bodies, introduced by Zhang in [41] (there they are referred to as "generalized $(n-k)$-intersection bodies"). These bodies represent a generalization of the notion of an intersection body. For completeness, we give the appropriate definitions below.

Definition. A star body K is said to be an *intersection body of a star body L*, if $\rho_K(\theta) = \mathrm{Vol}(L \cap \theta^{\perp})$ for every $\theta \in S^{n-1}$. K is said to be an *intersection body*, if it is the limit in the radial metric d_r of intersection bodies $\{K_i\}$ of star bodies $\{L_i\}$. This is equivalent (e.g., [15,32]) to $\rho_K = R^*(d\mu)$, where μ is a non-negative Borel measure on S^{n-1}, R^* is the dual transform (as in (2.3)) to the Spherical Radon Transform $R : C(S^{n-1}) \to C(S^{n-1})$, which is defined for $f \in C(S^{n-1})$ as:

$$R(f)(\theta) = \int_{S^{n-1}\cap\theta^{\perp}} f(\xi)\, d\sigma_{n-1}(\xi),$$

where σ_{n-1} the Haar probability measure on S^{n-2} (and we have identified S^{n-2} with $S^{n-1} \cap \theta^{\perp}$).

Let $G(n,m)$ denote the Grassmann manifold of all m-dimensional linear subspaces of $\mathbb{R}^n$. Generalizing the Spherical Radon Transform is the m-dimensional Spherical Radon Transform R_m, acting on spaces of continuous functions as follows:

$$R_m : C\big(S^{n-1}\big) \to C\big(G(n,m)\big),$$
$$R_m(f)(E) = \int_{S^{n-1}\cap E} f(\theta)\, d\sigma_m(\theta),$$

where σ_m is the Haar probability measure on S^{m-1} (and we have identified S^{m-1} with $S^{n-1} \cap E$). Notice that for a star-body L in $\mathbb{R}^n$:

$$R_m\big(\rho_L^m\big)(E) = \mathrm{Vol}(L \cap E)/\mathrm{Vol}(D_m) \quad \forall E \in G(n,m).$$

The dual transform is defined on spaces of *signed* Borel measures $\mathcal{M}$ by:

$$R_m^* : \mathcal{M}\big(G(n,m)\big) \to \mathcal{M}\big(S^{n-1}\big),$$
$$\int_{S^{n-1}} f R_m^*(d\mu) = \int_{G(n,m)} R_m(f)\, d\mu \quad \forall f \in C\big(S^{n-1}\big), \tag{2.3}$$

and for a measure μ with continuous density g, the transform may be explicitly written in terms of g (see [41]):

$$R_m^* g(\theta) = \int\limits_{\theta \in E \in G(n,m)} g(E)\, d\nu_m(E),$$

where ν_m is the Haar probability measure on $G(n-1, m-1)$.

Definition. A star body K is said to be a *k-Busemann–Petty body* if $\rho_K^k = R_{n-k}^*(d\mu)$, where μ is a non-negative Borel measure on $G(n, n-k)$. We shall denote the class of such bodies by BP_k^n.

Choosing $k = 1$, for which $G(n, n-1)$ is isometric to S^{n-1}/Z_2 by mapping H to $S^{n-1} \cap H^\perp$, and noticing that R is equivalent to R_{n-1} under this map, we see that BP_1^n is exactly the class of intersection bodies.

To conclude this section, we mention that we always work with the radial metric topology on the space of star-bodies. Equivalently, we always work with the maximum norm on the space of continuous functions on S^{n-1}. So whenever an expression of the following form appears:

$$f = \int f_\alpha \, d\mu(\alpha),$$

where f and $\{f_\alpha\}$ are continuous functions on S^{n-1}, the convergence of the integral should be understood in the maximum norm.

3. Combining isotropic measures with type/cotype 2

In this section we introduce a very simple yet effective framework, which demonstrates how to utilize isotropic measures associated with a convex body K, to give bounds on $M_2(K)$ and $M_2^*(K)$ in terms of the type-2 and cotype-2 constants of X_K and X_K^*. As an immediate corollary, we revive a couple of known (yet partially forgotten) lemmas on John's maximal volume ellipsoid position, one of which will be used in Section 4 to improve the bound on the isotropic constant of quotients of L_q. Another immediate corollary of this framework is that L_K is always bounded by $T_2(X_K)$.

Recall that a Borel measure μ on $\mathbb{R}^n$ is said to be *isotropic* if:

$$\int\limits_{\mathbb{R}^n} \langle x, \theta \rangle^2 \, d\mu(x) = |\theta|^2 \quad \forall \theta \in \mathbb{R}^n.$$

This is easily seen to be equivalent to:

$$\int\limits_{\mathbb{R}^n} \langle x, \theta_1 \rangle \langle x, \theta_2 \rangle \, d\mu(x) = \langle \theta_1, \theta_2 \rangle \quad \forall \theta_1, \theta_2 \in \mathbb{R}^n.$$

The main point of this section is the following easy yet useful observation:

Lemma 3.1. *Let $v_i \in \mathbb{R}^n$ and $\lambda_i > 0$, for $i = 1, \dots, m$, be such that $\mu = \sum_{i=1}^m \lambda_i \delta_{v_i}$ is an isotropic measure. Let $\{g_i\}_{i=1}^m$ be a sequence of independent real-valued standard Gaussian r.v.'s, and define the r.v. Λ_μ as*:

$$\Lambda_\mu = \sum_{i=1}^m g_i \sqrt{\lambda_i} v_i. \tag{3.1}$$

Then Λ_μ is an n-dimensional standard Gaussian.

Proof. Obviously Λ_μ is a zero mean Gaussian r.v., so it remains to show that its correlation matrix is the identity. Indeed, from the independence of the g_i's and the isotropicity of μ:

$$E\big(\langle \Lambda_\mu, \theta_1\rangle\langle \Lambda_\mu, \theta_2\rangle\big) = E\left(\sum_{i,j=1}^m g_i g_j \sqrt{\lambda_i}\sqrt{\lambda_j}\langle v_i, \theta_1\rangle\langle v_j, \theta_2\rangle\right) = E\left(\sum_{i=1}^m g_i^2 \lambda_i \langle v_i, \theta_1\rangle\langle v_i, \theta_2\rangle\right)$$
$$= \sum_{i=1}^m \lambda_i \langle v_i, \theta_1\rangle\langle v_i, \theta_2\rangle = \langle \theta_1, \theta_2\rangle. \qquad \square$$

By taking the Fourier transform of the densities on both sides of (3.1), or by projecting them onto an arbitrary direction, we get:

$$\exp\big(-|x|^2\big) = \prod_{i=1}^m \big(\exp\big(-\langle x, v_i\rangle^2\big)\big)^{\lambda_i}.$$

This formulation, which is easy to check directly, has been used by many authors (e.g., [2,40]), mostly with connection to John's decomposition of the identity. The advantage of Lemma 3.1 is that we may work directly on the Gaussian r.v.'s and use type and cotype estimates on $\|\Lambda_\mu\|$, as summarized in the following proposition.

Proposition 3.2. *Let K denote a convex body and let μ be any finite, compactly supported, isotropic measure. Then*:

$$\frac{1}{C_2(X_K)}\left(\int \|x\|_K^2 \, d\mu(x)\right)^{1/2} \leqslant \sqrt{n} M_2(K) \leqslant T_2(X_K)\left(\int \|x\|_K^2 \, d\mu(x)\right)^{1/2}.$$

Proof. First, assume that μ is a discrete isotropic measure supported on finitely many points, of the form $\mu = \sum_{i=1}^m \lambda_i \delta_{v_i}$. Then by Lemma 3.1, denoting $\{g_i\}_{i=1}^m$ and $\{g_i'\}_{i=1}^n$ two sequences of independent standard Gaussian r.v.'s on a common probability space $(\Omega, d\omega)$, we have:

$$\int_\Omega \left\|\sum_{i=1}^m g_i(\omega)\sqrt{\lambda_i} v_i\right\|_K^2 d\omega = \int_\Omega \left\|\sum_{i=1}^n g_i'(\omega) e_i\right\|_K^2 d\omega = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \|x\|_K^2 e^{-|x|^2/2} \, dx$$
$$= \frac{\int_0^\infty e^{-r^2/2} r^{n+1} \, dr}{(2\pi)^{n/2}} \int_{S^{n-1}} \|\theta\|_K^2 \, d\theta = n M_2(K)^2,$$

where the last equality is a standard calculation (e.g., [37]). But on the other hand, using the type-2 condition on X_K, we see that the initial expression on the left is bounded from above by:

$$T_2(X_K)^2 \sum_{i=1}^{m} \left\| \sqrt{\lambda_i} v_i \right\|_K^2 = T_2(X_K)^2 \int \|x\|_K^2 \, d\mu(x).$$

Taking square root, the type-2 upper bound follows for a discrete measure μ, and the cotype-2 lower bound follows similarly.

When μ is a general isotropic measure, we approximate μ by a series of discrete (not necessarily isotropic) measures $\mu_\epsilon = \sum_{i=1}^{m_\epsilon} \lambda_i^\epsilon \delta_{v_i^\epsilon}$, where $\epsilon > 0$ is a parameter which will tend to 0. Since the set of discrete finitely supported measures is dense in the space of compactly supported Borel measures on $\mathbb{R}^n$ in the w^*-topology, we may choose μ_ϵ so that as linear functionals, the values of μ and μ_ϵ on the following $n(n+1)/2 + 1$ continuous functions are ϵ close:

$$\left| \int x_i x_j \, d\mu_\epsilon(x) - \delta_{i,j} \right| = \left| \int x_i x_j \, d\mu_\epsilon(x) - \int x_i x_j \, d\mu(x) \right| < \epsilon,$$

for all $1 \leqslant i \leqslant j \leqslant n$ and:

$$\left| \int \|x\|_K^2 \, d\mu_\epsilon(x) - \int \|x\|_K^2 \, d\mu(x) \right| < \epsilon. \tag{3.2}$$

We see that μ_ϵ is chosen to be almost isotropic, but we do not know how to guarantee this in general. Now, repeating the proof of Lemma 3.1, we see that Λ_{μ_ϵ} in (3.1) is a Gaussian r.v. whose correlation matrix is almost the identity (up to an l_∞ error of ϵ w.r.t. the standard basis). Therefore sending ϵ to 0, Λ_{μ_ϵ} tends to an n-dimensional standard Gaussian r.v. almost surely, implying that $\int \| \sum_{i=1}^{m_\epsilon} g_i(\omega) \sqrt{\lambda_i^\epsilon} v_i^\epsilon \|_K^2 \, d\omega$ tends to $\int \| \sum_{i=1}^{n} g_i'(\omega) e_i \|_K^2 \, d\omega = n M_2(K)^2$. Since by the discrete case:

$$\int \left\| \sum_{i=1}^{m_\epsilon} g_i(\omega) \sqrt{\lambda_i^\epsilon} v_i^\epsilon \right\|_K^2 d\omega \leqslant T_2(X_K)^2 \int \|x\|_K^2 \, d\mu_\epsilon(x),$$

and $\int \|x\|_K^2 \, d\mu_\epsilon(x)$ tends to $\int \|x\|_K^2 \, d\mu(x)$ by (3.2), this completes the proof. □

One of the most useful isotropic measures associated to the geometry of a convex body K, comes from John's decomposition of the identity, when K is put in John's maximal volume ellipsoid position: if D_n is the ellipsoid of maximal volume inside K, there exist contact points $\{v_i\}$ of D_n and K and positive scalars $\{\lambda_i\}$, such that $\mu_K = \sum_{i=1}^{m} \lambda_i \delta_{v_i}$ is isotropic. Since $|v_i| = 1$, it immediately follows that $\sum_{i=1}^{m} \lambda_i = n$. Applying Proposition 3.2 with the measure μ_K, first with K and then with K°, we immediately have as a corollary the following two known inequalities. The first essentially appears in [33], and in [37] with a worse constant, and the second appears in [14]. Both in [14] and in [33], the proofs rely on operator theory, whereas in our approach the elementary geometric flavor is retained, and both proofs are unified into a single framework.

Corollary 3.3. *Let K be a convex body in John's maximal volume ellipsoid position. Then*:

$$\frac{M_2(K)}{b(K)} \geqslant \frac{1}{C_2(X_K)}, \qquad M_2^*(K)b(K) \leqslant T_2(X_K^*).$$

Proof. The $b(K)$ terms are simply normalizations to the case that D_n is indeed the ellipsoid of maximal volume inside K. It remains to notice that $|v_i| = \|v_i\|_K = \|v_i\|_K^* = 1$, as contact points between D_n and K. Since $\sum_{i=1}^m \lambda_i = n$, we have:

$$\left(\sum_{i=1}^m \lambda_i \left(\|v_i\|_K\right)^2\right)^{1/2} = \left(\sum_{i=1}^m \lambda_i \left(\|v_i\|_K^*\right)^2\right)^{1/2} = \sqrt{n}.$$

The assertions now clearly follow from Proposition 3.2. □

Remark 3.4. The other two inequalities:

$$\frac{M_2(K)}{b(K)} \leqslant T_2(X_K), \qquad M_2^*(K)b(K) \geqslant \frac{1}{C_2(X_K^*)},$$

are trivial and loose. The first follows from $M_2(K) \leqslant b(K)$, and the second from Urysohn's inequality:

$$M_2^*(K) \geqslant M^*(K) \geqslant \left(\frac{\mathrm{Vol}(K)}{\mathrm{Vol}(D_n)}\right)^{1/n} \geqslant \frac{1}{b(K)}.$$

By duality, we have:

Corollary 3.5. *Let K be a convex body in Lowner's minimal volume outer ellipsoid position. Then*:

$$\frac{M_2^*(K)}{a(K)} \geqslant \frac{1}{C_2(X_K^*)}, \qquad M_2(K)a(K) \leqslant T_2(X_K).$$

Corollary 3.5 shows that having type 2 implies having finite outer volume-ratio (this will be evident in the proof of the next theorem), so it is not surprising that we get the following useful bound on the isotropic constant, when placing the body in Lowner's outer ellipsoid position. What is a little more surprising, is that we manage to get the same bound by putting the body in the isotropic position, and directly applying Proposition 3.2 on the (properly normalized) uniform measure on K. The latter part may also be shown to follow from Theorem 1.4 in [12].

Theorem 3.6. *Let K be a convex body. Then*:

$$L_K \leqslant C \inf\left\{ T_2(X_L)\left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(K)}\right)^{1/n} \;\middle|\; K \subset L \text{ is a convex body}\right\}. \tag{3.3}$$

In addition, if $\mathrm{Vol}(K) = 1$ *and K is in Lowner's minimal volume outer ellipsoid position or in isotropic position, then*:

$$L_K \leqslant C \frac{T_2(X_K)}{\sqrt{n}M_2(K)}.$$

Proof. Since (3.3) is invariant under homothety, we may assume that $\mathrm{Vol}(K)=1$. Now let L be any convex body containing K, and assume that $T(L)$ is in Lowner's minimal volume outer ellipsoid position, where $T\in SL(n)$. By Corollary 3.5 and Jensen's inequality (as in (1.3)):

$$a\big(T(L)\big)\leqslant \frac{T_2(X_L)}{M_2(T(L))}\leqslant C\sqrt{n}\left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(K)}\right)^{1/n}T_2(X_L).$$

Using the characterization of L_K mentioned in the Introduction, we immediately have:

$$L_K^2\leqslant \frac{1}{n}\int\limits_{T(K)}|x|^2\,dx\leqslant \frac{1}{n}a\big(T(L)\big)^2\leqslant \left(C\left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(K)}\right)^{1/n}T_2(X_L)\right)^2.$$

Evidently, the above argument also proves the second part of the theorem when K is in Lowner's minimal volume outer ellipsoid position. When K is in isotropic position, we apply Proposition 3.2 to the isotropic measure $d\mu=1/L_K^2\chi_K\,dx$, yielding:

$$\sqrt{n}M_2(K)\leqslant T_2(X_K)\frac{1}{L_K}\left(\int\limits_K\|x\|_K^2\right)^{1/2}\leqslant\frac{T_2(X_K)}{L_K}.$$

The assertion therefore follows (even without a constant). □

Remark 3.7. For completeness, it is worthwhile to mention that a different form of Theorem 3.6 may be derived from a deeper result of Milman and Pisier, who showed in [36] that the volume-ratio of K is bounded from above by $CC_2(X_K)\log C_2(X_K)$ (this is an improvement over the initial bound showed in [10]). Using another deep result, the reverse Blaschke–Santalo inequality ([10], see (4.11)), this implies that the outer volume-ratio of K is bounded from above by $C'C_2(X_K^*)\log C_2(X_K^*)$, so the same argument as above gives:

$$L_K\leqslant C\inf\left\{C_2\big(X_L^*\big)\log C_2\big(X_L^*\big)\left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(K)}\right)^{1/n}\;\Big|\;K\subset L\text{ is a convex body}\right\}.$$

Since $C_2(X_L^*)\leqslant T_2(X_L)\leqslant C_2(X_L^*)\|\mathrm{Rad}(X_L)\|$, where Rad denotes the Rademacher projection (see [37]), we see that the two forms are very similar, but elementary examples show that neither form out-performs the other.

Since it is well known (e.g., [37]) that subspaces of L_p, for $p\geqslant 2$, have a type-2 constant of the order of $\sqrt{p}$ (this is a consequence of Kahane's inequality), we immediately have the following corollary of Theorem 3.6.

Corollary 3.8.

$$L_K\leqslant C\inf\left\{\sqrt{p}\left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(K)}\right)^{1/n}\;\Big|\;K\subset L,\ L\in SL_p^n,\ p\geqslant 2\right\}.$$

We conclude this section by giving another application of Proposition 3.2. In principle, it seems useful to apply it on any isotropic measure which is naturally associated to a convex body in certain special positions. Fortunately, in [16], Giannopoulos and Milman have derived a framework to generate such measures, by considering bodies in minimum quermassintegral positions. We will only give the following application for the minimal surface-area position, i.e. the position for which $\mathrm{Vol}(\partial T(K))$ is minimal for all $T \in SL(n)$, which was characterized by Petty in [39]. Recall that σ_K, the area measure of K is defined on S^{n-1} as:

$$\sigma_K(A) = \nu\bigl(\bigl\{x \in \partial K \bigm| n_K(x) \in A\bigr\}\bigr),$$

where $n_K(x)$ denotes an outer normal to K at x and ν is the $(n-1)$-dimensional surface measure on K.

Proposition 3.9. *Let K be a convex body in minimal surface-area position. Then*:

$$\frac{1}{C_2(X_K)} \leqslant \frac{M_2(K)}{(1/\mathrm{Vol}(\partial K)\int_{S^{n-1}} \|x\|_K^2\, d\sigma_K(x))^{1/2}} \leqslant T_2(X_K).$$

Proof. It was shown in [39] that K is in minimal surface-area position iff $n/\mathrm{Vol}(\partial K)\, d\sigma_K$ is isotropic. Applying Proposition 3.2 with σ_K yields the claimed inequalities. □

4. Sections and quotients of L_p

As seen in the previous section, it is actually pretty straightforward to obtain a bound on the isotropic constant of any convex body K for which we have control over $T_2(X_K)$, since in that case K has bounded outer volume-ratio. In particular, this applies for sections of L_p, at least for $p \geqslant 2$. In this section, we introduce a new technique involving dual mixed-volumes, which is well adapted to deal specifically with integral representations of $\|\cdot\|^t$. This is well suited for dealing with sections of L_p, since by a classical result of P. Lévy [27], $L \in SL_p^n$ for $p \geqslant 1$ iff there exists a non-negative Borel measure μ_L on S^{n-1} such that:

$$\|x\|_L^p = \int\limits_{S^{n-1}} \bigl|\langle x, \theta\rangle\bigr|^p\, d\mu_L(\theta), \tag{4.1}$$

for all $x \in \mathbb{R}^n$. This characterization extends to any $p > 0$, and it will enable us to extend the bound on L_K to the case $K \in SL_p^n$ for all $p > 0$. As we shall see, for a general convex body K, it is not the *volume-ratio* between $L \in SL_p^n$ containing K and K which matters, but rather some other natural parameter. Moreover, our new technique will enable us to pass to the dual, and recover Junge's bound on the isotropic constant of quotients of L_q. In Section 5, we continue to apply our technique to bound the isotropic constant of convex bodies contained in k-Busemann–Petty bodies.

Theorem 4.1. *Let K be a centrally symmetric convex body in isotropic position, and let D be a Euclidean ball normalized so that* $\mathrm{Vol}(D) = \mathrm{Vol}(K)$. *Then for any $p > 0$ and any $L \in SL_p^n$*:

$$\frac{C_1}{\sqrt{p_0}} \leqslant L_K \Big/ \left(\frac{\widetilde{V}_{-p}(L,K)}{\widetilde{V}_{-p}(L,D)}\right)^{1/p} \leqslant C_2\sqrt{p_0}, \tag{4.2}$$

where $p_0 = \max(1, \min(p,n))$.

Remark 4.2. By taking the limit in (4.1) as $p \to 0+$, we may define SL_0^n to be the class of n-dimensional star-bodies L for which:

$$\|x\|_L = \exp\left(\int_{S^{n-1}} \log\left|\langle x,\theta\rangle\right| d\mu_L(\theta) + C\right),$$

for some Borel probability measure μ_L and constant C and all $x \in \mathbb{R}^n$. In that case, Theorem 4.1 holds true for $p = 0$ as well (by passing to the limit), if we replace the expressions of the form $\widetilde{V}_{-p}(L_1,L_2)^{1/p}$ appearing in (4.2), by the limit as $p \to 0+$ assuming $\mathrm{Vol}(L_2) = 1$, namely $\exp(1/n\int_{S^{n-1}} \log(\rho_{L_2}(x)/\rho_{L_1}(x))\rho_{L_2}(x)^n\,dx)$.

Proof of Theorem 4.1. Let μ_L denote the Borel measure on S^{n-1} from (4.1) corresponding to L. Then for any star-body G:

$$\begin{aligned}
\widetilde{V}_{-p}(L,G) &= \frac{1}{n}\int_{S^{n-1}} \|x\|_L^p \|x\|_G^{-(n+p)}\,dx \\
&= \frac{1}{n}\int_{S^{n-1}}\int_{S^{n-1}} \left|\langle x,\theta\rangle\right|^p d\mu_L(\theta)\|x\|_G^{-(n+p)}\,dx \\
&= \frac{1}{n}\int_{S^{n-1}} d\mu_L(\theta)\int_{S^{n-1}} \left|\langle x,\theta\rangle\right|^p \|x\|_G^{-(n+p)}\,dx \\
&= \frac{n+p}{n}\int_{S^{n-1}} d\mu_L(\theta)\int_G \left|\langle x,\theta\rangle\right|^p\,dx.
\end{aligned} \tag{4.3}$$

Let us evaluate the expression $\int_G |\langle x,\theta\rangle|^p\,dx$. If G is of volume 1 and $p \geqslant 1$, then by Jensen's inequality:

$$\int_G \left|\langle x,\theta\rangle\right| dx \leqslant \left(\int_G \left|\langle x,\theta\rangle\right|^p dx\right)^{1/p} \quad \forall p \geqslant 1. \tag{4.4}$$

If G is in addition convex, then by a well-known consequence of a lemma by C. Borell [7], it follows that the linear functional $\langle\cdot,\theta\rangle$ has a ψ_1-type behaviour on G, and therefore:

$$\left(\int_G \left|\langle x,\theta\rangle\right|^p dx\right)^{1/p} \leqslant Cp\int_G \left|\langle x,\theta\rangle\right| dx \quad \forall p \geqslant 1. \tag{4.5}$$

If in addition $p \geqslant n$, it is well known that (e.g., [38, Lemma 4.1]):

$$\left(\int_G \left|\langle x,\theta\rangle\right|^p dx\right)^{1/p} \simeq \|\theta\|_G^* \quad \forall p \geqslant n. \tag{4.6}$$

Finally, if G is convex, of volume 1 and $0 < p < 1$, then it follows from the estimates in Corollaries 2.5 and 2.7 in [35] that:

$$\left(\int_G \left|\langle x,\theta\rangle\right|^p dx\right)^{1/p} \simeq \int_G \left|\langle x,\theta\rangle\right| dx \quad \forall p \in (0,1). \tag{4.7}$$

The expression in (4.2) is invariant under simultaneous homothety of K and D, so we may assume that $\mathrm{Vol}(K) = \mathrm{Vol}(D) = 1$. Since K is in isotropic position, we have $\int_K \langle x,\theta\rangle^2 dx = L_K^2$ for all $\theta \in S^{n-1}$, and by (4.4)–(4.7) it follows that for all $\theta \in S^{n-1}$:

$$A \leqslant \left(\int_K \left|\langle x,\theta\rangle\right|^p dx\right)^{1/p} \Big/ L_K \leqslant B p_0 \quad \forall p > 0. \tag{4.8}$$

It remains to notice that for a Euclidean ball D of volume 1, a straightforward computation (in the case $1 \leqslant p \leqslant n$) together with (4.6) and (4.7), gives that for all $\theta \in S^{n-1}$:

$$\left(\int_D \left|\langle x,\theta\rangle\right|^p dx\right)^{1/p} \simeq \sqrt{p_0} \quad \forall p > 0. \tag{4.9}$$

By (4.3), we have:

$$\left(\frac{\widetilde{V}_{-p}(L,K)}{\widetilde{V}_{-p}(L,D)}\right)^{1/p} = \left(\frac{\int_{S^{n-1}} d\mu_L(\theta) \int_K |\langle x,\theta\rangle|^p dx}{\int_{S^{n-1}} d\mu_L(\theta) \int_D |\langle x,\theta\rangle|^p dx}\right)^{1/p}.$$

Since $\mu_L \geqslant 0$, using (4.8) and (4.9), we get the required (4.2):

$$\frac{1}{C_2\sqrt{p_0}} \leqslant \left(\frac{\widetilde{V}_{-p}(L,K)}{\widetilde{V}_{-p}(L,D)}\right)^{1/p} \Big/ L_K \leqslant \frac{\sqrt{p_0}}{C_1}. \qquad \square$$

Remark 4.3. Notice that for $0 \leqslant p < 1$, the unit-ball of a subspace of L_p is no longer necessarily a convex body. We will see more examples where L is a non-convex star-body later on. In fact, using the results in [21] of Guédon, it is possible to extend Theorem 4.1 to $p > -1$, but then the constants C_1 and C_2 will depend on p. We do not proceed in this direction, because we are able to show in Section 5 that Theorem 4.1 is also valid for $p = -1$ (then SL_p^n is replaced by the class of intersection-bodies), and we are able to generalize this to k-Busemann–Petty bodies.

We can now extend Corollary 3.8 to the following more general result.

Theorem 4.4. *Let K be a centrally symmetric convex body in isotropic position with* $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$. *Then*:

$$L_K \leqslant C \inf\left\{ \frac{\sqrt{p_0}}{M_p(L)} \;\Big|\; K \subset L,\ L \in SL_p^n,\ p \geqslant 0 \right\},$$

where $p_0 = \max(1, \min(p, n))$.

Proof. If $K \subset L$, then obviously $\widetilde{V}_{-p}(L, K) \leqslant \widetilde{V}_{-p}(K, K) = \mathrm{Vol}(K)$. Applying Theorem 4.1 with $\mathrm{Vol}(D) = \mathrm{Vol}(K) = \mathrm{Vol}(D_n)$, (4.2) implies:

$$L_K \leqslant C_2 \sqrt{p_0} \left(\frac{\mathrm{Vol}(D_n)}{\frac{1}{n} \int_{S^{n-1}} \rho_L(x)^{-p}\, dx} \right)^{1/p} = C_2 \frac{\sqrt{p_0}}{M_p(L)}. \qquad \square \tag{4.10}$$

Using Jensen's inequality (1.3) and homogeneity, we immediately have the following corollary, which unifies the bounds on L_K for SL_p^n of Ball (the case $1 \leqslant p \leqslant 2$) and Junge (the case $p \geqslant 2$), and extends their results to $p \geqslant 0$:

Corollary 4.5. *For any centrally symmetric convex body K*:

$$L_K \leqslant C \inf\left\{ \sqrt{p_0} \left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(K)} \right)^{1/n} \;\Big|\; K \subset L,\ L \in SL_p^n,\ p \geqslant 0 \right\},$$

where $p_0 = \max(1, \min(p, n))$.

Remark 4.6. Notice that the proof of Theorem 4.1 does not use the assumption that the body D is a Euclidean ball: the only property used is the one in (4.9). In fact, for the right-hand inequality in (4.2), D may be chosen as any ψ_2-body in isotropic position. Recall that D is called a ψ_2-body (with constant $A > 1$), if for all $p \geqslant 1$:

$$\left(\int_D \left|\langle x, \theta \rangle\right|^p dx \right)^{1/p} \leqslant A\sqrt{p} \left(\int_D \left|\langle x, \theta \rangle\right|^2 dx \right)^{1/2} \quad \forall \theta \in S^{n-1}.$$

Bourgain has shown in [9] that if D is a ψ_2-body then $L_D \leqslant CA \log A$. Therefore, if D is a ψ_2-body of volume 1 in isotropic position, (4.9) may be replaced by:

$$\left(\int_D \left|\langle x, \theta \rangle\right|^p dx \right)^{1/p} \leqslant A^2 \log A \sqrt{p} \quad \forall \theta \in S^{n-1},\ \forall p \geqslant 1.$$

(4.10) then reads (when $\mathrm{Vol}(K) = \mathrm{Vol}(D) = \mathrm{Vol}(D_n)$):

$$L_K \leqslant C(A) \sqrt{p_0} \left(\frac{\mathrm{Vol}(D_n)}{\widetilde{V}_{-p}(L, D)} \right)^{1/p} = C(A) \sqrt{p_0} \left(\int_{S^{n-1}} \|x\|_L^p \rho_D(x)^{n+p}\, d\sigma(x) \right)^{-1/p}.$$

By Bourgain's result, if all linear functionals are Ψ_2, then L_K is bounded. Ironically, it follows from the proof of Theorem 4.1 that if all linear functionals have "bad" ψ_2 behaviour, e.g.,

$$\left(\int_K \left|\langle x,\theta\rangle\right|^q dx\right)^{1/q} \geqslant C\sqrt{q}\int_K \left|\langle x,\theta\rangle\right| dx \quad \forall\theta\in S^{n-1},$$

for a certain $q \geqslant 1$, then the bound on L_K improves ($L_K \leqslant C(\mathrm{Vol}(L)/\mathrm{Vol}(K))^{1/n}$ for all $L \in SL_q^n$ containing K, in the example above). Perhaps this may be used to our advantage?

We now turn to reproduce Junge's bound on L_K for quotients of L_q. As mentioned in the Introduction, for $1 < q \leqslant 2$, Junge's result is more general than ours and applies to all *subspaces* of quotients of L_q. Nevertheless, our proof provides a (formally) stronger bound, applies to the entire range $1 < q \leqslant \infty$, and retains the problem's geometric nature, avoiding unnecessary tools from Operator Theory. In addition, although derived independently, our proof is very similar to Bourgain's proof that $L_K \leqslant Cn^{1/4}\log(1+n)$, and the latter may be thought of as an extremal case of our proof, where our argument breaks down.

Theorem 4.7. *Let K be a centrally symmetric convex body in isotropic position with* $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$. *Then*:

$$L_K \leqslant C\inf\left\{\sqrt{p_0}M_p^*\big(T(L)\big) \;\middle|\; \begin{array}{l} K\subset L,\ L\in QL_q^n,\ T\in SL(n), \\ 1\leqslant q\leqslant\infty,\ 1/p+1/q=1 \end{array}\right\},$$

where $p_0 = \min(p,n)$ and $p = q^*$ is the conjugate exponent to q.

We postpone the proof of Theorem 4.7 for later. In order to see why this theorem implies Junge's bound for quotients of L_q, we will need the following lemma:

Lemma 4.8. *Let K be a convex body with* $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$.

(1) *If $K \in SL_p^n$ for $1 \leqslant p \leqslant \infty$, then there exists a position of K for which $M_p(K) \leqslant C\sqrt{p_0}$, where $p_0 = \min(p,n)$.*
(2) *If $K \in QL_q^n$ for $1 \leqslant q \leqslant \infty$, then there exists a position of K for which $M_p^*(K) \leqslant C\sqrt{p_0}$, for $p = q^* = q/(q-1)$ and p_0 as above.*

Applying the second part of the lemma to the body L from Theorem 4.7 and using homogeneity, we immediately have:

Corollary 4.9. *For any centrally symmetric convex body K*:

$$L_K \leqslant C\inf\left\{p_0\left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(K)}\right)^{1/n} \;\middle|\; \begin{array}{l} K\subset L,\ L\in QL_q^n, \\ 1\leqslant q\leqslant\infty,\ 1/p+1/q=1 \end{array}\right\},$$

where $p_0 = \min(p,n)$.

Proof of Lemma 4.8. We will prove part (1). Part (2) then follows easily by duality, using the reverse Blaschke–Santalo inequality [10]:

$$\left(\frac{\operatorname{Vol}(K)}{\operatorname{Vol}(D_n)}\right)^{1/n}\left(\frac{\operatorname{Vol}(K^\circ)}{\operatorname{Vol}(D_n)}\right)^{1/n} \geqslant c, \tag{4.11}$$

to ensure that the volume of K° is not too small.

The case $1 \leqslant p \leqslant 2$ is straightforward, since for this range it is well known that sections of L_p have finite volume-ratio (for instance, because they have cotype 2 and using [10], or by [5]). Therefore, in John's maximal volume ellipsoid position, $M_p(K) \leqslant b(K) \leqslant C$. We remark that it remains to prove the lemma for $2 \leqslant p \leqslant n$, since it is known that $M_p(K) \simeq M_n(K) \simeq b(K)$ for $p > n$ (e.g., [29]).

We will present three different proofs for the case $2 \leqslant p \leqslant n$, placing the body K in three different positions. We note that the first two proofs actually prove a stronger statement: for any $K \in SL_p^n$ there exists a position in which $M_p(K) \leqslant C\sqrt{p}/a(K)$. Since this formulation is volume free, we do not really need the reverse Blaschke–Santalo inequality to prove the dual second part of the lemma (for the range $1 \leqslant q \leqslant 2$). The third proof is an elementary consequence of Theorem 4.4, and appears also in Corollary 6.3.

(1) If $2 \leqslant p \leqslant n$, then $T_2(X_K) \leqslant C\sqrt{p}$ (by Kahane's inequality), so by Corollary 3.5, if K is in *Lowner's minimal volume outer ellipsoid position*, then $M_2(K)a(K) \leqslant C\sqrt{p}$. Notice that in Lowner's position, $b(K) \leqslant \sqrt{n}/a(K)$. Since $\operatorname{Vol}(K) = \operatorname{Vol}(D_n)$, we obviously have $a(K) \geqslant 1$, implying that $M_2(K) \leqslant C\sqrt{p}$ and $b(K) \leqslant \sqrt{n}$. We now use a known result from [29], stating that $M_p(K) \simeq \max(M_2(K), b(K)\sqrt{p}/\sqrt{n})$, which under our conditions implies $M_p(K) \leqslant C\sqrt{p}$.

(2) By approximation, we may assume that K is a section of l_p^m, for some large enough m. We will put K in the *Lewis position* [28], as used in [5]. In this position, there exists a sequence of m unit vectors $\{u_i\}$ and positive scalars $\{c_i\}$, such that $\|x\|_K^p = \sum_{i=1}^m c_i |\langle x, u_i\rangle|^p$ and such that $\mu = \sum_{i=1}^m c_i \delta_{u_i}$ is an isotropic measure (see Section 3). In particular, $\sum_{i=1}^m c_i = n$. An elementary computation shows that for $2 \leqslant p \leqslant n$:

$$M_p(K) = \left(\sum_{i=1}^m c_i \int_{S^{n-1}} \left|\langle\theta, u_1\rangle\right|^p d\sigma(\theta)\right)^{1/p} \simeq \left(\sum_{i=1}^m c_i\right)^{1/p} \frac{\sqrt{p}}{\sqrt{n}} = \frac{\sqrt{p}}{n^{1/2-1/p}}.$$

But in this position, Hölder's inequality shows that:

$$|x|^2 = \sum_{i=1}^m c_i \left|\langle x, u_i\rangle\right|^2 \leqslant \left(\sum_{i=1}^m c_i\right)^{1-2/p} \left(\sum_{i=1}^m c_i \left|\langle x, u_i\rangle\right|^p\right)^{2/p} = n^{1-2/p}\|x\|_K^2,$$

and therefore $a(K) \leqslant n^{1/2-1/p}$. It follows that $M_p(K) \leqslant C\sqrt{p}/a(K)$, as required.

(3) Put the body K in *isotropic position*, and apply Theorem 4.4 with $L = K$. Using the well-known fact that L_K is always bounded from below by a universal constant (e.g., [35]), we immediately have $M_p(K) \leqslant C\sqrt{p_0}(\operatorname{Vol}(K)/\operatorname{Vol}(D_n))^{1/n}$, and this is valid for all $p \geqslant 0$, with $p_0 = \max(1, \min(p, n))$. $\square$

Proof of Theorem 4.7. Let K be in isotropic position and assume $\mathrm{Vol}(K) = 1$. Fix $q > 1$ and let $L \in QL_q^n$ contain K. By duality, L°, the polar body to L, is a section of L_p, and so is $T(L^\circ)$ for any $T \in SL(n)$. Applying Theorem 4.1, the left- (!) hand side of (4.2) gives:

$$L_K \sqrt{p_0}/C_1 \geqslant \left(\frac{\widetilde{V}_{-p}(T(L^\circ), K)}{\widetilde{V}_{-p}(T(L^\circ), D)}\right)^{1/p} \geqslant \left(\frac{\widetilde{V}_{-p}(T(K^\circ), K)}{\widetilde{V}_{-p}(T(L^\circ), D)}\right)^{1/p}, \tag{4.12}$$

for D the Euclidean ball of volume 1. Evaluating the numerator on the right using the trivial $\|x\|_{T(K^\circ)}\|x\|_K \geqslant |\langle T^{-1}(x), x\rangle|$, we have that for any positive-definite $T \in SL(n)$:

$$\begin{aligned}
\big(\widetilde{V}_{-p}\big(T(K^\circ), K\big)\big)^{1/p} &= \left(\frac{1}{n}\int_{S^{n-1}} \|x\|^p_{T(K^\circ)}\|x\|_K^{-(n+p)}\,dx\right)^{1/p} \\
&\geqslant \left(\frac{1}{n}\int_{S^{n-1}} \big|\big\langle T^{-1}(x), x\big\rangle\big|^p \|x\|_K^{-(n+2p)}\,dx\right)^{1/p} \\
&= \left(\frac{n+2p}{n}\int_K \big|\big\langle T^{-1}(x), x\big\rangle\big|^p\,dx\right)^{1/p} \geqslant \int_K \big\langle T^{-1}(x), x\big\rangle\,dx \\
&= tr\big(T^{-1}\big)L_K^2 \geqslant \det\big(T^{-1}\big)^{1/n} n L_K^2 = n L_K^2,
\end{aligned}$$

where we have used Jensen's inequality, the fact that $\int_K x_i x_j\,dx = L_K^2 \delta_{i,j}$, and the arithmetic–geometric means inequality (since T is positive-definite). Together with (4.12), and cancelling out one L_K term, this gives:

$$L_K \leqslant \frac{\sqrt{p_0}}{C_1 n}\big(\widetilde{V}_{-p}\big(T(L^\circ), D\big)\big)^{1/p} = \frac{\sqrt{p_0}}{C_1 n}\mathrm{Vol}(D_n)^{-1/n} M_p\big(T(L^\circ)\big) \simeq \frac{\sqrt{p_0}}{\sqrt{n}} M_p^*\big(\big(T^{-1}\big)^*(L)\big),$$

for any $T \in SL(n)$ (since it can be factorized into a composition of a rotation and a positive-definite transformation, and M_p is invariant to rotations). Changing normalization from $\mathrm{Vol}(K) = 1$ to $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$, we have the desired:

$$L_K \leqslant C\sqrt{p_0} M_p^*\big(T(L)\big). \qquad \square$$

Remark 4.10. As already mentioned, the proof of Theorem 4.7 clearly resembles Bourgain's proof that $L_K \leqslant C n^{1/4} \log(1+n)$. In this respect, we mention that instead of using $\sqrt{p_0}$ on the left-hand side of (4.2) or $\sqrt{p}$ on the left-hand side of (4.12), it is easy to check that one may use A, if K is a Ψ_2 body with constant A (as defined in Remark 4.6). This implies that whenever $A < \sqrt{p}$, we get a better bound on L_K. Bourgain has shown that in the general case, one may always assume that $A \leqslant n^{1/4}$, but this does not seem to help us in our context.

To conclude this section, we mention that for a *general* convex body K (not necessarily a section of L_p), representations other than (4.1) of $\|\cdot\|_K$ as a spherical convolution of a kernel with a non-negative Borel measure on S^{n-1} are known. Repeating the relevant parts of the proof of Theorem 4.1 with $L = K$, it may be possible to bound some natural parameter of the body K other than L_K.

5. k-Busemann–Petty bodies

An analogous result to Theorem 4.1 for k-Busemann–Petty bodies is the following:

Theorem 5.1. *Let K be a centrally symmetric convex body in isotropic position, and let D be a Euclidean ball normalized so that* $\mathrm{Vol}(D) = \mathrm{Vol}(K)$. *Then for any integer $k = 1, \dots, n-1$ and any $L \in BP_k^n$:*

$$C_1 \leqslant L_K \Bigg/ \left(\frac{\widetilde{V}_k(L, D)}{\widetilde{V}_k(L, K)} \right)^{1/k} \leqslant C_2 \mathcal{L}_k. \tag{5.1}$$

Proof. By definition, if $L \in BP_k^n$ there exists a Borel measure μ_L on $G(n, n-k)$ such that:

$$\rho_L^k = R_{n-k}^*(d\mu_L). \tag{5.2}$$

Therefore, for any star-body G:

$$\begin{aligned} \widetilde{V}_k(L, G) &= \frac{1}{n} \int\limits_{S^{n-1}} \rho_L(x)^k \rho_G(x)^{n-k}\, dx \\ &= \mathrm{Vol}(D_n) \int\limits_{S^{n-1}} R_{n-k}^*(d\mu_L)(x) \rho_G(x)^{n-k}\, d\sigma(x) \\ &= \mathrm{Vol}(D_n) \int\limits_{G(n,n-k)} R_{n-k}\big(\rho_G^{n-k}\big)(E)\, d\mu_L(E) \\ &= \frac{\mathrm{Vol}(D_n)}{\mathrm{Vol}(D_{n-k})} \int\limits_{G(n,n-k)} \mathrm{Vol}(G \cap E)\, d\mu_L(E). \end{aligned} \tag{5.3}$$

The expression in (5.1) is invariant under simultaneous homothety of K and D, so we may assume that $\mathrm{Vol}(K) = \mathrm{Vol}(D) = 1$. It is known [1,3,35] that for a convex K in isotropic position and volume 1:

$$A \leqslant \mathrm{Vol}(K \cap E)^{1/k} L_K \leqslant B\mathcal{L}_k \quad \forall E \in G(n, n-k). \tag{5.4}$$

The proof of (5.4) is based on the fact that the function $f(x) = \mathrm{Vol}(K \cap \{E + x\})$ on $E^\perp$ is log-concave and isotropic, and its isotropic constant is $L_f = f(0)^{1/k} L_K$. It was shown in [1] that an isotropic log-concave function f on $\mathbb{R}^k$ satisfies $A \leqslant L_f \leqslant B\mathcal{L}_k$, implying (5.4).

It remains to notice that for a Euclidean ball D of volume 1, a straightforward computation shows that for any $k = 1, \dots, n-1$:

$$\mathrm{Vol}(D \cap E)^{1/k} \simeq 1 \quad \forall E \in G(n, n-k). \tag{5.5}$$

By (5.3), we have:

$$\left(\frac{\widetilde{V}_k(L, K)}{\widetilde{V}_k(L, D)} \right)^{1/k} = \left(\frac{\int_{G(n,n-k)} \mathrm{Vol}(K \cap E)\, d\mu_L(E)}{\int_{G(n,n-k)} \mathrm{Vol}(D \cap E)\, d\mu_L(E)} \right)^{1/k}.$$

Since $\mu_L \geqslant 0$, using (5.4) and (5.5), we get the required (5.1):

$$C_1 \leqslant \left(\frac{\widetilde{V}_k(L,K)}{\widetilde{V}_k(L,D)}\right)^{1/k} L_K \leqslant C_2 \mathcal{L}_k. \qquad \square$$

Remark 5.2. It is known [25] that the representation (5.2) exists for any star-body L whose radial function ρ_L is infinitely times differentiable on S^{n-1}, if we allow $\mu_L = \mu_{L,k}$ to be a *signed* measure on $G(n, n-k)$. Using $L = K$ for example, and repeating the argument in the proof of Theorem 5.1, we get that:

$$L_K \leqslant C\left(\frac{\int_{G(n,n-k)} |d\mu_{K,k}|(E)}{\int_{G(n,n-k)} d\mu_{K,k}(E)}\right)^{1/k} \mathcal{L}_k,$$

so it remains to evaluate the above ratio. Unfortunately, this approach does not seem promising, since for a general smooth function f on S^{n-1}, for which the representation $f = R^*_{n-k}(d\mu)$ is known to exist, it is easy to show that this ratio may be arbitrarily large for $k = 1$ and a fixed value of n.

We can now prove analogous results to Theorem 4.4 and Corollary 4.5.

Theorem 5.3. *Let K be a centrally symmetric convex body in isotropic position with* $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$. *Then*:

$$L_K \leqslant C \inf\left\{\mathcal{L}_k \widetilde{M}_k(L) \;\middle|\; K \subset L,\ L \in BP^n_k,\ k = 1, \dots, n-1\right\}.$$

Proof. If $K \subset L$, then obviously $\widetilde{V}_k(L,K) \geqslant \widetilde{V}_k(K,K) = \mathrm{Vol}(K)$. Applying Theorem 5.1 with $\mathrm{Vol}(D) = \mathrm{Vol}(K) = \mathrm{Vol}(D_n)$, (5.1) implies:

$$L_K \leqslant C_2 \mathcal{L}_k \left(\frac{\frac{1}{n}\int_{S^{n-1}} \rho_L(x)^k\, dx}{\mathrm{Vol}(D_n)}\right)^{1/k} = C_2 \mathcal{L}_k \widetilde{M}_k(L). \qquad \square \tag{5.6}$$

Using Jensen's inequality (1.3) and homogeneity, we immediately have the following corollary, which generalizes Ball's bound on L_K for SL^n_p with $1 \leqslant p \leqslant 2$, since in that range $SL^n_p \subset BP^n_k$ for $k = 1, \dots, n-1$ (as explained in the introduction):

Corollary 5.4. *For any centrally symmetric convex body K*:

$$L_K \leqslant C \inf\left\{\mathcal{L}_k \left(\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(K)}\right)^{1/n} \;\middle|\; K \subset L,\ L \in BP^n_k,\ k = 1, \dots, n-1\right\}.$$

Remark 5.5. As before, the proof of Theorem 5.1 does not utilize the assumption that D is a Euclidean ball. The only property of D used is the one stated in (5.5). By a result of Junge [23], this is satisfied by any 1-unconditional convex body in isotropic position. (5.6) then reads (when $\mathrm{Vol}(K) = \mathrm{Vol}(D) = \mathrm{Vol}(D_n)$):

$$L_K \leqslant C_2 \mathcal{L}_k \left(\frac{\widetilde{V}_k(L,D)}{\mathrm{Vol}(D_n)}\right)^{1/k} = C_2 \mathcal{L}_k \left(\int_{S^{n-1}} \rho_L(x)^k \rho_D(x)^{n-k}\, d\sigma(x)\right)^{1/k}.$$

As in the previous section, we may prove dual counterparts to Theorem 5.3 and Corollary 5.4. Before proceeding, we will need the following useful lemma:

Lemma 5.6. *For any compact set $A \subset \mathbb{R}^n$ and $m = 1, \dots, n$:*

$$\int_{G(n,m)} \mathrm{Vol}(A \cap E)\, d\nu(E) \leqslant \inf_{T \in SL(n)} \sup_{E \in G(n,m)} \mathrm{Vol}\big(T(A) \cap E\big),$$

where ν is the Haar probability measure on $G(n,m)$.

Proof. Notice that for any compact set $A \subset \mathbb{R}^n$ and $T \in SL(n)$:

$$\mathrm{Vol}(A \cap E) = D_T(E)\, \mathrm{Vol}\big(T(A) \cap T(E)\big),$$

where the Jacobian $D_T(E)$ does not depend on A. Now let D be the Euclidean ball of volume 1, fix $T \in SL(n)$, and denote $G = G(n,m)$ for short. Denote $M = \sup_{E \in G} \mathrm{Vol}(T(A) \cap E)$. Then:

$$\begin{aligned}
\int_G \mathrm{Vol}(A \cap E)\, d\nu(E) &= \int_G \mathrm{Vol}\big(T(A) \cap T(E)\big) D_T(E)\, d\nu(E) \\
&\leqslant M \int_G D_T(E)\, d\nu(E) \\
&= M \frac{\mathrm{Vol}(D_n)^{m/n}}{\mathrm{Vol}(D_m)} \int_G \mathrm{Vol}\big(D \cap T(E)\big) D_T(E)\, d\nu(E) \\
&= M \frac{\mathrm{Vol}(D_n)^{m/n}}{\mathrm{Vol}(D_m)} \int_G \mathrm{Vol}\big(T^{-1}(D) \cap E\big)\, d\nu(E).
\end{aligned}$$

Now, using polar coordinates, double integration and Jensen's inequality, we have:

$$\begin{aligned}
\int_G \mathrm{Vol}\big(T^{-1}(D) \cap E\big)\, d\nu(E) &= \mathrm{Vol}(D_m) \int_G \int_{S^{n-1} \cap E} \|\theta\|_{T^{-1}(D)}^{-m}\, d\sigma_E(\theta)\, d\nu(E) \\
&= \mathrm{Vol}(D_m) \int_{S^{n-1}} \|\theta\|_{T^{-1}(D)}^{-m}\, d\sigma(\theta) \\
&\leqslant \mathrm{Vol}(D_m) \bigg(\int_{S^{n-1}} \|\theta\|_{T^{-1}(D)}^{-n}\, d\sigma(\theta) \bigg)^{m/n} \\
&= \mathrm{Vol}(D_m) \bigg(\frac{\mathrm{Vol}(T^{-1}(D))}{\mathrm{Vol}(D_n)} \bigg)^{m/n} = \frac{\mathrm{Vol}(D_m)}{\mathrm{Vol}(D_n)^{m/n}}.
\end{aligned}$$

We therefore see that for any $T \in SL(n)$:

$$\int_{G(n,m)} \mathrm{Vol}(A \cap E)\, d\nu(E) \leqslant \sup_{E \in G} \mathrm{Vol}\big(T(A) \cap E\big),$$

which proves the assertion. □

Remark 5.7. An alternative way to prove Lemma 5.6 was suggested to us by the referee, to whom we are grateful. It makes use of a very interesting result by Grinberg [19], which was unknown to this author. In hope of interesting the unfamiliar reader, we bring it here. The dual affine quermassintegral of a compact set A, which was introduced by Lutwak in the 80s (see also [31]), is defined (up to normalization) as:

$$\Phi_{n-m}(A) = \bigg(\int_{G(n,m)} \mathrm{Vol}(A \cap E)^n\, d\nu(E)\bigg)^{1/n}.$$

It was shown in [19] that Φ_{n-m} is indeed invariant to volume preserving linear transformations: $\Phi_{n-m}(T(A)) = \Phi_{n-m}(A)$ for all $T \in SL(n)$. Using this, Lemma 5.6 is easily deduced from Jensen's inequality, since for any $T \in SL(n)$:

$$\int_{G(n,m)} \mathrm{Vol}(A \cap E)\, d\nu(E) \leqslant \bigg(\int_{G(n,m)} \mathrm{Vol}(A \cap E)^n\, d\nu(E)\bigg)^{1/n} = \Phi_{n-m}(A) = \Phi_{n-m}\big(T(A)\big)$$
$$\leqslant \sup_{E \in G(n,m)} \mathrm{Vol}\big(T(A) \cap E\big).$$

We mention another result from [19], stating that for a convex body K:

$$\Phi_{n-m}(K) \leqslant C_{m,n}\, \mathrm{Vol}(K)^{m/n},$$

where $C_{m,n}$ is determined by choosing $K = D_n$, and with equality iff K is a centrally symmetric ellipsoid. This may be used to give a universal bound for the expression appearing in the next Lemma 5.8, but we will need an estimate depending on L_K for the proof of Theorem 5.9.

Applying Lemma 5.6 on a convex body K of volume 1, and using (5.4) when $T(K)$ is in isotropic position, we immediately get the following lemma as a corollary:

Lemma 5.8. *For any centrally symmetric convex body K with* $\mathrm{Vol}(K) = 1$:

$$\bigg(\int_{G(n,n-k)} \mathrm{Vol}(K \cap E)\, d\nu(E)\bigg)^{1/k} \leqslant C\mathcal{L}_k / L_K,$$

where ν is the Haar probability measure on $G(n, n-k)$.

We can now formulate the dual counterpart to Theorem 5.3. Note that since $(L^\circ)^\circ \neq L$ for a general k-Busemann–Petty body, our formulation is a little different than before.

Theorem 5.9. *Let K be a centrally symmetric convex body in isotropic position with* $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$. *Then*:

$$L_K \leqslant C \inf\left\{ \frac{\mathcal{L}_{2k}^2}{\widetilde{M}_k(T(L))} \;\middle|\; \begin{array}{l} L \subset K^\circ,\ L \in BP_k^n, \\ T \in SL(n),\ k = 1, \ldots, \lfloor n/3 \rfloor \end{array} \right\}.$$

Proof. First, let us assume $\mathrm{Vol}(K) = 1$, and correct for this later. Fix $k = 1, \ldots, \lfloor n/3 \rfloor$ and let $L \in BP_k^n$ be contained in K°. As in the proof of Theorem 4.7, we note that $T(L) \in BP_k^n$ for any $T \in SL(n)$. Applying Theorem 5.1, the left-hand side of (5.1) gives:

$$L_K / C_1 \geqslant \left(\frac{\widetilde{V}_k(T(L), D)}{\widetilde{V}_k(T(L), K)} \right)^{1/k} \geqslant \left(\frac{\widetilde{V}_k(T(L), D)}{\widetilde{V}_k(T(K^\circ), K)} \right)^{1/k}, \tag{5.7}$$

for D the Euclidean ball of volume 1. Evaluating the denominator on the right using the trivial $\|x\|_{T(K^\circ)} \|x\|_K \geqslant |\langle T^{-1}(x), x \rangle| = |T^{-1/2}(x)|^2$ for any positive definite $T \in SL(n)$, we have that:

$$\begin{aligned}
\left(\widetilde{V}_k\left(T(K^\circ), K\right)\right)^{1/k} &= \left(\frac{1}{n} \int\limits_{S^{n-1}} \|x\|_{T(K^\circ)}^{-k} \|x\|_K^{-(n-k)} \, dx \right)^{1/k} \\
&\leqslant \left(\frac{1}{n} \int\limits_{S^{n-1}} \|x\|_{T^{1/2}(D_n)}^{-2k} \|x\|_K^{-(n-2k)} \, dx \right)^{1/k} \\
&= V_{2k}\left(T^{1/2}(D_n), K\right)^{1/k}.
\end{aligned}$$

Using property (2.1) of dual mixed-volumes, the latter expression is equal to $V_{2k}(D_n, T^{-1/2}(K))^{1/k}$. Denoting $G = G(n, n-2k)$, and using polar coordinates and double integration, we have:

$$\begin{aligned}
V_{2k}\left(D_n, T^{-1/2}(K)\right)^{1/k} &= \left(\mathrm{Vol}(D_n) \int\limits_G \int\limits_{S^{n-1} \cap E} \|\theta\|_{T^{-1/2}(K)}^{-(n-2k)} \, d\sigma_E(\theta) \, d\nu(E) \right)^{1/k} \\
&= \left(\frac{\mathrm{Vol}(D_n)}{\mathrm{Vol}(D_{n-2k})} \int\limits_G \mathrm{Vol}\left(T^{-1/2}(K) \cap E\right) d\nu(E) \right)^{1/k} \\
&\leqslant \frac{C}{n-2k} \left(\frac{\mathcal{L}_{2k}}{L_K} \right)^2,
\end{aligned}$$

where we have used Lemma 5.8 in the last inequality and (2.2). Together with (5.7), cancelling out one L_K term, and using $n - 2k \geqslant n/3$, this gives:

$$L_K \leqslant C' n^{-1} \frac{\mathcal{L}_{2k}^2}{\widetilde{V}_k(T(L), D)^{1/k}} \simeq n^{-1/2} \frac{\mathcal{L}_{2k}^2}{\widetilde{M}_k(T(L))}, \tag{5.8}$$

for any $T \in SL(n)$ (since it can be factorized into a composition of a rotation and a positive-definite transformation, and $\widetilde{M}_k$ is invariant to rotations). Now correcting for our initial assumption on $\mathrm{Vol}(K)$ and going back to $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$, we have the desired:

$$L_K \leqslant C \frac{\mathcal{L}_{2k}^2}{\widetilde{M}_k(T(L))}. \qquad \square$$

As in the previous section, it would be nice to know that for $L \in BP_k^n$, there exists a position in which we can bound $\widetilde{M}_k(T(L))$ from below by $(\mathrm{Vol}(L)/\mathrm{Vol}(D_n))^{1/n}$ times some function of k. Unfortunately, we cannot provide an analogue of Lemma 4.8 for general k-Busemann–Petty bodies, but for *convex* members we have the following lemma, which is stated again in Corollary 6.3:

Lemma 5.10. *Let K be an isotropic convex body with* $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$*, and assume that $K \in BP_k^n$ for some $k = 1, \dots, n-1$. Then*:

$$\widetilde{M}_k(K) \geqslant \frac{C}{\mathcal{L}_k}.$$

Proof. This is a trivial consequence of Theorem 5.3 applied with $L = K$, and using the well-known fact (e.g., [35]) that L_K is always bounded from below by a universal constant. $\square$

We will therefore require that the body L from Theorem 5.9 be convex, and denote by CBP_k^n the class of convex k-Busemann–Petty bodies in $\mathbb{R}^n$. Applying Lemma 5.10 to the body L, using the reverse Blaschke–Santalo inequality (4.11) and homogeneity, we immediately have:

Corollary 5.11. *For any centrally symmetric convex body K*:

$$L_K \leqslant C \inf\left\{ \mathcal{L}_{2k}^2 \mathcal{L}_k \left(\frac{\mathrm{Vol}(L^\circ)}{\mathrm{Vol}(K)} \right)^{1/n} \;\middle|\; \begin{array}{l} K \subset L^\circ,\ L \in CBP_k^n, \\ k = 1, \dots, \lfloor n/3 \rfloor \end{array} \right\}.$$

We will see some applications of Theorem 5.3 in the next section.

6. Applications

As applications, we state a couple of immediate consequences of Corollaries 4.5 and 4.9 about the isotropic constant of polytopes with few facets or vertices. Next, we give several corollaries of Theorem 5.3, and show how they may be used to bound the isotropic constant of new classes of bodies.

It is well known that any centrally symmetric polytope with $2m$ facets is a section of an m-dimensional cube, and by duality, any centrally symmetric polytope with $2m$ vertices is a projection of an m-dimensional unit ball of l_1. It is also well known that l_∞^m isomorphically embeds in L_p for $p = \log(1+m)$, and by duality, l_1^m is isomorphic to a quotient of L_q, for $q = p^*$ the conjugate exponent to p. With the same notations, it follows that a polytope with $2m$ facets is isomorphic to a section of L_p and that a polytope with $2m$ vertices is isomorphic to a quotient of L_q. The following is therefore an immediate consequence of Corollary 4.5 or Junge's theorem:

Corollary 6.1. *Let K be a convex centrally symmetric polytope with $2m$ facets. Then $L_K \leqslant C\sqrt{\log(1+m)}$.*

Since any convex body may be isomorphically approximated by a polytope with C^n facets (or vertices), we retrieve the well-known naive bound $L_K \leqslant C\sqrt{n}$. In this respect, the factor of $\sqrt{p}$ in Corollary 4.5 for sections of L_p seems more natural than the factor of p for quotients of L_q, appearing in Corollary 4.9 or Junge's theorem. Reproducing the above argument, an immediate consequence of Corollary 4.9 or Junge's theorem is:

Corollary 6.2. *Let K be a convex centrally symmetric polytope with $2m$ vertices. Then $L_K \leqslant C\log(1+m)$.*

As mentioned in the Introduction, Corollary 6.2 implies that Gluskin's probabilistic construction in [17] of two convex bodies K_1 and K_2 with Banach–Mazur distance of order n, satisfies $L_{K_1}, L_{K_2} \leqslant C\log(1+n)$. This is simply because the bodies K_1 and K_2 are constructed as random polytopes with (at most) $4n$ vertices.

Another easy corollary, which was already partially stated in Lemmas 4.8 and 5.10, may be deduced from Theorems 3.6, 4.4 and 5.3, if we use the well-known fact that L_K is always bounded from below. Together with Jensen's inequality (as in (1.3)), this reads as follows:

Corollary 6.3. *Let K be convex centrally symmetric isotropic body with $\mathrm{Vol}(K) = \mathrm{Vol}(D_n)$. Then*:

(1) $1 \leqslant M_2(K) \leqslant CT_2(X_K)$.
(2) *If $K \in SL_p^n$ $(p \geqslant 0)$, then $1 \leqslant M_p(K) \leqslant C\sqrt{p_0}$, where $p_0 = \max(1, \min(p, n))$.*
(3) *If $K \in BP_k^n$ $(k = 1, \dots, n-1)$, then $C/\mathcal{L}_k \leqslant \widetilde{M}_k(K) \leqslant 1$.*

Next, we proceed to deduce several consequences of Theorem 5.3. It is known that BP_k^n does not contain all convex bodies for $k < n-3$, and that BP_{n-1}^n already contains all star-bodies [11,25]. So definitely not all convex bodies are isometric to members of BP_k^n for $k < n-3$. Nevertheless, the following assumption might be true:

Outer Volume Ratio Assumption for BP_k^n. *There exist two universal constants $C, \epsilon > 0$, such that for any n and any convex body K in $\mathbb{R}^n$ there exists a star-body $L \in BP_k^n$ for $k = n^{1-\epsilon}$, such that $K \subset L$ and $(\mathrm{Vol}(L)/\mathrm{Vol}(K))^{1/n} \leqslant C$.*

Under this assumption, Theorem 5.3 would immediately imply that $\mathcal{L}_n \leqslant C\mathcal{L}_{n^{1-\epsilon}}$. Denoting $\delta = -1/\log(1-\epsilon)$, and iterating this inequality $\delta \log\log n$ times, we would have:

Corollary 6.4. *Under the Outer Volume Ratio Assumption for BP_k^n, we have*

$$\mathcal{L}_n \leqslant C_1\big(\log(1+n)\big)^{C_2\delta}$$

for $\delta > 0$ as above.

In addition, the advantage of working with BP_k^n when trying to find or build a body $L \in BP_k^n$ containing K, is that we need not worry about the convexity of L like in the case of SL_p^n. The convexity of K has already been used in Theorem 5.1 (in (5.4)), so we may now consider ρ_K as a function on S^{n-1} which we want to tightly bound from above using functions ρ_L from the given family BP^n_k. This is an especially attractive approach, as BP^n_k has the following nice characterization, first proved by Goodey and Weil in [18] for intersection-bodies (the case $k = 1$), and extended to general k by Grinberg and Zhang in [20]:

Theorem *(Grinberg and Zhang). A star-body K is a k-Busemann–Petty body iff it is the limit of $\{K_i\}$ in the radial metric d_r, where each K_i is a finite k-radial sums of ellipsoids $\{\mathcal{E}^i_j\}$:*

$$\rho^k_{K_i} = \rho^k_{\mathcal{E}^i_1} + \cdots + \rho^k_{\mathcal{E}^i_{m_i}},$$

or equivalently, if there exists a Borel measure μ on $SL(n)$ such that:

$$\rho^k_{K_i} = \int\limits_{SL(n)} \rho^k_{T(D_n)}\, d\mu(T).$$

In fact, even the "easiest" case $k = 1$ in Theorem 5.3 seems potentially useful, as we shall demonstrate below. Note that since the intersection-body L need not be convex (and therefore Corollary 6.3 does not apply to it), the mean-radius $\widetilde{M}(L)$ might be significantly smaller than the volume-radius $(\mathrm{Vol}(L)/\mathrm{Vol}(D_n))^{1/n}$. As demonstrated by Theorem 5.3, a smart way to bound ρ_K from above by ρ_L which is the sum of radial functions of ellipsoids, such that we have control over L's mean-radius, might provide a new bound on the isotropic constant. We give two examples of how such an approach might work. Unfortunately, we need to use some additional assumptions, which, although we believe to be true, we have not been able to prove. First, we need a new definition for a class of bodies.

Definition. Let K denote a star-body. We will work with the radial metric topology on the space of star-bodies. Introduce the closed set of volume preserving linear images of K,

$$B(K) = \bigl\{T(K) \bigm| T \in SL(n)\bigr\}.$$

The *radial sums of K*, denoted by $RS(K)$, is the closure in the radial metric of the family of all star-bodies L, such that there exists a non-negative Borel measure μ on $B(K)$, for which:

$$\rho_L = \int\limits_{B(K)} \rho_{K'}\, d\mu(K').$$

Similarly, if P is a closed set of star-bodies, then the *radial sums of P*, denoted $RS(P)$, is the closure in the radial metric of the family of all star-bodies L, such that there exists a non-negative Borel measure μ on $B(P) = \bigcup_{K\in P} B(K)$, for which:

$$\rho_L = \int\limits_{B(P)} \rho_{K'}(\theta)\, d\mu(K').$$

So for example $RS(D_n)$ is exactly the class of intersection-bodies, since $B(D_n)$ is the set of all ellipsoids of volume $\mathrm{Vol}(D_n)$, and by the aforementioned result of Goodey and Weil, the radial sums of this set are exactly the class of intersection-bodies. Another easy observation is that $RS(P)$ is closed under full-rank linear transformations, since for any linear T:

$$\rho_K = \rho_{K_1} + \rho_{K_2} \quad \Rightarrow \quad \rho_{T(K)} = \rho_{T(K_1)} + \rho_{T(K_2)}.$$

As a consequence, $RS(D_n) \subset RS(K)$ for any star-body K. To see this, first notice that $D_n \in RS(K)$, by choosing the Borel measure μ on $B(K)$ to be:

$$\mu(A) = \eta\big(\big\{T \in O(n) \bigm| T(K) \in A\big\}\big)$$

for every Borel set $A \subset B(K)$, where η is the appropriately normalized Haar measure on $O(n)$, the group of orthogonal rotations in $\mathbb{R}^n$. Since $RS(K)$ is closed under $SL(n)$, radial summation, and limit in the radial-metric, it follows that $RS(D_n) \subset RS(K)$. Therefore, for any non-intersection-body K, $RS(K)$ properly contains the class of intersection bodies.

There are many interesting questions that may be asked about radial sums of star-bodies, such as whether it is possible to characterize a minimal set P for which $RS(P)$ already contains all convex bodies, or, probably easier, all star-bodies. In particular it is not even clear to us whether P may be chosen as a singleton in either case. Our focus will be on the following two assumptions, which we believe to be true. The first is about the n-dimensional cube Q_n (of volume 1):

Outer Mean-Radius Assumption for the Cube Q_n. *For any $K \in B(Q_n)$, there exists an ellipsoid $\mathcal{E}$ containing K such that $\widetilde{M}(\mathcal{E})/\widetilde{M}(K) \leqslant C\log(1+n)$, for some universal constant $C > 0$.*

The second assumption is about $UC(n)$, the class of volume 1 convex bodies in $\mathbb{R}^n$ which are all unconditional with respect to the same fixed Euclidean structure. We shall say that a body is a cross-polytope if it is a linear-image of the unit ball of l_1^n.

Outer Mean-Radius Assumption for $UC(n)$. *For any $K \in B(UC(n))$, there exists a cross-polytope L containing K such that $\widetilde{M}(L)/\widetilde{M}(K) \leqslant C\log(1+n)$, for some universal constant $C > 0$.*

We will shortly give motivation for why these assumptions might be correct, but first, let us show an easy consequence of Theorem 5.3 under each assumption.

Corollary 6.5.

(1) *Under the Outer Mean-Radius Assumption for Q_n, for any convex body $K \in RS(Q_n)$, we have $L_K \leqslant C\log(1+n)$.*
(2) *Under the Outer Mean-Radius Assumption for $UC(n)$, for any convex body $K \in RS(UC(n))$, we have $L_K \leqslant C\log(1+n)$.*

As mentioned before, the families of convex bodies in $RS(Q_n)$ and $RS(UC(n))$ are potentially new classes of convex bodies, which might contain a big piece of the convex bodies compactum. Therefore, this new approach to bounding the isotropic constant might be applicable for a large family of convex bodies.

Proof. Let K be an isotropic convex body of volume $\mathrm{Vol}(D_n)$ in $RS(P)$, where P is either $\{Q_n\}$ or $UC(n)$. By approximation, we may assume that $\rho_K = \sum_i \mu_i \rho_{K_i}$, where $K_i \in B(P)$ and $\mu_i \geqslant 0$.

Notice that both the unit-ball of l_1^n and the Euclidean ball are intersection bodies, and this is preserved under volume preserving linear transformations. Therefore, by the Outer Mean-Radius Assumption for P, there exist intersection-bodies L_i such that $K_i \subset L_i$ and $\widetilde{M}(L_i)/\widetilde{M}(K_i) \leqslant C\log(1+n)$. Now define L to be the star-body for which $\rho_L = \sum_i \mu_i \rho_{L_i}$. It is obvious that L contains K, and that L is an intersection-body (since these are closed under non-negative radial summation, as follows from their definition). In addition, since the mean-radius $\widetilde{M}$ is additive under radial summation, it is clear that $\widetilde{M}(L)/\widetilde{M}(K) \leqslant C\log(1+n)$. But using Jensen's inequality (as in (1.3)), we have $\widetilde{M}(K) \leqslant \widetilde{M}_n(K) = 1$, and therefore $\widetilde{M}(L) \leqslant C\log(1+n)$. Using Theorem 5.3, the proof is complete. $\square$

We conclude by giving motivation for why the above two assumptions might be correct, and explain the difficulty in proving them. The next proposition demonstrates that the assumptions indeed hold when the bodies in question are in isotropic position, in which case the bounding bodies may be chosen to be in isotropic position as well.

Proposition 6.6.

(1) *Let D be the circumscribing Euclidean ball of Q_n. Then*:

$$\frac{\widetilde{M}(D)}{\widetilde{M}(Q_n)} \leqslant C\log(1+n).$$

(2) *Let K be an unconditional convex body in isotropic position, and let L be its circumscribing unit ball of l_1^n. Then*:

$$\frac{\widetilde{M}(L)}{\widetilde{M}(K)} \leqslant C\log(1+n).$$

Proof. (1) This is a standard calculation relating to the concentration of the norm $\|\cdot\|_{Q_n}$ on the sphere, which may be done using the standard concentration techniques from [37]. We prefer to quote a general result by Klartag and Vershynin from [26, Proposition 1.2], which states that for any convex body K, if $0 < l < Ck(K)$, where $k(K) = n(M(K)/b(K))^2$, then $\widetilde{M}_l(K) \simeq 1/M(K)$. Since for the volume 1 cube Q_n it is well known (e.g., [37]) that

$$M(Q_n) \simeq \frac{\sqrt{\log(1+n)}}{\sqrt{n}},$$

$b(Q_n) = 2$, and therefore $k(Q_n) \simeq \sqrt{\log(1+n)}$, it follows that for n large enough we may use the above result for $l = 1 < Ck(Q_n)$, to conclude that (for all n) $\widetilde{M}(Q_n) \simeq \sqrt{n}/\sqrt{\log(1+n)}$. Since $\widetilde{M}(D) = \sqrt{n}/2$, the claim follows.

(2) Let P_n be the unit ball of l_1^n of volume 1. It is well known (e.g., [6]) that there exist $C_1, C_2 > 0$, such that for any isotropic convex body K of volume 1, which is unconditional with respect to the given Euclidean structure, the following inclusions hold:

$$C_1 Q_n \subset K \subset C_2 P_n.$$

Therefore $\widetilde{M}(L)/\widetilde{M}(K) \leqslant \widetilde{M}(C_2 P_n)/\widetilde{M}(C_1 Q_n)$. We have already seen that $\widetilde{M}(Q_n) \simeq \sqrt{n}/\sqrt{\log(1+n)}$. We may estimate $\widetilde{M}(P_n)$ in the same manner, or alternatively, use Corollary 6.3 to deduce that $\widetilde{M}(P_n) \simeq \sqrt{n}$. Therefore $\widetilde{M}(L)/\widetilde{M}(K) \leqslant C \log(1+n)$. $\square$

Unfortunately, the techniques described above fail when used upon $T(K)$, where K is in isotropic position but T is an almost degenerate mapping. In particular, it is a bad idea to try to bound $T(K)$ using $T(L)$, where L is the optimal bounding body for K. Indeed, let us try to evaluate $\widetilde{M}(T(D))/\widetilde{M}(T(Q_n))$, where as in Proposition 6.6, D is the circumscribing Euclidean ball of Q_n. Using (2.1), we have:

$$\frac{\widetilde{M}(T(D))}{\widetilde{M}(T(Q_n))} = \frac{\widetilde{V}_1(T(D), D_n)}{\widetilde{V}_1(T(Q_n), D_n)} = \frac{\widetilde{V}_1(D, T^{-1}(D_n))}{\widetilde{V}_1(Q_n, T^{-1}(D_n))}.$$

Denoting $\mathcal{E} = T^{-1}(D_n)$, we see that:

$$\frac{\widetilde{M}(T(D))}{\widetilde{M}(T(Q_n))} = \frac{\int_{S^{n-1}} \rho_D(\theta)\rho_{\mathcal{E}}(\theta)^{n-1}\, d\sigma(\theta)}{\int_{S^{n-1}} \rho_{Q_n}(\theta)\rho_{\mathcal{E}}(\theta)^{n-1}\, d\sigma(\theta)},$$

and this is clearly invariant under homothety of $\mathcal{E}$. Now let us define $\mathcal{E}(\xi, a, b)$ for $\xi \in S^{n-1}$ and $a, b > 0$ as the ellipsoid whose corresponding norm is defined as:

$$\|x\|^2_{\mathcal{E}(\xi,a,b)} = \frac{\langle x, \xi\rangle^2}{a^2} + \frac{|x|^2 - \langle x, \xi\rangle^2}{b^2}.$$

It was shown in [18] that by appropriately choosing $a = a(\epsilon)$ very large and $b = b(\epsilon)$ very small, and setting $\mathcal{E}(\xi, \epsilon) = \mathcal{E}(\xi, a(\epsilon), b(\epsilon))$, the family $\rho^{n-1}_{\mathcal{E}(\xi,\epsilon)}$ is an approximation of unity on S^{n-1} at ξ (as $\epsilon > 0$ tends to 0). This means that for every $f \in C(S^{n-1})$:

$$\int\limits_{S^{n-1}} f(\theta)\rho^{n-1}_{\mathcal{E}(\xi,\epsilon)}(\theta)\, d\sigma(\theta) \to f(\xi) \quad \text{as } \epsilon \to 0.$$

Hence, we see that by choosing $T = T(\xi)$ to be very degenerate, we may arbitrarily approximate:

$$\frac{\widetilde{M}(T(D))}{\widetilde{M}(T(Q_n))} \simeq \frac{\rho_D(\xi)}{\rho_{Q_n}(\xi)},$$

and the latter ratio may be chosen to be any number between 1 and $\sqrt{n}$ by an appropriate choice of $\xi \in S^{n-1}$. This example demonstrates the difficulty in proving the Outer Mean-Radius Assumptions.

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