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Journal of Algebra 318 (2007) 723–736

JOURNAL OF
Algebrawww.elsevier.com/locate/jalgebra

On liftable and weakly liftable modules

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Received 1 February 2007

Available online 23 October 2007

Communicated by Luchezar L. Avramov

Abstract

Let T be a Noetherian ring and f a nonzerodivisor on T . We study concrete necessary and sufficient conditions for a module over $R = T/(f)$ to be weakly liftable to T , in the sense of Auslander, Ding, and Solberg. We focus on cyclic modules and obtain various positive and negative results on the lifting and weak lifting problems. For a module over T we define the loci for certain properties: liftable, weakly liftable, having finite projective dimension and study their relationships.

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Keywords: Lifting; Weak lifting of modules

1. Introduction

In this note, all rings are commutative, Noetherian with identity, and all modules are finitely generated. Let $T \rightarrow R$ be a ring homomorphism. An R -module M is said to *lift* (or liftable) to T if there is a T -module M' such that $M = M' \otimes_T R$ and $\text{Tor}_i^T(M', R) = 0$ for all $i > 0$. M is said to *weakly lift* (or weakly liftable) to T if it is a direct summand of a liftable module. When $R = T/(f)$ where f is a nonzerodivisor in T , a situation which will be our main focus, then the Tor condition for lifting simply says that f must be a nonzerodivisor on M' . The lifting questions began with:

Question 1.1 (*Grothendieck's lifting problem*). Let (T, m, k) be a complete regular local ring and $R = T/(f)$ where $f \in m - m^2$. Does an R -module always lift to T ?

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Note that if T is equicharacteristic, then the answer is obviously “yes”: in that case $T \cong R[[f]]$, and we can simply choose $M' = M[[f]]$. The significance of this question was first publicly realized by Nastold, who observed in [Na] that Serre’s multiplicity conjectures could be solved completely (i.e., in the case of ramified regular local ring) if we can always lift in the sense of Grothendieck. Hochster [Ho1] gave a negative answer to Grothendieck’s lifting problem (see Example 3.5). However, he pointed out that a positive answer to the lifting problem for prime cyclic modules, and even less would be enough for Serre’s conjectures. Specifically, he posed the following, which was indeed the starting point for this note:

Question 1.2 (*Hochster’s lifting problem*). Let (T, m, k) be a complete regular local ring and $R = T/(f)$ where $f \in m - m^2$. Let $P \in \text{Spec}(R)$.

- (1) When can $M = R/P$ lift to T ?
- (2) When does there exist an R -module M , liftable to T , such that $\text{Supp}(M) = \text{Supp}(R/P)$?

A negative answer to part (1) of the above question will be given in Section 4. A more general version to the questions above, first addressed in [PS], is:

Question 1.3. Let (T, m, k) be a regular local ring and $R = T/(f)$ where f is a regular element in m . Let M be an R module such that $\text{pd}_R M < \infty$. When can M lift to T ?

Over the years, a number of interesting results on the lifting problems have been published (see [PS, BE, Hof, ADS, Jo1, Jo2, Yo]). They are almost exclusively homological in nature. In this note, we will focus our attention on *concrete* sufficient and necessary conditions to weak liftability. In [ADS], Auslander, Ding and Solberg have made clear that understanding weak lifting is essential to understanding lifting. Many of our results are ideal-theoretic, not homological. We have several motivations for this approach. Firstly, in the context of Hochster’s lifting questions, when R is itself a regular local ring, if one has to find a negative example, most homological obstructions would not work (R is “homologically too nice”). In any case, to have any hope of answering part (2) of Question 1.2 one needs to know “What annihilates a liftable module?”. Secondly, for the more general lifting question, it would be very desirable to tell whether one can weakly lift a module just from its presentation. We are able to give some modest answers to these problems and shed some lights on why they are non-trivial. One thing is clear from our work: the modules which are not liftable are abundant (in some sense, they can be parametrized by a Zariski open set).

Section 2 reviews basic notations and important results we would use, including Hochster’s characterization of approximately Gorenstein rings. In Section 3 we study some general necessary and conditions for weak liftability that involves the annihilator of the module M (Theorem 3.2). As applications, we revisit Hochster’s counterexample to Grothendieck’s lifting question and show that it gives a lot more, namely an ideal that is not an annihilator of any weakly liftable module (see 3.5).

In Section 4 we focus on weak liftings of cyclic modules. We collect some simple but useful characterization of weakly liftable cyclic modules in Lemma 4.1. Many applications follow. We revisit Jorgensen’s example of an unliftable module with finite projective dimension and give a simple proof in 4.3, as well as a big class of such modules in 4.4. We also reprove a result related to modular representation of cyclic groups in 4.5. A negative example to part (1) of Hochster’s lifting question above is given in 4.6. Lastly, we prove very concrete characterizations of weak

liftability for Gorenstein ideals of dimension 0 and Cohen–Macaulay, generically Gorenstein ideals of dimension 1 in Theorem 4.9.

In Section 5 we formulate a comparative study of liftable, weakly liftable and finite projective dimension properties. We define a locus for each property in a quite general way: by fixing a module over T and asking what hypersurfaces R would make the module satisfy that property. Our definitions are inspired by the notions of “support sets” or “support varieties” of modules, invented and studied recently by Avramov, Buchweitz [AB] and Jorgensen [Jo3]. We show in many cases that weakly liftable and liftable are “open condition” (see 5.2, 5.3). This explains in a conceptual way the existence of many examples of modules with finite projective dimension but cannot lift: they can be parametrized by a Zariski open set in a certain affine space (see 5.4). Examples 5.5 and 5.6 show that computing these loci is quite non-trivial, and in particular the liftable locus may depend on the arithmetic of the residue field.

The author would like to thank Melvin Hochster, whose valuable insights and advices initiated and inspired most of this work.

2. Notations and preliminary results

In this note, all rings are commutative, Noetherian with identity, and all modules are finitely generated. Let R be a ring and M, N be R -modules. If N is a submodule of M , N is called a *pure* (respectively, *cyclically pure*) if for every R -module E (respectively, every cyclic R -module E), the induced map $N \otimes E \rightarrow M \otimes E$ is injective. If M/N is of finite presentation, then it is not hard to show that N is a pure submodule of M if and only if N is a direct summand of M (see [Ma, Theorem 7.14]).

A more interesting question is when cyclic purity implies purity, especially when $N = R$. This was answered completely in [Ho2]. Recall that a local ring (R, m, k) is called *approximately Gorenstein* if for any integer N , there is an ideal $I \subset m^N$ such that R/I is Gorenstein. A Noetherian ring R is called *approximately Gorenstein* if the localization at any maximal ideal of R is approximately Gorenstein. Then:

Proposition 2.1. (See [Ho2, Proposition 1.4].) *Let R be a Noetherian ring. The following are equivalent:*

- (1) R is approximately Gorenstein.
- (2) For every module extension $R \hookrightarrow M$, cyclic purity implies purity.

Hochster’s paper also provided very concrete characterizations of approximately Gorenstein ring. For our purpose, the following result would be enough:

Theorem 2.2. (See [Ho2, Theorem 1.7].) *Let R be a locally excellent Noetherian ring and suppose that R satisfies one of the conditions below:*

- (1) R is generically Gorenstein (i.e., the quotient ring of R is Gorenstein).
- (2) For any prime $P \in \text{Ass}(R)$ and maximal ideal $m \supset P$, $\dim(R/P)_m \geq 2$.

Then R is approximately Gorenstein.

Let (R, m, k) be a local ring. Let M, N be R -modules such that $l(M \otimes N) < \infty$. One can define the Poincaré series for M, N as

$$P_{M,N}^R(t) = \sum_i l(\text{Tor}_i^R(M, N))t^i.$$

When $N = k$, we shall simply write $P_M^R(t)$.

The result below is essential for our study of weak lifting. It is from [ADS, Proposition 3.2]:

Proposition 2.3. *Consider $R = T/(f)$, where f is a nonzerodivisor on T , which is a Noetherian algebra over a local ring. The following are equivalent:*

- (1) M is weakly liftable to T .
- (2) $\text{syz}_1^T(M)/f \text{syz}_1^T(M) \cong M \oplus \text{syz}_1^R(M)$, where $\text{syz}_1^R(M)$ is induced from the free resolution defining $\text{syz}_1^T(M)$.
- (3) M is liftable to $R_2 = T/(f^2)$.

Remark. Throughout this paper, when we consider the lifting in the situation $R = T/(f)$, we will always assume the condition: “ T is a Noetherian algebra over a local ring.” Since this covers algebras over fields or DVRs and all local rings, it is not a serious restriction.

Finally, we would like to make a definition, mainly for notational conveniences (see 3.2).

Definition 2.4. Let J, L be ideals of a ring R . One defines:

$$\text{int}_L(J) := \{x \in R \mid \exists a_i \in J^i, i = 1, \dots, n: x^n + a_1x^{n-1} + \dots + a_n \in L\}.$$

Lemma 2.5. *It is easy to see that*

$$\text{int}_L(J) = \text{int}_L(J + L) \subseteq \text{rad}(J + L).$$

Lemma 2.6. *If M is a T module and $I = \text{Ann}_T(M)$ then for any ideal J of T :*

$$\text{Ann}(M/JM) \subseteq \text{int}_I(J).$$

Proof. See [Ma, Theorem 2.1]. \square

3. Some general remarks on weak lifting

In this section we study several necessary conditions for a module over $R = T/(f)$ to be weakly liftable to T . Our main purpose is to find concrete obstructions to weak liftability of M . Note that an obstruction to weak lifting is naturally an obstruction to lifting.

To state the first result, let us recall the change of rings exact sequence for Tor. Let $R = T/(f)$, where f is a nonzerodivisor on T . Let M, N be R -modules. Then we have the long exact sequence of Tors

$$\begin{aligned} \cdots \rightarrow \operatorname{Tor}_n^R(M, N) &\rightarrow \operatorname{Tor}_{n+1}^T(M, N) \rightarrow \operatorname{Tor}_{n+1}^R(M, N) \\ &\rightarrow \operatorname{Tor}_{n-1}^R(M, N) \rightarrow \operatorname{Tor}_n^T(M, N) \rightarrow \operatorname{Tor}_n^R(M, N) \rightarrow \cdots \\ &\rightarrow \operatorname{Tor}_0^R(M, N) \rightarrow \operatorname{Tor}_1^T(M, N) \rightarrow \operatorname{Tor}_1^R(M, N) \rightarrow 0. \end{aligned}$$

In the long exact sequence above, let α_i be the connecting map $\operatorname{Tor}_{i+2}^R(M, N) \rightarrow \operatorname{Tor}_i^R(M, N)$.

Proposition 3.1. *Let T be a Noetherian algebra over a local ring. Let f be a nonzerodivisor in T and $R = T/(f)$. Let M be an R -module. The following are equivalent:*

- (1) M is weakly liftable.
- (2) The map $\theta : 0 \rightarrow M \rightarrow \operatorname{syz}_1^T M / f \operatorname{syz}_1^T M$ splits.
- (3) For any R -module N , the map $\alpha_0 : \operatorname{Tor}_2^R(M, N) \rightarrow \operatorname{Tor}_0^R(M, N)$ is 0.
- (4) For any R -module N and any integer $i \geq 0$ the map $\alpha_i : \operatorname{Tor}_{i+2}^R(M, N) \rightarrow \operatorname{Tor}_i^R(M, N)$ is 0.

Proof. The equivalence of (1) and (2) is from [ADS]. That (4) implies (3) is obvious. It remains to show that (2) and (3) are equivalent and (2) implies (4). For that we need to understand how the maps α_0 arises. Let

$$0 \rightarrow \operatorname{syz}_1^T M \rightarrow T^a \rightarrow M$$

be the projective covering of M with respect to T . Tensoring with $R = T/(f)$, since $\operatorname{Tor}_1^T(T, R) = 0$ and $\operatorname{Tor}_1^T(M, R) = M$, we get

$$0 \rightarrow M \rightarrow \operatorname{syz}_1^T M / f \operatorname{syz}_1^T M \rightarrow R^a \rightarrow M.$$

Breaking down this exact sequence we have

$$0 \rightarrow M \rightarrow \operatorname{syz}_1^T M / f \operatorname{syz}_1^T M \rightarrow \operatorname{syz}_1^R M \rightarrow 0.$$

Tensoring the above exact sequence with N over R gives the connecting map $\operatorname{Tor}_1^R(\operatorname{syz}_1^R M, N) \rightarrow M \otimes_R N$, which is α_0 . From this discussion we can see that (3) is equivalent to the assertion that the injection $\theta : M \hookrightarrow \operatorname{syz}_1^T M / f \operatorname{syz}_1^T M$ remains injective when we tensor with any R -module N . But this is equivalent to θ splits (see [Ma, Theorem 7.14]). Also, if θ splits then all the maps $\operatorname{Tor}_{i+1}^R(\operatorname{syz}_1^R M, N) \rightarrow \operatorname{Tor}_i^R(M, N)$ must also be 0, which shows that (2) implies (4). \square

The following theorem gives necessary conditions for an ideal to be the annihilator of a weakly liftable module.

Theorem 3.2. *Let T be a Noetherian algebra over a local ring. Let f be a nonzerodivisor in T and $R = T/(f)$. Let M be an R -module and $I = \operatorname{Ann}_T(M)$. If M is weakly liftable to T then:*

- (1) $(I^2 : f) \subseteq I$.
- (2) $(JI : f) \subseteq \operatorname{int}_I(J)$ for all ideals J of T .
- (3) $(JI : f) \subseteq \operatorname{rad}(I + J)$ for all ideals J of T .

We begin with some lemmas. Let us try to understand concretely what weak liftability imposes on the annihilator of a module. Let M be an R -module and we pick a free covering of M as a T -module:

$$0 \rightarrow W \rightarrow G \rightarrow M \rightarrow 0.$$

Here $G = T^n$. Let $I = \text{Ann}_T(M)$. By the above proposition, the map θ :

$$0 \rightarrow G/W \xrightarrow{h} W/fW$$

which takes $x + W$ to $fx + fW$ splits.

Lemma 3.3. *Let T, R, M, G, W be as above. If M is weakly liftable to T then for any ideal $J \subseteq T$ we have $(JW : f) \subseteq (JG + W)$.*

Proof. We use the simple fact that for T -modules $P \subseteq Q$ such that P is a direct summand of Q , then for any ideals J of T , $P/J P$ injects into $Q/J Q$ (in other words, P is a cyclically pure submodule of Q).

Applying that to G/W and W/fW we have $G/(W + JG)$ injects into $W/(fW + JW)$ (with the map induced from h), which is equivalent to $(fW + JW) : f \subseteq (W + JG)$, or equivalently, $(JW : f) \subseteq (W + JG)$ \square

Lemma 3.4. *Let T, R, M, G, W, I be as above. Then for any ideal J in T we have $(JI : f) \subseteq \text{Ann}_T(G/(JW : f))$.*

Proof. Let $v \in (JI : f)$. So $vf \in JI$. Hence $vfG \subseteq JIG$. But I kills G/W , so $IG \subseteq W$. It implies that $vfG \subseteq JW \Rightarrow vG \subseteq (JW : f) \Rightarrow v \in \text{Ann}_T(G/(JW : f))$. \square

Now we can prove Theorem 3.2.

Proof of Theorem 3.2. By the previous lemmas we have

$$\begin{aligned} (JI : f) &\subseteq \text{Ann}_T(G/(JW : f)) \subseteq \text{Ann}_T(G/(JG + W)) \\ &= \text{Ann}_T((G/W)/(J(G/W))) = \text{Ann}_T(M/JM). \end{aligned}$$

The last term is I if $J = I$, and it is contained in $\text{int}_I(J)$ otherwise (by 2.5). Finally, by 2.6 we have $\text{int}_I(J) \subseteq \text{rad}(I + J)$, as required. \square

As an application we will revisit Hochster's counterexample to Grothendieck lifting question (see [Ho1]).

Example 3.5. Let $T = \mathbb{Z}_{(2)}[[x, y, z, a, b, c]]$. Let $f = 2$ and $R = T/(f)$. Let $I = (2, x^2, y^2, z^2, a^2, b^2, c^2, xa + yb + zc)$ and $g = xayb + ybzc + zcxa$. Because of the relation

$$2g = (xa + yb + zc)^2 - x^2a^2 + y^2b^2 + z^2c^2.$$

It follows that $g \in (I^2 : f)$. But is not hard to show $g \notin I$. By 3.2, not only T/I is not liftable to T , as Hochster showed, but I cannot be the annihilator of any R -module which is weakly liftable to T .

Next, we present a simple corollary of 3.2:

Corollary 3.6. *Let (T, m, k) be a local ring and $R = T/(f)$ where f is a nonzerodivisor in T . Suppose M, N are R -modules such that $M \otimes N$ is of finite length and M is weakly liftable to T . Then $P_{M,N}^T(t) = (t + 1)P_{M,N}^R(t)$. If T is regular, M is weakly liftable to T and $\dim M < \dim R$, then $(t + 1)^2 \mid P_M^T(t)$.*

Proof. By Theorem 3.2, the change of rings long exact sequence for Tor would break down into short exact sequences:

$$0 \rightarrow \text{Tor}_i^R(M, N) \rightarrow \text{Tor}_{i+1}^T(M, N) \rightarrow \text{Tor}_{i+1}^R(M, N) \rightarrow 0$$

for all $i \geq 0$. The first statement is immediate. As for the second, first note that $\text{pd}_R M < \infty$. Since $\dim M < \dim R$, $P_M^R(-1) = \chi^R(M, k) = 0$. So $(t + 1) \mid P_M^R(t)$, this fact and the first statement finish the proof. \square

As an application, we will show that weakly liftable Cohen–Macaulay or Gorenstein ideals of small heights often are complete intersections:

Corollary 3.7. *Let (T, m, k) be a regular local ring and $R = T/(f)$ where f is a nonzerodivisor in T . Let I be an ideal in R such that R/I is weakly liftable to T . If $\text{height}(I) = 1$ and R/I is Cohen–Macaulay then I is principal. If $\text{height}(I) = 2$ and R/I is Gorenstein then I is generated by two elements.*

Proof. Let J be the preimage of I in T . By Corollary 3.6 we have $(t + 1)^2 \mid P_{T/J}^T(t)$. In the first case $P_{T/J}^T(t)$ has to be equal to $(t + 1)^2$ (because $\text{pd}_T T/J = 2$). In the second case $P_{T/J}^T(t)$ has to be equal to $(t + 1)^3$ (because $\text{pd}_T T/J = 3$ and the last Betti number is 1 since T/J is Gorenstein). In both cases we must conclude that J is a complete intersection, and so is I . \square

Example 3.8. Let $T = k[[x_1, \dots, x_n]]$, $f = x_1$ and $R = k[[x_2, \dots, x_n]]$. Then any R -module is liftable to T and the above corollary says that in R , a height 1 Cohen–Macaulay ideal has to be principal and a height 2 Gorenstein ideal has to be 2-generated. So there is little hope to strengthen the previous result.

4. Weakly liftable cyclic modules

In the case of cyclic modules, the statements of the previous section can be simplified or strengthened. Let us recall the basic setup. Let T be a Noetherian algebra over a local ring and f be a nonzerodivisor in T . Let $R = T/(f)$ and I be an ideal in T which contains f . We will focus on finding conditions for T/I to be weakly liftable (as an R -module) to T .

Lemma 4.1. *Let T, f, R, I be as above. Fix $\mathbf{v} = (f, f_1, \dots, f_n)$ a set of generators for I . The following are equivalent:*

- (1) $M = T/I$ is weakly liftable to T .
- (2) The T -linear map $h : T/I \rightarrow I/fI$ which takes $1 + I$ to $f + fI$ splits.
- (3) The T -linear map $g : T/I \rightarrow I/I^2$ which takes $1 + I$ to $f + I^2$ splits.
- (4) For any presentation of I :

$$T^m \xrightarrow{X} T^{n+1} \xrightarrow{v} I \rightarrow 0.$$

Let $\mathbf{r}, \mathbf{r}_1, \dots, \mathbf{r}_n$ be the rows of X . There exist $x_1, \dots, x_n \in T$ such that

$$\mathbf{r} - x_1\mathbf{r}_1 + \dots + x_n\mathbf{r}_n \in IT^m$$

and they imply the following equivalent conditions:

- (5) $(IJ : f) \subseteq (J + I)$ for any ideal J .
- (6) $(IJ : f) \subseteq J$ for any ideal $J \supseteq I$.
- (7) (If T is local) $(IJ : f) \subseteq J$ for any irreducible ideal J .

If in addition, T/I is approximately Gorenstein, then all the conditions (1) to (6) (and (7) in the local case) are equivalent.

Remark. The last assertion (when T/I is approximately Gorenstein) was first suggested in [Ho1, p. 462].

Proof of Lemma 4.1. The equivalence of (1) and (2) is a restatement of 3.2. If (2) holds, then $I/fI = T/I \oplus N$ for some T -module N . Tensoring with T/I we get $I/I^2 = T/I \oplus N/IN$, which gives (3). Now assume (3) which says the map g splits. But g is a composition of

$$T/I \xrightarrow{h} I/fI \rightarrow I/I^2,$$

so h also splits.

For the equivalence of (3) and (4), let $Z = \text{Im}(X)$ be the first syzygy of I . Tensoring the exact sequence

$$0 \rightarrow Z \rightarrow T^{n+1} \rightarrow I \rightarrow 0$$

with T/I we get

$$0 \rightarrow (Z \cap IT^{n+1})/IZ \rightarrow Z/IZ \rightarrow (T/I)^{n+1} \rightarrow I/I^2 \rightarrow 0$$

which shows that $Z/(Z \cap IT^{n+1})$ is a first syzygy of I/I^2 (as a module over T/I). So there is no new relations, and I/I^2 admits the following presentation:

$$\bar{T}^m \xrightarrow{\bar{X}} \bar{T}^{n+1} \xrightarrow{\bar{v}} I/I^2 \rightarrow 0.$$

Here $\bar{}$ denotes mod I . Then (3) means exactly that there exist $x_1, \dots, x_n \in T$ such that

$$\bar{\mathbf{r}} = \bar{x}_1\bar{\mathbf{r}}_1 + \dots + \bar{x}_n\bar{\mathbf{r}}_n.$$

Next, (1) implies (5) is a restatement of Lemma 3.3. The equivalence of (5) and (6) is trivial. The only thing to check now is equivalence of (6) and (7). Clearly (6) implies (7). Suppose (6) fails and we have an ideal J such that $(IJ : f) \not\subseteq J$. Pick $x \notin J$ such that $xf \in IJ$. Choose a maximal ideal J_1 containing J such that $x \notin J_1$. Then J_1 is irreducible, and (7) fails as well.

Finally, suppose that in addition T/I is approximately Gorenstein. Condition (4) says that the map g , viewed as a T/I -module extension, is cyclically pure. Then Proposition 2.1 implies that T/I is a pure submodule of I/I^2 via g , so (3) holds. That finishes our proof. \square

Example 4.2. We give an example to show that if T/I is not approximately Gorenstein, the last assertion of Lemma 4.1 would fail even in simplest cases. Let $T = \mathbb{Q}[[x, y]]$, $m = (x, y)$, $I = m^2$ and $f = x^2 + y^2$. Clearly $\text{Im} : f \subset m$ and $I^2 : f \subset I$. Let J be any ideal lying strictly between I and m . Then $J = m^2 + (ux + vy)$, with $u, v \in \mathbb{Q}$. We want to show that $IJ : f \subset J$. Pick $g \in m$ such that $fg \in IJ = m^4 + (ux + vy)m^2$. Let g' be the linear part of g , then clearly $g'f \in (ux + vy)m^2$. Since f is irreducible in T , $g' \in (ux + vy)$, thus $g \in J$. So condition (6) of Lemma 4.1 is satisfied. However $T/I = T/m^2$ is not weakly liftable to T . One can see it by using Theorem 4.4 or simply observing that $\text{pd}_{T/(f)} T/m^2 = \infty$.

It is now quite easy to show that one of the main examples in a paper by Jorgensen (Example 3.3 in [Jo1]) gives a cyclic module of finite projective dimension but is unliftable:

Example 4.3. Let k be a field, $T = k[[x_1, x_2, x_3, x_4]]$, $f = x_1x_2 - x_3^2$, $R = T/(f)$, $I = (f, i_1, i_2, i_3, i_4)$, where:

$$i_1 = -x_2x_3 + x_2x_4, \quad i_2 = x_1x_3 + x_2x_3, \quad i_3 = -x_2^2 - x_3x_4, \quad i_4 = x_1^2 - x_2^2 + x_3^2 - x_4^2.$$

Finally, let $J = (x_1, x_3, x_4, x_2^2) \supset I$. It can be shown using Macaulay that $\text{pd}_R T/I = 3$. But $-x_3b_1 + x_4b_2 + x_1b_3 = x_2f$, so $x_2 \in (JI : f)$. Obviously $x_2 \notin J$, so T/I is not even weakly liftable.

The above example suggests the following:

Theorem 4.4. Let $T = \bigoplus_{n \geq 0} T_n$ be a graded ring with $T_0 = k$ is a field. Let I be a T -ideal generated by homogeneous elements of degree a . Let $f \in I$ be a homogeneous nonzerodivisor of degree a such that $(f) \subsetneq I$. Assume that I admits a free presentation

$$F \xrightarrow{X} G \xrightarrow{Y} I \rightarrow 0$$

such that all the entries of the matrix X has degree $b < a$. Then T/I as a module over $R = T/(f)$ is not weakly liftable to T .

Proof. As f must be a k -linear combination of the generators of I , we may as well assume that $Y = (f, f_1, \dots, f_n)$. Then let $\mathbf{r}, \mathbf{r}_1, \dots, \mathbf{r}_n$ be the rows of X . By part (4) of 4.1 there exist $x_1, \dots, x_n \in T$ such that

$$\mathbf{r} - x_1\mathbf{r}_1 + \dots + x_n\mathbf{r}_n \in IT^m.$$

Counting degree, there must be $y_1, \dots, y_n \in k$ such that

$$\mathbf{r} = y_1 \mathbf{r}_1 + \dots + y_n \mathbf{r}_n.$$

But this means that (f) is a direct summand of I as T -modules. This is impossible unless $(f) = I$, so we are done. \square

As another application, we would prove the following, which is relevant to the theory of modular representation of cyclic groups (see [The]). We give a brief explanation. Let D be a discrete valuation ring whose maximal ideal is generated by a prime number p . Let C_p be the cyclic group of order p . Let $A = D/p^2$ and $k = D/pD$. One wishes to study the $AC_p \cong A[X]/(X^p - 1)$ -modules. Let M be such a module. Then M/pM is a $kC_p \cong k[X]/(X^p - 1) \cong k[X]/(X - 1)^p$ module. The indecomposable modules over kC_p must be of the form $S_i = k[X]/(X - 1)^i$. So M/pM is a direct sum of S_i 's. The interesting questions is which i may occur? Clearly this corresponds to when is S_i liftable to AC_p , or equivalently, weakly liftable to DC_p (by 2.3). In view of this, the following corollary is a special case of Theorem 5.5 in [The].

Corollary 4.5. *Let (D, m, K) be a discrete valuation ring whose maximal ideal is generated by a prime number p . Let $T = D[X]/(X^p - 1)$, $R = T/(p) \cong K[X]/(X^p - 1) \cong K[X]/(X - 1)^p$. Let $S_i = K[X]/(X - 1)^i$ ($1 \leq i \leq p$) be R -modules. Then S_i is weakly liftable to T if and only if $i \in \{1, p - 1, p\}$.*

Proof. Clearly $S_p = R$ lifts and S_1 lifts (take $S = T/(X - 1)$, then S is a lift of S_1). We assume $1 < i < p$. Note that $S_i = T/(p, (X - 1)^i)$. Over T , the ideal $I = (p, (X - 1)^i)$ has a presentation

$$T^2 \xrightarrow{X} T^2 \xrightarrow{\mathbf{v}} I \rightarrow 0.$$

Here $\mathbf{v} = (p, (X - 1)^i)$ and X has 2 rows: $\mathbf{r} = ((X - 1)^p, g(X))$ where $g(X) = \frac{(X^p - 1) - (X - 1)^p}{p}$ and $\mathbf{r}_1 = (-p, (X - 1)^{p-i})$. By Theorem 4.1 (equivalence of (1) and (4)), T/I is weakly liftable if and only if $g(X)$ is a multiple of $(X - 1)^{p-i} \pmod{I}$. Rewriting:

$$g(X) = \frac{((X - 1 + 1)^p - 1) - (X - 1)^p}{p} = \sum_{j=1}^{p-1} \frac{\binom{p}{j}}{p} (X - 1)^j.$$

One can see that it happens if and only if $p - i = 1$. \square

Next we gives an example in which $R = T/(f)$ is a ramified regular local ring of dimension 11 and a prime cyclic module of R that is not weakly liftable. This shows that there is a negative example to part (1) of Question 1.2.

Example 4.6. Let $T = V[[x, y, z, a, b, c, u, v, w, t]]$, in which $(V, 2V)$ is a DVR. Let $f = 2$ and $R = T/(f)$ and let $\bar{}$ denote mod f . Abusing notation, we do not use $\bar{}$ for the indeterminates. Let $I = (2, tu - x^2, tv - y^2, tw - z^2, xa + yb + zc)$. Since t is not nilpotent modulo I , we can pick a minimal prime \bar{P} over \bar{I} which does not contain \bar{t} . It is easy to see that actually, $\bar{P} = \bar{I} : t^\infty = \bar{I} : t$. Using Macaulay 2, we can actually calculate $P = (I, ua^2 + vb^2 + wc^2, uayz + vbzx + wcx y)$. For our purpose, we only need to see that $\bar{P} \subset (u, v, w, x^2, y^2, z^2, xa + yb + zc)$. Now,

let P be the preimage of \bar{P} in T , $J = (P, t, a^2, b^2, c^2)$ and $g = xayb + ybzc + zcxa$. Because of the relation

$$2g = (xa + yb + zc)^2 + (tu - x^2)a^2 + (tv - y^2)b^2 + (tw - z^2)c^2 - t(ua^2 + vb^2 + wc^2).$$

It follows that $fg \in PJ$. It suffices to show $g \notin J$. We can do so modulo $2, u, v, w, t$. It is enough to show $g \notin (x^2, y^2, z^2, a^2, b^2, c^2, xa + yb + zc)$. But this is true by 3.5. By 5.3 we can replace f by $2 + f'$ with $f' \in mP$ to get an example where R is an honest ramified regular local ring.

Remark. Similar examples surely exist for all characteristics.

Lemma 4.1 still leaves much to be desired when one wants to show some module to be weakly liftable, since checking cyclic purity involves infinitely many ideals J . To really take advantage of the conditions, we need a few lemmas:

Lemma 4.7. *Let (T, m, k) be a local ring and $I \subseteq J_1 \subseteq J_2$ be ideals in T . Assume that T/J_1 is 0-dimensional and Gorenstein (in other words, J_1 is irreducible). Then $IJ_2 : f \subseteq J_2$ if $IJ_1 : f \subseteq J_1$.*

Proof. The assumption implies that the map $T/J_1 \rightarrow I/IJ_1$ induced by multiplication by f is injective. Since T/J_1 is injective, the map splits, thus tensoring with T/J_2 preserves injectivity. \square

Lemma 4.8. *Let (T, m, k) be a local ring and $I \subseteq J$ be ideals in T . Assume that T/J is 0-dimensional and Gorenstein. Let $u \in T$ represent the generator of the socle of T/J . Then $IJ : f \subseteq J$ if and only if $fu \notin IJ$.*

Proof. Again, we consider the map $T/J \rightarrow I/IJ$ induced by multiplication by f . As T/J is Gorenstein, this map is injective if and only if it splits if and only if the image of the socle element is nonzero. \square

Theorem 4.9. *Let $R = T/(f)$ where (T, m, k) is a local ring and f is a nonzerodivisor in T . Let T/I be an R -module (so $f \in I$).*

(1) *Suppose that T/I is 0-dimensional and Gorenstein. Let $u \in T$ represent the generator of the socle of T/I . Then T/I is weakly liftable if and only if $uf \notin I^2$.*

(2) *Suppose that T/I is 1-dimensional, Cohen–Macaulay and generically Gorenstein. Let $J \subset T$ represent the canonical ideal of T/I . Let $u \in T$ represent the generator of the socle of T/J . Then T/I is weakly liftable if and only if $uf \notin IJ + I^{(2)}$.*

Proof. (1) By Lemmas 4.1 and 4.7.

(2) Let $S = T/I$. Then since S is generically Gorenstein, its canonical module ω_S is isomorphic to an ideal of height 1. Let \bar{J} be that ideal in S (here J is an ideal in T and $\bar{}$ denotes modulo I). Since S is Cohen–Macaulay and \bar{J} is the canonical ideal of S , S/\bar{J} is 0-dimensional and Gorenstein. Let \bar{x} be a nonzerodivisor in S . Then $x\bar{J} \cong J \cong \omega_S$ so $x\bar{J}$ must also be an irreducible ideal. Note that xu represent the generator of $\text{Soc}(S)$. By Lemmas 4.1 and 4.7 we only need to check that $xuf \notin I(I + xJ)$ for any x such that \bar{x} is a nonzerodivisor in S . This is equivalent to $uf \notin IJ + (I^2 : x)$ for all such x , or $uf \notin IJ + I^{(2)}$ as desired. \square

5. The (non) liftable and weakly liftable loci

This section is a comparative study of liftable, weakly liftable and finite projective dimension properties. Throughout the section we will assume that (T, m, k) is a local ring, and M is a T -module. Let $I \subset \text{Ann}_T(M)$ be an ideal in T and fix a minimal system of generators (f_1, \dots, f_n) for I . Then there is a map $\alpha : I \rightarrow k^n \cong I/mI$ induced by (f_1, \dots, f_n) . For a property \mathcal{P} we define the \mathcal{P} -locus of M in I as

$$\mathcal{L}_{\mathcal{P}}(I, M) := \{f \in I \mid M \text{ satisfies } \mathcal{P} \text{ as a module over } T/(f)\}$$

and the geometric \mathcal{P} -locus of M in I as

$$\mathcal{V}_{\mathcal{P}}(I, M) := \alpha(\mathcal{L}_{\mathcal{P}}(I, M)).$$

If $I = \text{Ann}_T(M)$ we shall simply write $\mathcal{L}_{\mathcal{P}}(M)$ and $\mathcal{V}_{\mathcal{P}}(M)$. The set $\mathcal{V}_{\mathcal{P}}(M)$ parametrizes the hypersurfaces R for which M , as an R module, has property \mathcal{P} . For $\mathcal{P} = \{\text{not liftable}\}$ (respectively not weakly liftable, not finite projective dimension) we will write \mathcal{L}_{nl} (respectively \mathcal{L}_{nwl} , \mathcal{L}_{npd}) (by convention 0 is in all of these sets) and \mathcal{V}_{nl} (respectively \mathcal{V}_{nwl} , \mathcal{V}_{npd}). It is more convenient to work with the negative properties, as they turns out to be “closed” conditions.

Remark. When (f_1, \dots, f_n) form a regular sequence on T , then $\mathcal{V}_{npd}(I, M)$ agrees with the “support variety” of M as defined in [AB]. When $I = \text{Ann}(M)$, $\mathcal{V}_{npd}(M)$ agrees with the “support set” of M defined in [Jo3].

We first observe that:

Proposition 5.1. *Suppose T is a regular local ring and M is a T -module. Let $I = \text{Ann}_T(M)$. Then*

$$I \supseteq \mathcal{L}_{nl}(M) \supseteq \mathcal{L}_{nwl}(M) \supseteq \mathcal{L}_{npd}(M) \supseteq mI$$

and

$$k^n \supseteq \mathcal{V}_{nl}(M) \supseteq \mathcal{V}_{nwl}(M) \supseteq \mathcal{V}_{npd}(M).$$

Proof. The only thing needs to be proved is $\mathcal{L}_{npd}(M) \supseteq mI$. Let us assume $f \in mI$ and $R = T/(f)$. By a result of Shamash [Sha], in this situation:

$$P_M^T(t) = (1 - t^2)P_M^R(t)$$

which clearly shows that the $P_M^R(t)$ cannot be finite series (otherwise $P_M^T(t)$ would have negative terms!). \square

Proposition 5.2. *$\mathcal{L}_{nl}(T/I)$ is an ideal.*

Proof. First, let $f \in \mathcal{L}_{nl}(T/I)$ and $a \in T$. We want to show $af \in \mathcal{L}_{nl}(T/I)$. Assume it is not true, so there exists a T -ideal J such that af is a nonzerodivisor on $\bar{T} = T/J$ and $J + (af) = I$. The

first condition shows that f is also a nonzerodivisor on \bar{T} , and the second shows that $\bar{f}\bar{T} \subset \bar{a}\bar{f}\bar{T}$. By Nakayama’s Lemma, \bar{a} is an unit in \bar{T} , so \bar{T} is also a lift of T/I with respect to f .

Secondly, let $f, g \in \mathcal{L}_{nl}(T/I)$. Similarly, suppose $f + g \notin \mathcal{L}_{nl}(T/I)$, we seek a contradiction. Again, there exists a T -ideal J such that $f + g$ is a nonzerodivisor on $\bar{T} = T/J$ and $J + (f + g) = I$. Since $f, g \in I$ we must have, in \bar{T} , $\bar{f} = (\bar{f} + \bar{g})e_1$ and $\bar{g} = (\bar{f} + \bar{g})e_2$. Adding the two equations and using that $f + g$ is a nonzerodivisor on \bar{T} , we get $e_1 + e_2 = 1$ in \bar{T} . This forces e_1 or e_2 to be a unit in \bar{T} , but then \bar{T} must be a lift of T/I with respect to either f or g . \square

Proposition 5.3. *If T/I is approximately Gorenstein, then $\mathcal{L}_{nwl}(T/I)$ is an ideal.*

Proof. We first construct a sequence $\{L_i\}$ of irreducible ideals in T/I such that $L_{i+1} \subsetneq L_i \forall i$ and $\{L_i\}$ in T/I are cofinal with the powers of the maximal ideal in $\bar{T} = T/I$. Just pick L_1 as any irreducible ideal in \bar{T} . Then there is a power of \bar{m} , $\bar{m}^l \subset L_1$. By assumption we can pick an irreducible ideal $L_2 \subset \bar{m}^l$, and so on. Let J_i be the preimage of L_i in T . By 4.1 and 4.7 we have $f \in \mathcal{L}_{nwl}(T/I)$ if and only if $IJ_i : f \not\subseteq J_i$ for some i (since any irreducible ideal would contain some J_i). Let $I_i := \{f \in I \mid IJ_i : f \not\subseteq J_i\}$. By 4.8, $I_i = (IJ_i : s_i) \cap I$, here s_i represent the socle element of J_i . So each I_i is an ideal in T . But 4.7 and the fact that $J_{i+1} \subseteq J_i$ shows that $I_i \subseteq I_{i+1}$. Hence the sequence of ideals $\{I_i\}$ must stabilize, and since $\mathcal{L}_{nwl}(T/I) = \bigcup_1^\infty I_i$ we are done. \square

Example 5.4. Proposition 5.2 implies that $\mathcal{V}_{nl}(T/I)$ is an affine space. So as long as $\mathcal{V}_{npd}(T/I)$ is not a linear algebraic set, then there should be quite a few example of finite projective dimension, unliftable cyclic modules: they form the non-empty Zariski open set $\mathcal{V}_{nl}(T/I) \setminus \mathcal{V}_{npd}(T/I)$ in $\mathcal{V}_{nl}(T/I)$. Such nonlinear $\mathcal{V}_{npd}(T/I)$ are known to be quite common, see the examples at the end of [Jo3].

Example 5.5. Theorem 4.9 gives explicit formula for $\mathcal{L}_{nwl}(T/I)$ in some cases. Specifically, using the notations of Theorem 4.9 we have $\mathcal{L}_{nwl}(T/I) = I^2 : u$ when T/I is Gorenstein of dimension 0 and $\mathcal{L}_{nwl}(T/I) = (IJ + I^{(2)}) : u$ if T/I is Cohen–Macaulay, generically Gorenstein of dimension 1.

Example 5.6. Let $T = k[[X, Y, Z]]/(X^2 + Y^2 + Z^2)$, here k is a field. Let x, y, z be the images of X, Y, Z , respectively, and let $m = (x, y, z)$. We claim that $\mathcal{L}_{nl}(T/m) = m^2$ if $k = \mathbb{C}$ and $\mathcal{L}_{nl}(T/m) = m$ if $k = \mathbb{Q}$.

First, let $k = \mathbb{C}$. Choose any element $f = ax + by + cz$ with $a, b, c \in \mathbb{C}$. We have to show $f \notin \mathcal{L}_{nl}(T/m)$, in other words, T/m is liftable to T as a $T/(f)$ -module. Let $I_1 = (x, y + iz)$, $I_2 = (y, z + ix)$, $I_3 = (z, x + iy)$. Note that they are prime ideals of height 1 in T . We claim that one of these ideals together with f will generate m . Let $V_i = \alpha(I_i)$ (so for example V_1 is generated by the vectors $(1, 0, 0)$ and $(0, 1, i)$). Then the planes V_1, V_2, V_3 intersect at only the origin in \mathbb{C}^3 so one of them, say V_1 , cannot contain the vector (a, b, c) . This shows that $(I_1, f) = m$. But f is clearly a nonzerodivisor on T/I_1 , and so T/m is liftable.

Next, assume $k = \mathbb{Q}$. It suffices to show that $x \in \mathcal{L}_{nl}(T/m)$, as then $y, z \in \mathcal{L}_{nl}(T/m)$ by symmetry and hence $m = (x, y, z) \subseteq \mathcal{L}_{nl}(T/m)$ by Proposition 5.2. Suppose T/I is a lift of T/m as a module over $T/(x)$. Then $I + (x) = m$. So there are $a, b \in T$ such that $y - ax, z - bx \in I$. But since $x^2 + y^2 + z^2 = 0$ this forces $x^2(1 + a^2 + b^2) \in I$. Since $k = \mathbb{Q}$, $(1 + a^2 + b^2)$ must be a unit, hence $x^2 \in I$. But then x cannot be a nonzerodivisor on T/I .

Finally, observe that $\mathcal{L}_{nwl}(T/m) = m^2$ in both cases. Indeed, by the previous example, since the socle element of T/m is 1, we have $\mathcal{L}_{nwl}(T/m) = m^2 : 1 = m^2$.

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