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Note on the rank of quadratic twists of Mordell equations

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Abstract

Let E be the elliptic curve given by a Mordell equation $y^2 = x^3 - A$ where $A \in \mathbb{Z}$. Michael Stoll found a precise formula for the size of a Selmer group of E for certain values of A. For $D \in \mathbb{Z}$, let E_D denote the *quadratic twist* $Dy^2 = x^3 - A$. We use Stoll's formula to show that for a positive square-free integer $A \equiv 1$ or 25 mod 36 and for a nonnegative integer k, we can compute a lower bound for the proportion of square-free integers D up to X such that $\operatorname{rank} E_D(\mathbb{Q}) \leqslant 2k$. We also compute an upper bound for a certain average rank of quadratic twists of E.

Keywords: Quadratic twists of elliptic curves; Average Mordell–Weil ranks

1. Introduction

Let $E/\mathbb{Q}$ be the elliptic curve $y^2 = x^3 - A$ where A is a nonzero integer. Then, let us denote by E_D the *quadratic twist* $y^2 = x^3 - AD^3$ for each nonzero square-free integer D.

For a nonnegative integer k, let

$$\delta_k := \frac{3^{k+1} - 2}{3^{k+1} - 1},$$

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and let $T(X)$ denote the set of all positive square-free integers less than X. In this paper, we shall prove the following result:

Theorem 2.2. *Let A be a positive square-free integer such that $A \equiv 1$ or $25 \bmod 36$. Then, for a nonnegative integer k,*

$$\liminf_{X} \frac{\#\{D \in T(X)\colon \operatorname{rank} E_D(\mathbb{Q}) \leqslant 2k\}}{\#T(X)} \geqslant \frac{\delta_k}{8} \cdot \prod_{p \mid A} \frac{p}{(p-1)(p+1)}. \tag{1}$$

In particular,

$$\liminf_{X} \frac{\#\{D \in T(X)\colon E_D(\mathbb{Q}) = \{O\}\}}{\#T(X)} \geqslant \frac{1}{16} \cdot \prod_{p \mid A} \frac{p}{(p-1)(p+1)}. \tag{2}$$

One of the earliest (known) examples of elliptic curves $E/\mathbb{Q}$ with a positive proportion of square-free integers D such that $\operatorname{rank} E_D(\mathbb{Q}) = 0$ is the elliptic curve given by $y^2 = x^3 - x$, proved by the work of Heath-Brown in [5], 1994. Note that a similar result was already available in the late eighties. There are two results known in 1988 which together simply imply that the elliptic curve $E\colon y^2 = x^3 - 1$ has positive proportion of quadratic twists of rank 0. In 1985, Frey proved in [2, Proposition, p. 237] that if D is a square-free integer such that $D \equiv 1 \bmod 4$, then

$$\#\mathrm{Cl}\big(\mathbb{Q}\big(\sqrt{-D}\big)\big)[3] = 1 \quad \text{if and only if} \quad \mathsf{Sel}^{(3)}(E_D, \mathbb{Q}) = \{0\},$$

where $\mathsf{Sel}^{(3)}(E_D, \mathbb{Q})$ is *the* 3*-Selmer group of* $E_D/\mathbb{Q}$. In 1988, Nakagawa and Horie proved in [7] Theorem 1.3 stated in this paper, which is a refined result of the famous theorem of Davenport and Heilbronn. Their theorem implies that there is a positive proportion of positive square-free integers D such that $D \equiv 1 \bmod 4$ and $\#\mathrm{Cl}(\mathbb{Q}(\sqrt{-D}))[3] = 1$. Therefore, it follows that there is a positive proportion of (positive) square-free integers D such that

$$\mathsf{Sel}^{(3)}(E_D, \mathbb{Q}) = \{0\} \quad \text{and, hence,} \quad \operatorname{rank} E_D(\mathbb{Q}) = 0.$$

Let us introduce our second result. For two positive integers m and N, let us denote by $N_2^+(X, m, N)$ the set of positive *fundamental discriminants* $\Delta < X$ such that $\Delta \equiv m \bmod N$.

Theorem 3.1. *Let $E/\mathbb{Q}$ be an elliptic curve given by $y^2 = x^3 - A$ for some $A \in \mathbb{Z}$ such that $A \equiv 1$ or $25 \bmod 36$ is a square-free integer.*

If $A > 0$, then

$$\limsup_{X \to \infty} \frac{\sum_{D \in N_2^+(X,1,12A)} \operatorname{rank}(E_D(\mathbb{Q}))}{\#N_2^+(X, 1, 12A)} \leqslant 1.$$

If $A < 0$, then

$$\limsup_{X\to\infty} \frac{\sum_{D\in N_2^+(X,1,12|A|)} \operatorname{rank}(E_D(\mathbb{Q}))}{\#N_2^+(X,1,12|A|)} \leqslant \frac{4}{3}.$$

Recall that $T(X)$ denotes the set of all positive square-free integers less than X. Assuming the Birch and Swinnerton-Dyer Conjecture (the Modularity Conjecture), and a form of the Riemann hypothesis, Goldfeld proved in [3] that

$$\limsup_{X\to\infty} \frac{\sum_{|D|\in T(X)} \operatorname{rank} E_D(\mathbb{Q})}{2\cdot \#T(X)} \leqslant 3.25. \tag{3}$$

In [4], this upper bound is reduced to 1.5 by Heath-Brown. In [3], Goldfeld conjectured that the average in (3) should be $1/2$, which is known as Goldfeld's conjecture. In [5], Heath-Brown (unconditionally) computes an upper bound for the average rank of quadratic twists $y^2 = x^3 - D^2x$ over odd integers D, and Gang Yu in [11] computes a certain average rank of quadratic twists of infinitely many elliptic curves with rational 2-torsion points. At the moment of writing, these two results were the only unconditional results, known to the author, on the average rank of quadratic twists of an elliptic curve.

1.1. *Stoll's formula*

Let $\zeta \in \overline{\mathbb{Q}}$ be a primitive third root of unity, and let $\lambda := 1 - \zeta$. Let K denote the cyclotomic field extension $\mathbb{Q}(\zeta)$. Let E/K be an elliptic curve given by $y^2 = x^3 - A$ where $A \in \mathbb{Z}$. We denote simply by ζ the endomorphism on E given by $(x, y) \mapsto (\zeta x, y)$ which is defined over K. Let us denote the endomorphism $1 - \zeta$ on E simply by λ, and let $E[\lambda]$ denote the kernel of λ. The endomorphism λ induces the Kummer sequence

$$0 \to E[\lambda] \to E \xrightarrow{\lambda} E \to 0. \tag{4}$$

Let M_K denote the set of all places of K, and let K_v denote the completion of K with respect to a place $v \in M_K$. Note that (4) induces the following injective homomorphism into the first cohomology group of the $\operatorname{Gal}(\overline{F}/F)$-module $E[\lambda]$ where F is the number field K or a completion K_v:

$$\delta_F : E(F)/\lambda E(F) \to \mathsf{H}^1\bigl(F, E[\lambda]\bigr). \tag{5}$$

When $F = K_v$ for some $v \in M_K$, let us denote δ_F also by δ_v.

Note that for each $v \in M_K$, there is the restriction map $\operatorname{res}_v : \mathsf{H}^1(K, E[\lambda]) \to \mathsf{H}^1(K_v, E[\lambda])$ (see [8, Chapter X, Section 4]). *The λ-Selmer group of E/K* is

$$\mathsf{Sel}^{(\lambda)}(E, K) := \bigl\{\xi \in \mathsf{H}^1\bigl(K, E[\lambda]\bigr)\colon \operatorname{res}_v(\xi) \in \operatorname{Im}\delta_v \text{ for all } v \in M_K\bigr\}, \tag{6}$$

and it contains the image of $E(K)/\lambda E(K)$.

Theorem 1.1. (Stoll [9, Corollary 2.1]) *Let A be a rational integer. Let E/K be the elliptic curve given by $y^2 = x^3 - A$.*

Suppose that the following conditions[1] *are satisfied*:

(a) $-A \equiv 2 \bmod 3$.
(b) *For all places $v \neq \lambda$ of K of bad reduction for E/K, the integer $-A$ is nonsquare in K_v^*, e.g., $-A$ is square-free, and $-A \equiv 3 \bmod 4$.*

Then, $\dim_{\mathbb{F}_3} \mathsf{Sel}^{(\lambda)}(E, K) =$

$$\begin{aligned}
&1 + 2\dim_{\mathbb{F}_3} \mathrm{Cl}\big(\mathbb{Q}\big(\sqrt{-A}\big)\big)[3] \quad \textit{if } -A \equiv 2, 8 \bmod 9 \textit{ and } A < 0,\\
&2\dim_{\mathbb{F}_3} \mathrm{Cl}\big(\mathbb{Q}\big(\sqrt{-A}\big)\big)[3] \quad \textit{if } -A \equiv 2, 8 \bmod 9 \textit{ and } A > 0,\\
&2\dim_{\mathbb{F}_3} \mathrm{Cl}\big(\mathbb{Q}\big(\sqrt{3A}\big)\big)[3] \quad \textit{if } -A \equiv 5 \bmod 9 \textit{ and } A < 0,\\
&1 + 2\dim_{\mathbb{F}_3} \mathrm{Cl}\big(\mathbb{Q}\big(\sqrt{3A}\big)\big)[3] \quad \textit{if } -A \equiv 5 \bmod 9 \textit{ and } A > 0.
\end{aligned}$$

In particular, these numbers give a bound on $\operatorname{rank} E(\mathbb{Q})$.

Lemma 1.2. *If n is a nonzero integer and p is a prime number > 3 such that $n \equiv 3 \bmod 4$ and $\mathrm{ord}_p(n) \equiv 1 \bmod 2$, then $n \notin (K_{\mathfrak{p}}^*)^2$ where $K_{\mathfrak{p}}$ is the completion at any prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ lying over p or 2.*

Let A be a square-free integer $\equiv 1 \bmod 12$. If D is a square-free integer coprime to A such that $D \equiv 1 \bmod 12$, then the elliptic curve $y^2 = x^3 - AD^3$ satisfies conditions (a) *and* (b) *in Theorem* 1.1.

Proof. Let n be a nonzero integer $\equiv 3 \bmod 4$. Note that $2\mathcal{O}_K$ is a prime ideal, and that $\{a + 2b\colon a, b \in R\}$ where $R := \{0, 1, \zeta, \zeta^2\}$ forms a complete residue class modulo $4\mathcal{O}_K$. Since $(a + 2b)^2 \equiv a^2 \bmod 4\mathcal{O}_K$, the integer n is not a square mod $4\mathcal{O}_K$. In particular, n is not a square in the completion $K_{\mathfrak{p}}$ where $\mathfrak{p} = 2\mathcal{O}_K$. If p is a prime number > 3 such that $\mathrm{ord}_p(n) \equiv 1 \bmod 2$, then p is unramified in $\mathcal{O}_K$ and, hence, $\mathrm{ord}_{\mathfrak{p}}(n) \equiv 1 \bmod 2$ for all prime ideals $\mathfrak{p}$ dividing $p\mathcal{O}_K$. Thus, n is not a square in $K_{\mathfrak{p}}^*$.

Let A and D be square-free integers coprime to each other such that A, and $D \equiv 1 \bmod 12$. Then, the following set is precisely the set of places of bad reduction of E_D/K:

$$\{v \in M_K\colon E/K \text{ has bad reduction at } v\} \cup \{v \in M_K\colon v \mid D\}. \tag{7}$$

Note that $-AD^3 \equiv 2 \bmod 3$ and, hence, condition (a) is satisfied. Let p be a prime number > 3, and let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ lying over p at which E_D/K has bad reduction. Suppose that E/K has bad reduction at $\mathfrak{p}$. Since A is square-free, and $p > 3$, the prime number p must divide A. Suppose that $\mathfrak{p}$ divides D. Then, p must divide D. Note also that $-AD^3 \equiv 3 \bmod 4$ and that $\mathrm{ord}_p(-AD^3) \equiv 1 \bmod 2$ for any prime number $p > 3$ dividing

[1] In [10], which is a sequel of [9], Stoll improved the conditions so that more possibilities of values of A can be considered for Theorem 2.2.

A or D since A and D are coprime to each other. By the first statement of this lemma, $-AD^3$ is not a square in $K_{\mathfrak{p}}^*$ for any prime ideal $\mathfrak{p}$ dividing AD or 2. Therefore, if $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$ lying over a prime number $p \neq 3$ at which E_D/K has bad reduction, then p must divide AD or 2 and, hence, $-AD^3$ is not a square in $K_{\mathfrak{p}}^*$. □

In this paper, we will focus on quadratic twists of the elliptic curve given by a Mordell equation, but the reader might have noticed from the formula in Theorem 1.1 that if A is replaced with AD^2 for an integer D such that $D \equiv 1 \bmod 9$, and such that the elliptic curve E^D: $y^2 = x^3 - AD^2$ satisfies the conditions required for the formula, then the size of the Selmer group of E^D/K equals that of the Selmer group of E/K. Since $y^2 = x^3 - AD^2$ forms a family of cubic twists, we can use the formula to obtain the following result on the distribution of Mordell–Weil rank of cubic twists of E: If A is a positive square-free integer such that $A \equiv 1$ or $25 \bmod 36$ and $\dim_{\mathbb{F}_3} \mathrm{Cl}(\mathbb{Q}(\sqrt{-A}))[3] = 0$, then there is a positive real number $\epsilon < 1$ such that

$$\#\{0 < D < X\colon\ D \text{ cube-free, } \operatorname{rank} E^D(\mathbb{Q}) = 0\} \gg \frac{X}{(\log X)^{\epsilon}}. \tag{8}$$

To compute the lower bound in (8), we construct a set of prime numbers with positive Dirichlet density, and show that whenever D is a positive integer divisible only by prime numbers contained in this set, the Mordell–Weil rank of E^D is 0. This observation is generalized for *superelliptic curves over global fields* in [1]. To my knowledge, the only known example of an elliptic curve with infinitely many cubic twists of Mordell–Weil rank 0 is $x^3 + y^3 = D$ proved by D. Lieman [6].

1.2. The refined result of Davenport–Heilbronn

In [7], Nakagawa and Horie proved a refined result of Davenport and Heilbronn. Let N and m be positive integers. Let $N_2^-(X, m, N)$ be the set of fundamental discriminants Δ such that $-X < \Delta < 0$, and $\Delta \equiv m \bmod N$, and let $N_2^+(X, m, N)$ be the set of fundamental discriminants Δ such that $0 < \Delta < X$, and $\Delta \equiv m \bmod N$. Let $h_3(\Delta)$ denote $\#\mathrm{Cl}(F)[3]$ where F is the quadratic extension of $\mathbb{Q}$ with discriminant Δ.

Let us describe the property for N and m, which we require for Theorem 1.3.

Condition (∗∗). If an odd prime number p is a common divisor of m and N, then $p^2 \mid N$ and $p^2 \nmid m$. Further, if N is even, then $4 \mid N$ and $m \equiv 1 \bmod 4$, or $16 \mid N$ and $m \equiv 8$ or $12 \bmod 16$.

Theorem 1.3. (Nakagawa–Horie [7]) *Let N and m be positive integers satisfying Condition* (∗∗). *Then,*

$$\lim_{X\to\infty} \frac{1}{\#N_2^+(X, m, N)} \sum_{\Delta \in N_2^+(X,m,N)} h_3(\Delta) = \frac{4}{3}, \tag{9}$$

$$\lim_{X\to\infty} \frac{1}{\#N_2^-(X,m,N)} \sum_{\Delta\in N_2^-(X,m,N)} h_3(\Delta) = 2. \tag{10}$$

2. Proof of Theorem 2.2

Let S be a subset of $\mathbb{Z}$, and for a positive integer x, let $S(x)$ denote the set of integers n contained in S such that $|n| < x$. Let $\mathbb{N}$ be the set of positive integers. Let h be a set-theoretic function: $\mathbb{N} \to \mathbb{N}$ such that the images of h are powers of 3. For a nonnegative integer k, let

$$S_k(x) := \{a \in S(x)\colon\ h(a) \leqslant 3^k\} \quad \text{and} \quad \delta_k(x) := \frac{\#S_k(x)}{\#S(x)}. \tag{11}$$

Lemma 2.1. *If* $\lim_{x\to\infty} \frac{1}{\#S(x)} \sum_{a\in S(x)} h(a) = B$ *for some positive real number* B*, then for a nonnegative integer* k*,*

$$\liminf_{x\to\infty} \delta_k(x) \geqslant \frac{3^{k+1} - B}{3^{k+1} - 1}.$$

Proof. Note that

$$\begin{aligned}
\frac{1}{\#S(x)} \sum_{a\in S(x)} h(a) &= \frac{1}{\#S(x)} \left(\sum_{a\in S_k(x)} h(a) + \sum_{a\notin S_k(x)} h(a) \right) \\
&\geqslant \frac{1}{\#S(x)} \left(\sum_{a\in S_k(x)} 1 + \sum_{a\notin S_k(x)} 3^{k+1} \right) \\
&= \delta_k(x) + 3^{k+1}\bigl(1 - \delta_k(x)\bigr).
\end{aligned}$$

Hence, there is $\epsilon(x)$ such that $B + \epsilon(x) \geqslant \delta_k(x) + 3^{k+1}(1 - \delta_k(x))$ and $\lim_{x\to\infty} \epsilon(x) = 0$. It follows that

$$\delta_k(x) \geqslant \frac{3^{k+1} - B - \epsilon(x)}{3^{k+1} - 1},$$

which implies the result. □

Recall from Section 1 the constant δ_k for nonnegative integers k, and that $T(X)$ denotes the set of positive square-free integers $D < X$.

Theorem 2.2. *Let* A *be a positive square-free integer such that* $A \equiv 1$ *or* 25 mod 36. *Then, for a nonnegative integer* k*,*

$$\liminf_X \frac{\#\{D \in T(X)\colon\ \operatorname{rank} E_D(\mathbb{Q}) \leqslant 2k\}}{\#T(X)} \geqslant \frac{\delta_k}{8} \cdot \prod_{p|A} \frac{p}{(p-1)(p+1)}. \tag{12}$$

In particular,

$$\liminf_{X} \frac{\#\{D \in T(X)\colon\ E_D(\mathbb{Q}) = \{O\}\}}{\#T(X)} \geqslant \frac{1}{16} \cdot \prod_{p|A} \frac{p}{(p-1)(p+1)}. \tag{13}$$

Proof. Let $D \in N_2^+(X/4A, 1, 12A)$. Then, D is a square-free integer coprime to A such that $D \equiv 1 \bmod 12$. By Lemma 1.2, $-AD^3$ satisfies conditions (a) and (b) in Theorem 1.1. Recall that E_D is given by $y^2 = x^3 - AD^3$. Note that if $D \equiv 1 \bmod 12$, then $D^3 \equiv 1 \bmod 9$. Since $-AD^3 \equiv -A \equiv 2$ or $8 \bmod 9$, by Theorem 1.1,

$$\operatorname{rank} E_D(\mathbb{Q}) \leqslant \dim_{\mathbb{F}_3} \mathsf{Sel}^{(\lambda)}(E_D, K) = 2\dim_{\mathbb{F}_3} \mathrm{Cl}\big(\mathbb{Q}\big(\sqrt{-AD^3}\,\big)\big)[3] = 2\log_3 h_3(-4AD).$$

Let $m := 48A^2 - 4A$, and note that there is a one-to-one correspondence between $N_2^+(X/4A, 1, 12A)$ and $N_2^-(X, m, 48A^2)$ given by $D \mapsto -4AD$. Then it follows that for a nonnegative integer k,

$$\begin{aligned}&\big\{\Delta \in N_2^-\big(X, m, 48A^2\big)\colon\ h_3(\Delta) \leqslant 3^k\big\}\\ &\quad\hookrightarrow \big\{D : -4AD \in N_2^-\big(X, m, 48A^2\big),\ \operatorname{rank} E_D(\mathbb{Q}) \leqslant 2k\big\}.\end{aligned} \tag{14}$$

Let $h := h_3$, and $B := 2$. Then, by Lemma 2.1 and Theorem 1.3, given $\epsilon > 0$,

$$\frac{1}{\#N_2^-(X, m, 48A^2)} \#\big\{\Delta \in N_2^-\big(X, m, 48A^2\big)\colon\ h_3(\Delta) \leqslant 3^k\big\} \geqslant \delta_k - \epsilon \tag{15}$$

for all sufficiently large X. Note that $\{D\colon\ -4AD \in N_2^-(X, m, 48A^2)\}$ is contained in $T(X/4A)$. Then, it follows that given $\epsilon > 0$, for all sufficiently large X,

$$\begin{aligned}&\frac{1}{\#T(X/4A)} \#\big\{D \in T(X/4A)\colon\ \operatorname{rank} E_D(\mathbb{Q}) \leqslant 2k\big\}\\ &\quad\geqslant \frac{1}{\#T(X/4A)}\ \#\big\{D\colon\ -4AD \in N_2^-\big(X, m, 48A^2\big),\ \operatorname{rank} E_D(\mathbb{Q}) \leqslant 2k\big\}\\ &\quad\geqslant \frac{1}{\#T(X/4A)}\ \#\big\{\Delta \in N_2^-\big(X, m, 48A^2\big)\colon\ h_3(\Delta) \leqslant 3^k\big\} \quad \text{by (14)}\\ &\quad\geqslant \frac{\#N_2^-(X, m, 48A^2)}{\#T(X/4A)} \cdot (\delta_k - \epsilon) \quad \text{by (15)}.\end{aligned} \tag{16}$$

By [7, Proposition 2], we find

$$\lim_{X\to\infty} \frac{\#N_2^-(X, m, 48A^2)}{\#T(X/4A)} = \frac{1}{8} \prod_{p|A} \frac{p}{(p-1)(p+1)}, \tag{17}$$

and this proves (12).

Let $E'/\mathbb{Q}$ be an elliptic curve given by $y^2 = x^3 + B$ such that B is an integer not equal to -432, 1, a cube, or a square. Then, it is well known that the torsion subgroup of $E'(\mathbb{Q})$ is trivial and, hence, for all but finitely many square-free integers D, the torsion subgroup of $E_D(\mathbb{Q})$ is trivial. Therefore, (13) follows from (12) with $k = 0$. $\square$

3. Proof of Theorem 3.1

Let A be a square-free integer such that $A \equiv 1$ or 25 mod 36. Let $m := 48A^2 - 4A$ if $A > 0$, and $m := -4A$ if $A < 0$. Note that $A \equiv 1$ or 7 mod 9, and that $-AD^3 \equiv -A \equiv 2$ or 8 mod 9 for $D \in N_2^+(X/4|A|, 1, 12|A|)$ since $D \equiv 1$ mod 12 implies $D^3 \equiv 1$ mod 9. Recall that E_D is given by $y^2 = x^3 - AD^3$. By Lemma 1.2, if $D \in N_2^+(X/4|A|, 1, 12|A|)$, then E_D satisfies conditions (a) and (b) in Theorem 1.1 and, hence,

$$\dim_{\mathbb{F}_3} \mathsf{Sel}^{(\lambda)}(E_D, K) = \begin{cases} 2\log_3 h_3(-4AD) & \text{if } A > 0; \\ 1 + 2\log_3 h_3(-4AD) & \text{if } A < 0. \end{cases}$$

If $A > 0$, then there is a one-to-one correspondence between $N_2^+(X/4A, 1, 12A)$ and $N_2^-(X, m, 48A^2)$ given by $D \mapsto -4AD$. If $A < 0$, then there is a one-to-one correspondence between $N_2^+(X/4|A|, 1, 12|A|)$ and $N_2^+(X, m, 48A^2)$ given by $D \mapsto -4AD$. Note that if n is a positive integer which is a power of 3, then $\log_3 n \leqslant \frac{1}{2}(n-1)$. Then, it follows that if $A > 0$, then

$$\begin{aligned} \frac{\sum_{D \in N_2^+(X/4A,1,12A)} \dim_{\mathbb{F}_3} \mathsf{Sel}^{(\lambda)}(E_D, K)}{\# N_2^+(X/4A, 1, 12A)} &= \frac{\sum_{-4AD \in N_2^-(X,m,48A^2)} \dim_{\mathbb{F}_3} \mathsf{Sel}^{(\lambda)}(E_D, K)}{\# N_2^-(X, m, 48A^2)} \\ &= \frac{\sum_{\Delta \in N_2^-(X,m,48A^2)} 2\ \log_3 h_3(\Delta)}{\# N_2^-(X, m, 48A^2)} \\ &\leqslant \frac{\sum_{\Delta \in N_2^-(X,m,48A^2)} 2\ \frac{1}{2}(h_3(\Delta) - 1)}{\# N_2^-(X, m, 48A^2)} \\ &\to 1 \quad \text{as } X \to \infty \text{, by Theorem 1.3.} \end{aligned}$$

If $A < 0$, then

$$\begin{aligned} \frac{\sum_{D \in N_2^+(X/4|A|,1,12|A|)} \dim_{\mathbb{F}_3} \mathsf{Sel}^{(\lambda)}(E_D, K)}{\# N_2^+(X/4|A|, 1, 12|A|)} &\leqslant \frac{\sum_{\Delta \in N_2^+(X,m,48A^2)} 1 + 2\frac{1}{2}(h_3(\Delta) - 1)}{\# N_2^+(X, m, 48A^2)} \\ &\to \frac{4}{3} \quad \text{as } X \to \infty. \end{aligned}$$

Since the λ-Selmer rank over K bounds from above the Mordell–Weil rank over $\mathbb{Q}$, we have proved

Theorem 3.1. *Let $E/\mathbb{Q}$ be an elliptic curve given by $y^2 = x^3 - A$ where A is a square-free integer such that $A \equiv 1$ or $25 \bmod 36$.*

If $A > 0$, then

$$\limsup_{X\to\infty} \frac{\sum_{D\in N_2^+(X,1,12A)} \operatorname{rank}(E_D(\mathbb{Q}))}{\#N_2^+(X,1,12A)} \leqslant 1.$$

If $A < 0$, then

$$\limsup_{X\to\infty} \frac{\sum_{D\in N_2^+(X,1,12|A|)} \operatorname{rank}(E_D(\mathbb{Q}))}{\#N_2^+(X,1,12|A|)} \leqslant \frac{4}{3}.$$

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