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Quasi-pseudo-metrization of topological preordered spaces

E. Minguzzi

Dipartimento di Matematica Applicata "G. Sansone", Università degli Studi di Firenze, Via S. Marta 3, I-50139 Firenze, Italy

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ABSTRACT

We establish that every second countable completely regularly preordered space $(E, \mathscr{T}, \leqslant)$ is quasi-pseudo-metrizable, in the sense that there is a quasi-pseudo-metric p on E for which the pseudo-metric $p \vee p^{-1}$ induces $\mathscr{T}$ and the graph of $\leqslant$ is exactly the set $\{(x, y)\colon\ p(x, y) = 0\}$. In the ordered case it is proved that these spaces can be characterized as being order homeomorphic to subspaces of the ordered Hilbert cube. The connection with quasi-pseudo-metrization results obtained in bitopology is clarified. In particular, strictly quasi-pseudo-metrizable ordered spaces are characterized as being order homeomorphic to order subspaces of the ordered Hilbert cube.

1. Introduction

A fundamental theorem by Urysohn and Tychonoff establishes that every second countable regular space (T_3-space) is metrizable. This work aims at generalizing this result for topological spaces endowed with a preorder $\leqslant$. In this case one would like to prove the existence of a function $p : E \times E \to [0, +\infty)$ which encodes both the topology and the preorder, where the latter is obtained through the condition $x \leqslant y$ iff $p(x, y) = 0$. Clearly, function p cannot be a metric in the usual sense, in fact we shall need the more general notion of quasi-pseudo-metric.

Topological preordered spaces appear in various fields, for instance in the study of dynamical systems [1], general relativity [2], microeconomics [3] and computer science [4]. Quasi-pseudo-metrizable preordered spaces are among the most well-behaved topological preordered spaces, thus it is important to establish if one can just work with quasi-pseudo-metrizable preordered spaces in the mentioned applications. Topological preordered spaces are connected to bitopological spaces but the latter spaces are less directly connected with the said applications. This is so because, generically, a flow on a manifold, a causal order on spacetime, or a preference of an agent in microeconomics, to make a few examples, are represented by a preorder which does not necessarily come from a nicely behaved bitopological space. The category of topological preordered spaces is in this respect more interesting, and so far much less investigated, than that of bitopological spaces.

Some definitions will help us to make our mathematical problem more precise. A *topological preordered space* is a triple $(E, \mathscr{T}, \leqslant)$ where $(E, \mathscr{T})$ is a topological space and $\leqslant$ is a preorder, namely a reflexive and transitive relation. Our terminology for topological preordered spaces will follow Nachbin [5]. Two topological preordered spaces E_1, E_2, are *preorder homeomorphic* if there is a homeomorphism $\varphi : E_1 \to E_2$ such that $x \leqslant y$ if and only if $\varphi(x) \leqslant \varphi(y)$. A subset $S \subset E$ of a topological preordered space E is a *subspace* once it is endowed with the induced topology $\mathscr{T}_S$ and preorder $\leqslant_S$ defined by: for $x, y \in S$, $x \leqslant_S y$ if $x \leqslant y$. The topological preordered space E_1 is preorder embedded in E_2 if it is preorder homeomorphic with a subspace of E_2.

E-mail address: ettore.minguzzi@unifi.it.

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A preorder is an *order* if it is antisymmetric. With $i(x) = \{y: x \leqslant y\}$ and $d(x) = \{y: y \leqslant x\}$ we denote the increasing and decreasing hulls. The topological preordered space is *semiclosed preordered* if $i(x)$ and $d(x)$ are closed for every $x \in E$, and it is *closed preordered* if the graph of the preorder

$$G(\leqslant) = \{(x, y): x \leqslant y\},$$

is closed. A subset $S \subset E$, is called *increasing or upper* if $i(S) = S$ and *decreasing or lower* if $d(S) = S$. It is called *monotone* if it is increasing or decreasing. A subset C is convex if it is the intersection of a decreasing and an increasing set in which case it follows $C = d(C) \cap i(C)$. In this work it is understood that the set inclusion is reflexive, $S \subset S$.

A topological preordered space is a *normally preordered space* if it is semiclosed preordered and for every closed decreasing set A and closed increasing set B which are disjoint, $A \cap B = \emptyset$, it is possible to find an open decreasing set U and an open increasing set V which separate them, namely $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

A topological preordered space is a *regularly preordered space* if it is semiclosed preordered, (a) for every closed decreasing set A and closed increasing set B of the form $B = i(x)$ which are disjoint, $A \cap B = \emptyset$, it is possible to find an open decreasing set U and an open increasing set V which separate them, namely $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$, and (b) for every closed decreasing set A of the form $A = d(x)$ and closed increasing set B which are disjoint, $A \cap B = \emptyset$, it is possible to find an open decreasing set U and an open increasing set V which separate them, namely $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

An *isotone* function is a function $f : E \to \mathbb{R}$ such that $x \leqslant y \Rightarrow f(x) \leqslant f(y)$.

In a normally preordered space if A is closed decreasing, B is closed increasing and $A \cap B = \emptyset$, there is some continuous isotone function $f : E \to [0, 1]$, such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$ (this is the preorder analog of Urysohn separation lemma, see [5, Theor. 1]). Normally preordered spaces are closed preordered spaces, and closed preordered spaces are semiclosed preordered spaces.

A topological preordered space is *convex* at $x \in E$, if for every open neighborhood $O \ni x$, there are an open decreasing set U and an open increasing set V such that $x \in U \cap V \subset O$. A topological preordered space E is *convex* if it is convex at every point [5–7]. Notice that according to this terminology the statement “the topological preordered space E is convex” differs from the statement “the subset E is convex” (which is always true). The terminology is not uniform in the literature, for instance Lawson [8] calls *strongly order convexity* what we call convexity.

A quasi-uniformity [5,9] is a pair $(X, \mathcal{U})$ such that $\mathcal{U}$ is a filter on $X \times X$, whose elements contain the diagonal $\Delta = \{(x, y): x = y\}$, and such that if $V \in \mathcal{U}$ then there is $W \subset \mathcal{U}$, such that $W \circ W \subset V$. A quasi-uniformity is a uniformity if $V \in \mathcal{U}$ implies $V^{-1} \in \mathcal{U}$. To any quasi-uniformity $\mathcal{U}$ corresponds a dual quasi-uniformity $\mathcal{U}^{-1} = \{U: U^{-1} \in \mathcal{U}\}$.

From a quasi-uniformity $\mathcal{U}$ it is possible to construct a topology $\mathscr{T}(\mathcal{U})$ in such a way that a base for the filter of neighborhoods at x is given by the sets of the form $U(x)$ where $U(x) = \{y: (x, y) \in U\}$ with $U \in \mathcal{U}$. In other words $O \in \mathscr{T}(\mathcal{U})$ if for every $x \in O$ there is $U \in \mathcal{U}$ such that $U(x) \subset O$.

Given a quasi-uniformity $\mathcal{U}$ the family $\mathcal{U}^*$ given by the sets of the form $V \cap W^{-1}$, $V, W \in \mathcal{U}$, is the coarsest uniformity containing $\mathcal{U}$. The symmetric topology of the quasi-uniformity is $\mathscr{T}(\mathcal{U}^*)$. Moreover, the intersection $\bigcap \mathcal{U}$ is the graph of a preorder on X (see [5]), thus given a quasi-uniformity one naturally obtains a topological preordered space $(X, \mathscr{T}(\mathcal{U}^*), \bigcap \mathcal{U})$. The topology $\mathscr{T}(\mathcal{U}^*)$ is Hausdorff if and only if the preorder $\bigcap \mathcal{U}$ is an order [5].

A *completely regularly preordered* space (Tychonoff-preordered space), is a topological preordered space for which the following two conditions hold:

(i) $\mathscr{T}$ coincides with the initial topology generated by the set of continuous isotone functions $g : E \to [0, 1]$,
(ii) $x \leqslant y$ if and only if for every continuous isotone function $f : E \to [0, 1]$, $f(x) \leqslant f(y)$.

Convex normally preordered spaces are completely regularly preordered spaces, and completely regularly preordered spaces are convex closed preordered spaces [5]. Nachbin proves [5, Prop. 8] that a topological preordered space $(E, \mathscr{T}, \leqslant)$ comes from a quasi-uniformity $\mathcal{U}$, in the sense that $\mathscr{T} = \mathscr{T}(\mathcal{U}^*)$ and $G(\leqslant) = \bigcap \mathcal{U}$ if and only if it is a completely regularly preordered space, and proves that every Hausdorff quasi-uniformizable space admits a closed order compactification (the Nachbin compactification).

For the discrete preorder $G(\leqslant) = \Delta$, the definitions of normally preordered space, completely regularly preordered space, regularly preordered space, closed preordered space, reduce respectively to normal space, completely regular space, regular space (T_3-space), Hausdorff space.

A *bitopological space* is a triple $(E, \mathscr{P}, \mathscr{Q})$ where E is a set and $\mathscr{P}, \mathscr{Q}$, are two topologies on E. It is possible to associate to every topological preordered space a bitopological space by taking as $\mathscr{P}$ the topology made of all the upper sets $\mathscr{T}^\sharp$, and as $\mathscr{Q}$ the topology made of all the lower sets $\mathscr{T}^\flat$. Bitopological spaces were introduced by Kelly [10] and subsequently investigated in [11,12].

A *quasi-pseudo-metric* [10,11] on a set X is a function $p : X \times X \to [0, +\infty)$ such that for $x, y, z \in X$

(i) $p(x, x) = 0$,
(ii) $p(x, z) \leqslant p(x, y) + p(y, z)$.

The quasi-pseudo-metric is called *quasi-metric* [13] if (i) is replaced with (i′): $p(x, y) = 0$ iff $x = y$. Other variations exist in the literature. For instance, if (i) is replaced by (i″) $p(x, y) = p(y, x) = 0$ iff $x = y$, we get what is sometimes referred to as *Albert's quasi-metric* [14].

The quasi-pseudo-metric is called *pseudo-metric* if $p(x, y) = p(y, x)$. If a quasi-metric is such that $p(x, y) = p(y, x)$ then it is a *metric* in the usual sense. Sometimes quasi-pseudo-metrics are called *semi-metrics* [5] but for other authors semi-metrics are quite different objects [15]. If p is a quasi-pseudo-metric then q, defined by

$$q(x, y) = p(y, x),$$

is a quasi-pseudo-metric called *conjugate* of p. This structure, called *quasi-pseudo-metric space*, is denoted (X, p, q) and we might equivalently use the notation p^{-1} for q.

From a quasi-pseudo-metric space (E, p, q) we can construct a quasi-uniformity $\mathcal{U}$ and the associated topological preordered space $(E, \mathscr{T}(\mathcal{U}^*), \bigcap \mathcal{U})$ following Nachbin [5], or a bitopological space $(X, \mathscr{P}, \mathscr{Q})$ following Kelly [10].

Nachbin defines the quasi-uniformity $\mathcal{U}$ as the filter generated by the countable base

$$W_n = \{(x, y) \in X \times X\colon\ p(x, y) < 1/n\}, \tag{1}$$

thus the graph of the preorder is $G(\leqslant) = \bigcap \mathcal{U} = \{(x, y)\colon\ p(x, y) = 0\}$ and the topology $\mathscr{T}(\mathcal{U}^*)$ is that of the pseudo-metric $p + q$. In particular this topology is Hausdorff if and only if $p + q$ is a metric i.e. $p(x, y) + p(y, x) = 0 \Rightarrow x = y$, which is the case if and only if the preorder $\leqslant$ is an order. Clearly, every topological preordered space obtained in this way is a completely regularly preordered space as it comes from a quasi-uniformity.

Remark 1.1. Clearly, the pseudo-metrics $p \vee q$, $p + q$, $(p^2 + q^2)^{1/2}$, induce the same topology. Nevertheless, we shall preferably use $p + q$ because in the proof of Theorem 3.4 we shall take advantage of the linearity of this expression. The choice $d = p + q$ has also been made by Kelly [10, p. 87].

Given a quasi-pseudo-metric p we shall denote $P(x, r) = \{y\colon\ p(x, y) < r\}$ the p-ball of radius r centered at x, and analogously $Q(x, r) = \{y\colon\ q(x, y) < r\}$ will be the q-ball for the conjugate metric q. If $d = p + q$, the d-balls will be denoted $D(x, r) = \{y\colon\ d(x, y) < r\}$.

From a quasi-pseudo-metric space (E, p, q) Kelly constructs a bitopological space $(X, \mathscr{P}, \mathscr{Q})$ as follows: $\mathscr{P}$ is the topology having as base the sets of the form $P(x, r)$ for arbitrary $x \in X$ and $r > 0$. Analogously, $\mathscr{Q}$ is the topology having as base the sets of the form $Q(x, r)$ for arbitrary $x \in X$ and $r > 0$.

Remark 1.2. A base of open neighborhoods at $z \in X$ is given by the sets of the form $P(z, \epsilon)$, $\epsilon > 0$. Indeed, if $z \in \{y\colon\ p(x, y) < r\}$, that is $p(x, z) < r$, then there is ϵ such that $\{w\colon\ p(z, w) < \epsilon\} \subset \{y\colon\ p(x, y) < r\}$. This follows from $p(x, w) \leqslant p(x, z) + p(z, w)$, as choosing $\epsilon = r - p(x, z)$, we get $p(x, w) < r$.

2. Quasi-pseudo-metrizability and preorders

We give and motivate the following definitions.

Definition 2.1. A topological preodered space $(E, \mathscr{T}, \leqslant)$ is *quasi-pseudo-metrizable* if there is a pair of conjugate quasi-pseudo-metrics p, q, said *admissible*, such that $\mathscr{T}$ is the topology generated by the pseudo-metric $p + q$, and the graph of the preorder is given by $G(\leqslant) = \{(x, y)\colon\ p(x, y) = 0\}$.

A topological preodered space $(E, \mathscr{T}, \leqslant)$ is *strictly quasi-pseudo-metrizable* if it is convex semiclosed preordered and there is a pair of conjugate quasi-pseudo-metrics p, q, such that the topology associated to p is the upper topology $\mathscr{T}^{\sharp}$, and the topology associated to q is the lower topology $\mathscr{T}^{\flat}$.

A (strictly) quasi-pseudo-metrizable preordered space is a *(strictly) quasi-pseudo-metrized preordered space* if a choice of conjugate metrics complying with the previous requirement is made.

It must be noted that we call *strictly quasi-pseudo-metrizable space* what, taking as reference the literature on bitopological spaces, one would simply call *quasi-pseudo-metrizable space*. The point is that in the topological preordered space version of a topological property one has usually two or more possibilities and the stronger can often be interpreted as the bitopological version of the property. For instance, Lawson [8] defines the *strictly completely regularly ordered spaces* which admit the bitopological interpretation of *pairwise complete regularity* [16], in contrast with Nachbin's completely regularly ordered spaces which do not admit a bitopological interpretation.

Proposition 2.2. *Let $(E, \mathscr{T}, \leqslant)$ be quasi-pseudo-metrizable preordered space and let p, q be a pair of admissible conjugate quasi-pseudo-metrics. The function $p : E \times E \to \mathbb{R}$ is continuous in the product topology $\mathscr{T} \times \mathscr{T}$ on E. Moreover, it is Lipschitz with respect to $d = p + q$, in the sense that*

$$\left|p(x, y) - p(w, z)\right| \leqslant d(x, w) + d(y, z). \tag{2}$$

(This inequality holds also for d replaced by $p \vee q$ or $(p^2+q^2)^{1/2}$.) For a fixed $x \in E$, the function $q(x,\cdot)$ is isotone and the function $p(x,\cdot) = q(\cdot, x)$ is anti-isotone.

Proof. The repeated application of the triangle inequality gives

$$p(x,y) \leqslant p(x,w) + p(w,z) + p(z,y) \leqslant d(x,w) + p(w,z) + d(y,z),$$
$$p(w,z) \leqslant p(w,x) + p(x,y) + p(y,z) \leqslant d(x,w) + p(x,y) + d(y,z),$$

thus $|p(x,y) - p(w,z)| \leqslant d(x,w) + d(y,z)$. By assumption, d generates $\mathscr{T}$ thus p is continuous in the product topology $\mathscr{T} \times \mathscr{T}$.

If $y \leqslant z$ then $p(y,z) = q(z,y) = 0$ and $q(x,y) \leqslant q(x,z) + q(z,y) = q(x,z)$ namely $q(x,\cdot)$ is isotone. If $y \leqslant z$ then $p(y,z) = 0$ and $p(x,z) \leqslant p(x,y) + p(y,z) = p(x,y)$ namely $p(x,\cdot)$ is anti-isotone. □

Proposition 2.3. *Every strictly quasi-pseudo-metrizable preordered space is a quasi-pseudo-metrizable preordered space. Every quasi-pseudo-metrizable preordered space is a completely regularly preordered space.*

Proof. Eq. (2) can be obtained as in the proof of Proposition 2.2 and written $|p(x,y) - p(w,z)| \leqslant p(x,w) + q(x,w) + p(y,z) + q(y,z)$, from which it follows that p is continuous in the product topology $\sup(\mathscr{T}^\sharp, \mathscr{T}^\flat) \times \sup(\mathscr{T}^\sharp, \mathscr{T}^\flat)$ (and analogously for q). Thus $p+q$ is $\sup(\mathscr{T}^\sharp, \mathscr{T}^\flat) \times \sup(\mathscr{T}^\sharp, \mathscr{T}^\flat)$-continuous, which implies that the topology $\mathscr{D}$ of the pseudo-metric $d = p+q$ is coarser than $\sup(\mathscr{T}^\sharp, \mathscr{T}^\flat)$. However, since $p, q \leqslant d$ the p-balls and q-balls centered at a point are $\mathscr{D}$-neighborhoods of the point, thus by Remark 1.2, $\sup(\mathscr{T}^\sharp, \mathscr{T}^\flat)$ is coarser than $\mathscr{D}$, thus $\mathscr{D} = \sup(\mathscr{T}^\sharp, \mathscr{T}^\flat)$. Clearly, $\sup(\mathscr{T}^\sharp, \mathscr{T}^\flat)$ is coarser than $\mathscr{T}$, but since E is convex, $\sup(\mathscr{T}^\sharp, \mathscr{T}^\flat) = \mathscr{T}$ which implies $\mathscr{D} = \mathscr{T}$.

Since $(E, \mathscr{T}, \leqslant)$ is semiclosed preordered, $i(x)$ is closed thus $i(x) = cl_{\mathscr{T}^\flat} x$. It follows that $y \in i(x)$ iff every q-ball centered at y includes x which is equivalent to "for all $n \geqslant 1$, $x \in \{w : q(y,w) < 1/n\}$", which in turn is equivalent to $p(x,y) = 0$. We conclude that $y \in i(x)$ iff $p(x,y) = 0$.

For the last statement, every quasi-pseudo-metrizable topological preordered space comes from a quasi-uniformity and hence is a completely regularly preordered space. □

The problem of quasi-pseudo-metrization of a bitopological space was considered already in Kelly's work [10] and has been extensively studied over the years [17–24]. As we shall see in a moment, the solution to this problem can be used to infer results on the quasi-pseudo-metrizability of a topological preordered space. The quasi-pseudo-metrizability of a *topological* space has also been investigated [25–29] but it is less interesting for our purposes because just one topology cannot bring information on a non-trivial preorder.

For bitopological spaces Kelly [10, Theor. 2.8] obtained a generalization of Urysohn's metrization theorem which in our topological preordered space framework reads as follows.

Theorem 2.4 *(Kelly). Let $(E, \mathscr{T}, \leqslant)$ be a convex regularly preordered space and assume that both $\mathscr{T}^\sharp$ and $\mathscr{T}^\flat$ are second countable, then $(E, \mathscr{T}, \leqslant)$ is strictly quasi-pseudo-metrizable.*

Unfortunately, this theorem is not so easily applied to topological preordered spaces because the second countability of $\mathscr{T}$ does not imply the second countability of the coarser topologies $\mathscr{T}^\sharp$ and $\mathscr{T}^\flat$. This type of difficulty is met for the various quasi-pseudo-metrizability results that can be found in the literature on bitopological spaces. Nevertheless, we shall show that by weakening the thesis it is indeed possible to prove

Theorem 2.5. *The following conditions are equivalent for a topological preordered space $(E, \mathscr{T}, \leqslant)$*

(a) *$(E, \mathscr{T}, \leqslant)$ is a second countable completely regularly preordered space,*
(b) *$(E, \mathscr{T}, \leqslant)$ is separable and quasi-pseudo-metrizable.*

Let us recall that in a pseudo-metric space, separability, second countability and the Lindelöf property are equivalent [15, Theor. 16.11].

Suppose that $(E, \mathscr{T}, \leqslant)$ is separable and quasi-pseudo-metrizable. Then as $\mathscr{T}$ is induces from the pseudo-metric $p+q$, $(E, \mathscr{T})$ is second countable, and by Nachbin's characterization cited above (paragraph of Eq. (1)), $(E, \mathscr{T}, \leqslant)$ is completely regularly preordered. Thus we have proved (b) ⇒ (a), and it remains to prove (a) ⇒ (b). To this end, we shall make use of the following result due to Nachbin [5, Theor. 8], which generalizes the well-known metrization theorem of Alexandroff and Urysohn.

Theorem 2.6 *(Nachbin). A quasi-uniformizable preordered space (i.e. completely regularly preordered space) comes from a quasi-pseudo-metric if and only if the quasi-uniformity admits a countable base.*

Given a preorder $\leqslant$ we obtain an equivalence relation $\sim$ through "$x \sim y$ if $x \leqslant y$ and $y \leqslant x$". In the next proof $E/\sim$ is the quotient space, $\mathscr{T}/\sim$ is the quotient topology, and $\lesssim$ is defined by, $[x] \lesssim [y]$ if $x \leqslant y$ for some representatives. The quotient preorder is by construction an order. The triple $(E/\sim, \mathscr{T}/\sim, \lesssim)$ is a topological ordered space and $\pi : E \to E/\sim$ is the continuous quotient projection.

Proof of Theorem 2.5, (a) ⇒ (b). As a first step let us show that there is a countable family $\mathcal{C}$ of continuous isotone functions $c_k : E \to [0, 1]$, $k \geqslant 1$, such that $x \leqslant y$ if and only if $\forall k$, $c_k(x) \leqslant c_k(y)$. Indeed, defined for every continuous isotone function c, $U_c = \{(x, y)\colon c(x) \leqslant c(y)\}$, we have by complete preorder regularity $G(\leqslant) = \bigcap_c U_c$. Note that since c is continuous U_c is closed in the product topology. But E is second countable thus $E \times E$ is second countable and hence any arbitrary intersection of closed sets can be reduced to a countable intersection [32, p. 180]. Therefore, there is a countable family $\mathcal{C}$ of continuous isotone functions c_k such that $G(\leqslant) = \bigcap_{c \in \mathcal{C}} U_c$ which is the thesis.

Since $(E, \mathscr{T}, \leqslant)$ is completely regularly preordered it is convex, thus by [33, Prop. 2.3] every open set is saturated with respect to π, namely if $O \in \mathscr{T}$ then $\pi^{-1}(\pi(O)) = O$, which implies that π is open (actually a quasi-homeomorphism). Since $(E, \mathscr{T})$ is second countable and π is open, we have that $(E/\sim, \mathscr{T}/\sim)$ is second countable.

Since $(E, \mathscr{T}, \leqslant)$ is a completely regularly preordered space then $(E/\sim, \mathscr{T}/\sim, \lesssim)$ is a completely regularly ordered space (immediate from the definitions) hence closed ordered which implies that $(E/\sim, \mathscr{T}/\sim)$ is Hausdorff. But again, since $(E/\sim, \mathscr{T}/\sim, \lesssim)$ is a completely regularly ordered space, $(E/\sim, \mathscr{T}/\sim)$ is Tychonoff. By Urysohn's theorem $(E/\sim, \mathscr{T}/\sim)$ is metrizable with a metric $\tilde{\rho}$.

Now, the strategy is to construct the quasi-uniformity as the weak quasi-uniformity $\mathcal{W}$ of a countable family $\mathcal{P}$ of continuous isotone functions with values in $[0, 1]$. Let us recall that if $f : E \to [0, 1]$ is a function then the sets $\{(x, y)\colon f(x) - f(y) < 1/k\}$ for every natural $k \geqslant 1$ form a (countable) base for a quasi-uniformity on E. If $\mathcal{P}$ counts more than one function then $\mathcal{W}$ is given by the smallest filter containing all the single quasi-uniformities. The quasi-uniformity $\mathcal{W}$ admits a countable base if $\mathcal{P}$ is countable, indeed a base is given by all the finite intersections of the base elements generating the single function quasi-uniformities.

As a first step we include the family $\mathcal{C}$ into $\mathcal{P}$, in this way we obtain that $\bigcap \mathcal{W} = G(\leqslant)$ and that this equation cannot be spoiled by the inclusion in $\mathcal{P}$ of arbitrary families of continuous isotone functions. Therefore, we have only to show that we can find a countable family $\mathcal{Q}$ of continuous isotone functions on E with values in $[0, 1]$, such that the weak quasi-uniformity of that family induces a topology as fine as $\mathscr{T}$ (since every (anti)isotone function on E passes to the quotient, with some abuse of notation, we will denote in the same way the functions on E or on $E/\sim$). Let $\tilde{\rho}$ be the metric on $E/\sim$ mentioned above. For every $n \geqslant 1$ we consider a covering on $E/\sim$ by open $\tilde{\rho}$-balls of radius $1/n$, then for every point $[x] \in E/\sim$ we construct a pair of functions $f^{(n)}_{[x]}, g^{(n)}_{[x]} : E/\sim \to [0, 1]$, the former continuous and isotone and the latter continuous and anti-isotone such that $f^{(n)}_{[x]}([x]) = g^{(n)}_{[x]}([x]) = 1$ and $\min(f^{(n)}_{[x]}, g^{(n)}_{[x]})([y]) = 0$ whenever $\tilde{\rho}([x], [y]) \geqslant 1/n$ (they exist by definition of completely regularly ordered space). The open sets $V^{(n)}([x]) = (E/\sim)\backslash\{[y]\colon \min(f^{(n)}_{[x]}, g^{(n)}_{[x]})([y]) = 0\}$ give an open covering of $E/\sim$ and each of these sets is contained in an open ball of radius $1/n$. By the Lindelöf property implied by second countability [15, Theor. 16.9] there is a countable subcovering which corresponds to points $[x^{(n)}_i]$ and functions $f^{(n)}_{[x^{(n)}_i]}, g^{(n)}_{[x^{(n)}_i]}$. We add for each n and i the (lifted) continuous isotone functions $f^{(n)}_{[x^{(n)}_i]}$ and $1 - g^{(n)}_{[x^{(n)}_i]}$ to $\mathcal{P}$ in such a way that the weak quasi-uniformity $\mathcal{W}$ satisfies $\mathscr{T} = \mathscr{T}(\mathcal{W}^*)$.

Indeed, if $O \ni x$, with $O \in \mathscr{T}$ then $\pi(O) \ni [x]$ and we have already proved that $\pi(O) \in \mathscr{T}/\sim$ and $\pi^{-1}(\pi(O)) = O$. There is some n such that the open $\tilde{\rho}$-ball of radius $2/n$ centered at $[x]$ is contained in $\pi(O)$. Since the sets $\{V^{(n)}([x^{(n)}_i])\}_i$ give a covering there is some i such that $[x] \in V^{(n)}([x^{(n)}_i]) \subset \pi(O)$, where the inclusion follows from the fact that the set $V^{(n)}([x^{(n)}_i])$ is contained in a $\tilde{\rho}$-ball of radius $1/n$. In particular, $f^{(n)}_{[x^{(n)}_i]}(x) > 0$ and $g^{(n)}_{[x^{(n)}_i]}(x) > 0$. Let $j \geqslant 1$ be such that $f^{(n)}_{[x^{(n)}_i]}(x) > 1/j$ and $g^{(n)}_{[x^{(n)}_i]}(x) > 1/j$ and let $U \cap V^{-1} \in \mathcal{W}^*$ be given by

$$U = \bigl\{(x, y)\colon f^{(n)}_{[x^{(n)}_i]}(x) - f^{(n)}_{[x^{(n)}_i]}(y) < 1/j\bigr\},$$

$$V = \bigl\{(x, y)\colon \bigl(1 - g^{(n)}_{[x^{(n)}_i]}(x)\bigr) - \bigl(1 - g^{(n)}_{[x^{(n)}_i]}(y)\bigr) < 1/j\bigr\},$$

to which corresponds a neighborhood of x in the topology $\mathscr{T}(\mathcal{W}^*)$ given by $(U \cap V^{-1})(x) = \{y\colon f^{(n)}_{[x^{(n)}_i]}(x) - f^{(n)}_{[x^{(n)}_i]}(y) < 1/j$ and $g^{(n)}_{[x^{(n)}_i]}(x) - g^{(n)}_{[x^{(n)}_i]}(y) < 1/j\} \subset \{y : f^{(n)}_{[x^{(n)}_i]}(y) > 0$ and $g^{(n)}_{[x^{(n)}_i]}(y) > 0\} = \pi^{-1}(V^{(n)}([x^{(n)}_i])) \subset O$. This last inclusion proves that $\mathscr{T} = \mathscr{T}(\mathcal{W}^*)$.

We have shown that $(E, \mathscr{T}, \leqslant)$ is quasi-uniformizable where the quasi-uniformity admits a countable base thus $(E, \mathscr{T}, \leqslant)$ is quasi-pseudo-metrizable. □

It should be noted that in (a) ⇒ (b) we do not assume that E is regularly preordered. This does not mean that the assumption is stronger than expected because a completely regularly preordered space need not be regularly preordered [30, Example 1]. This is a crucial difference with respect to the usual discrete-preorder version.

We do not use preorder regularity in Theorem 2.5 because this condition is not necessary in order to obtain (b), namely a separable quasi-pseudo-metrizable space need not be regularly preordered. An example has been given in [30, Example 1]. This example shows also that there are separable quasi-pseudo-metrizable spaces which are not strictly quasi-pseudo-metrizable. Indeed, the latter spaces are perfectly normally preordered because of a result due to Patty [11, Theor. 2.3] and hence regularly preordered.

A comparison with the discrete-preorder version is clarified by the following result.

Theorem 2.7. *Every second countable convex regularly preordered space is a completely regularly preordered space.*

Proof. In [31, Theor. 5.3] it has been proved that every second countable regularly preordered space is (perfectly) normally preordered. Since every convex normally preordered space is a completely regularly preordered space the thesis follows. □

Lemma 2.8. *Let $(E, \mathscr{T}, \leqslant)$ be a second countable completely regularly preordered space, then there is a countable family $\mathcal{F}$ of continuous isotone functions, $k \geqslant 1$, $f_k : E \to [0, 1]$ such that* (i) *$\mathscr{T}$ is the initial topology generated by $\mathcal{F}$, and* (ii) *$x \leqslant y$ if and only if for every $k \geqslant 1$, $f_k(x) \leqslant f_k(y)$.*

Proof. An inspection of the proof of Theorem 2.5 shows that we have already proved that there is a countable family $\mathcal{P}$ of continuous isotone functions, $k \geqslant 1$, $f_k : E \to [0, 1]$ such that (i) $\mathscr{T}$ is the initial topology generated by $\mathcal{P}$, and (ii) $x \leqslant y$ if and only if for every $k \geqslant 1$, $f_k(x) \leqslant f_k(y)$. □

3. The ordered Hilbert cube

In this section we investigate the ordered Hilbert cube and its connection with (strict) quasi-pseudo-metrization.

Theorem 3.1. *The property of being a quasi-pseudo-metrizable preordered space is hereditary.*

Proof. Assume E is quasi-pseudo-metrizable and let p, q be a pair of conjugate quasi-pseudo-metrics. Let S be a subspace then $x \leqslant_S y$ if and only if $p_S(x, y) = 0$ where $p_S = p|_{S\times S}$. Furthermore the induced topology $\mathscr{T}_S$ has a base of neighborhoods given by the d-balls intersected with S, $d = p + q$, thus by the d_S-balls, where $d_S = p_S + q_S = d|_{S\times S}$. □

In general it is not true that every open increasing (decreasing) set on the subspace S is the intersection of an open increasing (resp. decreasing) set on E with S. If this is the case S is called a *preorder subspace* [34,35,16]. In a closed preordered space every compact subspace S is a preorder subspace [31, Prop. 2.6].

Theorem 3.2. *The property of being a strictly quasi-pseudo-metrizable preordered space is hereditary with respect to preorder subspaces.*

Proof. It is well known that convexity and the semiclosed preordered space property are hereditary. For the remainder of the proof it suffices to define $p_S = p|_{S\times S}$, $q_S = q|_{S\times S}$ where S is a preorder subspace. Indeed, if $V \subset S$ is open increasing in S, there is V' open increasing in E such that $V = V' \cap S$. Let $x \in V$ then there is some $\epsilon > 0$ such that $P(x, \epsilon) \subset V'$ which implies $P_S(x, \epsilon) \subset V'$, where $P_S(x, \epsilon)$ is the p_S-ball of radius ϵ centered at x. The proof in the decreasing case is analogous. □

Lemma 3.3. *If $(E, \mathscr{T}, \leqslant)$ is a quasi-pseudo-metrizable preordered space, then it admits a quasi-pseudo-metric bounded by* 1. *If $(E, \mathscr{T}, \leqslant)$ is a strictly quasi-pseudo-metrizable preordered space, then it admits a quasi-pseudo-metric bounded by* 1 (*in the sense of strict quasi-pseudo-metric spaces i.e. it generates $\mathscr{T}^\sharp$ with the conjugate that generates $\mathscr{T}^\flat$*).

Proof. The function $h : [0, +\infty) \to [0, 1]$, $h(a) = \min(a, 1)$, is non-decreasing and sublinear, $h(a + b) \leqslant h(a) + h(b)$. If p is a quasi-pseudo-metric then $p_1 = h(p)$ satisfies the triangle inequality by the sublinearity of h and satisfies also $p_1(x, x) = h(p(x, x)) = h(0) = 0$ thus it is a quasi-pseudo-metric. Defined $q_1(x, y) = p_1(y, x) = h(q(x, y))$, we have that $d_1 = p_1 + q_1$ is a pseudo-metric which generates the same topology of $d = p + q$ (they have the same balls with radius smaller than 1) and furthermore, $p_1(x, y) = 0$ iff $p(x, y) = 0$.

The proof in the strict case is similar. The quasi-pseudo-metrics p_1 and q_1 are defined in the same way from p and q, since p_1 shares with p the same balls of radius less than 1, and since q_1 shares with q the same balls of radius less than 1, the thesis follows. □

Let $(E_n, \mathscr{T}_n, \leqslant_n)$, $n \in \mathbb{N}$, be topological preordered spaces and let $(E, \mathscr{T}, \leqslant)$ be the topological preordered space in which $(E, \mathscr{T})$ is the product space $E = \prod_{n\in\mathbb{N}} E_n$ endowed with the product topology and $\leqslant$ is the product preorder: $x \leqslant y$ if for all $n \in \mathbb{N}$, $x_n \leqslant_n y_n$. We have the following

Theorem 3.4. *The product topological preordered space $E=\prod_n E_n$ is quasi-pseudo-metrizable if and only if each E_n is quasi-pseudo-metrizable.*

Proof. If E is quasi-pseudo-metrizable E_n is quasi-pseudo-metrizable because it is preorder homeomorphic with a subset S of E obtained by fixing all the coordinates x_k of x to some value in E_k but for $k=n$. One can then use Theorem 3.1.

For the converse, let $p_n: E_n\times E_n\to[0,1]$ be a quasi-pseudo-metric for E_n bounded by 1 and endow E with the quasi-pseudo-metric

$$p(x,y)=\sum_{n=1}^{\infty} p_n(x_n,y_n)/2^n.$$

The proof that p is a quasi-pseudo-metric is straightforward. Let $q(x,y)=p(y,x)$ and analogously for q_n, $n\in\mathbb{N}$. The pseudo-metric $d=p+q$ reads $d(x,y)=\sum_{n=1}^{\infty} d_n(x_n,y_n)/2^n$ where $d_n=p_n+q_n$ is the pseudo-metric which generates the topology $\mathscr{T}_n$. According to [36, Theor. 14, Chap. 4] d generates the product topology $\mathscr{T}$. Finally, $p(x,y)=0$ if and only if for all $n\in\mathbb{N}$, $p_n(x_n,y_n)=0$ which is equivalent to: for all $n\in\mathbb{N}$, $x_n\leqslant_n y_n$, that is, $x\leqslant y$. □

One must be careful in trying to generalize the previous theorem to the strict case. It is known that the countable product of quasi-pseudo-metrizable spaces in the bitopological sense is quasi-pseudo-metrizable in the bitopological sense [10,17]. This fact does not imply the existence of a simple corresponding theorem in the strict quasi-pseudo-metrization case for topological preordered spaces. The reason is that the product bitopology can be different from the bitopology induced by the product topology and product preorder.

For I-spaces [34] (compare [35]), namely for topological preordered spaces for which the increasing and decreasing hulls of open sets are open, it is possible to prove a useful strict case generalization.

Theorem 3.5. *If the product topological preordered space $E=\prod_n E_n$ is strictly quasi-pseudo-metrizable, then each factor E_n is strictly quasi-pseudo-metrizable, furthermore if E is also an I-space then so are the factors E_n. If each factor E_n is a strictly quasi-pseudo-metrizable I-space, then E is a strictly quasi-pseudo-metrizable I-space. Finally, in this last case the upper topology on E is the product of the upper topologies of the factors, and analogously for the lower topology.*

Proof. Each E_n is preorder homeomorphic with a subset S of E obtained by fixing all the coordinates x_k of x to some value in E_k but for $k=n$. The subset S just defined is actually a preorder subspace because if $V\subset S$ is open increasing then (omitting the preorder homeomorphism of S with E_n) $\pi_n^{-1}(V)$ is open increasing and $\pi_n^{-1}(V)\cap S=V$, and analogously for the open decreasing sets. By Theorem 3.2 E is strictly quasi-pseudo-metrizable thus S and hence E_n is strictly quasi-pseudo-metrizable. Furthermore, if O is an open set of E_n then $\pi_n^{-1}(O)$ is an open set of E and if E is an I-space $i(\pi_n^{-1}(O))=\pi_n^{-1}(i_{E_n}(O))$ is open, thus $\pi_n(\pi_n^{-1}(i_{E_n}(O)))=i_{E_n}(O)$ is open because the projection maps are open [15, Theor. 8.6]. The proof that $d_{E_n}(O)$ is open is analogous. We conclude that each E_n is an I-space.

For the converse, let us prove that E is convex. Let O be an open set in the product topology and let $x\in O$. There are open sets $O_{i_1}\subset E_{i_1},\dots,O_{i_s}\subset E_{i_s}$, $x_{i_k}\in O_{i_k}$, such that $\prod_{n=1}^{\infty} W_n\subset O$, where $W_n=O_{i_k}$ if $n=i_k$ for some $1\leqslant k\leqslant s$, or $W_n=E_n$ otherwise. Recalling that each topological preordered space E_i is convex, the sets O_{i_k} can be chosen to be intersections $O_{i_k}=U_{i_k}\cap V_{i_k}$ where U_{i_k} is open decreasing and V_{i_k} is open increasing in E_{i_k}. Evidently defined $U'=\prod_{n=1}^{\infty} Y_n$ where $Y_n=U_{i_k}$ if $n=i_k$ for some $1\leqslant k\leqslant s$, or $Y_n=E_n$ otherwise, and $V'=\prod_{n=1}^{\infty} Z_n$ where $Z_n=V_{i_k}$ if $n=i_k$ for some $1\leqslant k\leqslant s$, or $Z_n=E_n$ otherwise, we have $x\in U'\cap V'\subset\prod_{n=1}^{\infty} W_n\subset O$ which proves that E is convex because U' is open decreasing in E and V' is open increasing in E.

Let us prove that E is semiclosed preordered. Indeed, if $x\in E$, using the definition of product order, $i(x)=\bigcap_n E\backslash\pi_n^{-1}(E_n\backslash i_{E_n}(x_n))$ from which we obtain that $i(x)$ is closed in E because each $i_{E_n}(x_n)$ is closed in E_n. Analogously, $d(x)$ is closed.

Let $p_n: E_n\times E_n\to[0,1]$ be a quasi-pseudo-metric for E_n bounded by 1 and endow E with the quasi-pseudo-metric $p(x,y)=\sum_{n=1}^{\infty} p_n(x_n,y_n)/2^n$. The proof that p is a quasi-pseudo-metric is straightforward. Let $V\subset E$ be an open increasing set and let $x\in V$ then by definition of product topology there are open sets $O_{i_1}\subset E_{i_1},\dots,O_{i_s}\subset E_{i_s}$, $x_{i_k}\in O_{i_k}$, such that defined $G=\prod_{n=1}^{\infty} W_n$, where $W_n=O_{i_k}$ if $n=i_k$ for some $1\leqslant k\leqslant s$, or $W_n=E_n$ otherwise, we have $G\subset V$. The sets $i_{E_{i_k}}(O_{i_k})$ are open and increasing by the I-space assumption. We define the open set on E, $Q=\prod_{n=1}^{\infty} R_n$ where $R_n=i_{E_{i_k}}(O_{i_k})$ if $n=i_k$ for some $1\leqslant k\leqslant s$, or $R_n=E_n$ otherwise. Using the definition of product preorder, $Q=i(G)$ (note that every base element on E has the form of G, as we proved that Q is open, this formula shows, among the other things, that E is an I-space).

By strict quasi-pseudo-metrizability of E_{i_k} there are numbers $r_{i_k}>0$ such that $P_{i_k}(x_{i_k},r_{i_k})\subset i_{E_{i_k}}(O_{i_k})$ where $P_{i_k}(x_{i_k},r_{i_k})$ is a p_{i_k}-ball centered at x_{i_k}. Let ϵ be the minimum of $r_{i_k}/2^{i_k}$ for $k=1,\dots,s$.

Let us prove that $P(x,\epsilon)\subset Q\subset V$. The last inclusion follows from the fact that V is increasing and $G\subset V$. For the former inclusion, if $y\in P(x,\epsilon)$ then $p_{i_k}(x_{i_k},y_{i_k})/2^{i_k}<\epsilon\leqslant r_{i_k}/2^{i_k}$ thus $y_{i_k}\in P_{i_k}(x_{i_k},r_{i_k})\subset i_{E_{i_k}}(O_{i_k})$. If we define $w\in E$ to be that point such that $w_n\in O_n$, $y_n\in i_{E_n}(w_n)$ for $n=i_k$, $k=1,\dots,s$, and $w_n=y_n$ otherwise, we have $y=i(w)$ and $w\in\prod_{n=1}^{\infty} W_n=G$, thus $y\in i(G)=Q$ which is the thesis.

The inclusion $P(x,\epsilon) \subset V$ proves that p generates $\mathscr{T}^\sharp$. The proof that q generates $\mathscr{T}^\flat$ is analogous.

The inclusion $Q \subset V$ proves that $\mathscr{T}^\sharp$ coincides with the product of the upper topologies on E_n. Analogously, the product of the lower topologies on E_n gives $\mathscr{T}^\flat$. □

The canonical quasi-pseudo-metric for the real line $\mathbb{R}$ with the usual order is $m(x, y) = \max(x - y, 0)$. With this choice $\mathbb{R}$ becomes a strict quasi-pseudo-metric I-space. The interval $[0,1]$ is a preorder subspace of the real line, thus the quasi-pseudo-metric on $\mathbb{R}$ induces on the interval $[0,1]$ a quasi-pseudo-metric which is bounded by 1, and which makes $[0,1]$ a strict quasi-pseudo-metrized space which is actually an I-space. From the previous theorem we get

Proposition 3.6. *The Hilbert cube $H = [0,1]^{\mathbb{N}}$ once endowed with the product topology and the product order is a strict quasi-pseudo-metric ordered I-space with quasi-pseudo-metric $p(x, y) = \sum_{n=1}^{\infty} \max(x_n - y_n, 0)/2^n$.*

Theorem 3.7. *The following conditions are equivalent for a topological ordered space $(E, \mathscr{T}, \leqslant)$*

(a) *$(E, \mathscr{T}, \leqslant)$ is a second countable completely regularly ordered space,*
(b) *$(E, \mathscr{T}, \leqslant)$ is order embeddable in the ordered Hilbert cube H.*

Proof. (b) ⇒ (a). Since H is the countable product of Hausdorff second countable spaces it is second countable [15, Theor. 16.2]. As E is homeomorphic to a subset S of H it is second countable. Furthermore, the subspace S is quasi-pseudo-metrizable because this property is hereditary and hence it is a completely regularly ordered space. As E is order homeomorphic with S the thesis follows.

(a) ⇒ (b). Let $\mathcal{F}$ be the family of continuous isotone functions $f_k : E \to [0,1]$ given by Lemma 2.8. They separate points because if it were $f_k(x) = f_k(y)$ for all k, then we would infer from $f_k(x) \leqslant f_k(y)$ for all k, $x \leqslant y$, and from $f_k(y) \leqslant f_k(x)$ for all k, $y \leqslant x$, from which it follows $x = y$. By the embedding Lemma [15, Theor. 8.12] the function $f : E \to H$ whose components are the functions $f_k : E \to [0,1]$, is an embedding. Actually, it is a preorder embedding because $x \leqslant y$ if and only if for all k, $f_k(x) \leqslant f_k(y)$, which is equivalent to $f(x) \leqslant f(y)$, as H is endowed with the product order. □

At least in the compact case it is possible to infer that the topological preordered space is strictly quasi-pseudo-metrizable through the following

Theorem 3.8. *Every compact quasi-pseudo-metrized preordered space is a strictly quasi-pseudo-metrized preordered space (with the same quasi-pseudo-metric).*

Proof. We already know that the p-balls $P(x,r) = \{y\colon\ p(x, y) < r\}$ are increasing and open because of the continuity of p. We have to prove that every open increasing set is the union of p-balls. The proof for the decreasing case will be analogous. Let V be an open increasing set and let $x \in V$, we have $V \supset i(x) = \{y\colon\ p(x, y) = 0\} = \bigcap_{i=1}^{\infty} C_i$, $C_i = \{y\colon\ p(x, y) \leqslant 1/i\}$, where C_i are closed sets. Thus $\emptyset = (M\backslash V) \cap (\bigcap_{i=1}^{\infty} C_i) = \bigcap_{i=1}^{\infty}[(M\backslash V) \cap C_i]$, but the sets $(M\backslash V) \cap C_i$ are closed, compact and, if non-empty, they satisfy the finite intersection property which contradicts the previous empty intersection [36, Theor. 1, Chap. 5]. Thus some of them must be empty, that is for some i, $C_i \subset V$, which reads $P(x, 1/i) \subset V$. □

Proposition 3.9. *Let $(E, \mathscr{T}, \leqslant)$ be a strictly quasi-pseudo-metrizable preordered space which is separable then on E the topologies $\mathscr{T}$, $\mathscr{T}^\sharp$ and $\mathscr{T}^\flat$ are second countable.*

Proof. Separability with respect to one topology implies separability with respect to any coarser topology thus E is separable with respect to $\mathscr{T}^\sharp$ and $\mathscr{T}^\flat$. The function $d = p + q$ is a pseudo-metric compatible with the topology $\mathscr{T}$. By [15, Theor. 16.11], separability with respect to $\mathscr{T}$ implies second countability of $\mathscr{T}$. Let us prove the second countability of $\mathscr{T}^\sharp$, the proof for $\mathscr{T}^\flat$ being similar. Let $\{c_1, c_2, \ldots\}$ be a countable set which is dense according to $\mathscr{T}$ and define

$$U_{nm} = \{x\colon\ p(c_n, x) < 1/m\}, \quad n = 1, 2, \ldots,\ m = 1, 2, \ldots$$

so that $\{U_{nm}\colon\ n = 1, 2, \ldots,\ m = 1, 2, \ldots\}$ is countable. We claim it is a base indeed let $y \in V \in \mathscr{T}^\sharp$. By Remark 1.2 there is some m such that $P(y, 1/m) \subset V$. Consider the set $D(y, 1/(2m))$. This set is open in the topology $\mathscr{T}$ thus there is some n such that $d(y, c_n) < 1/(2m)$ which implies $p(y, c_n) < 1/(2m)$ and $p(c_n, y) < 1/(2m)$. The set U_{n2m} is therefore such that $y \in U_{n2m}$ and $U_{n2m} \subset V$ (as $p(y,x) \leqslant p(y, c_n) + p(c_n, x) < 1/m$). □

The next result clarifies that the difference between non-strict and strict quasi-pseudo-metrizable spaces is that both can be identified with subspaces of the ordered Hilbert cube but the latter types can also be identified with *order* subspaces of the ordered Hilbert cube.

Theorem 3.10. *The following conditions are equivalent for a topological ordered space* $(E, \mathscr{T}, \leqslant)$

(a) $(E, \mathscr{T}, \leqslant)$ *is a separable strictly quasi-pseudo-metrizable space,*
(b) $(E, \mathscr{T}, \leqslant)$ *is order embeddable as an order subspace of the ordered Hilbert cube* H.

Proof. (b) $\Rightarrow$ (a). If E can be identified with an order subspace of the ordered Hilbert cube H then E is strictly quasi-pseudo-metrizable since this property is hereditary with respect to order subspaces. Furthermore, H has a second countable topology thus the topology of E is second countable and hence separable.

(a) $\Rightarrow$ (b). Let p be a quasi-pseudo-metric bounded by 1 for the strict quasi-pseudo-metrizable space E. Let $\{c_1, c_2, \dots\}$ be a countable set which is dense according to $\mathscr{T}$ and define $f_n(x) = 1 - p(c_n, x)$, $g_n(x) = p(x, c_n)$, which are both continuous and isotone with value in $[0, 1]$ (Prop. 2.2). The countable family of functions f_n is denoted $\mathcal{F}$ and the countable family of functions g_n is denoted $\mathcal{G}$. We define $\mathcal{R} = \mathcal{F} \cup \mathcal{G}$. We are going to prove that (i1) the coarsest topology on E which makes all the elements of $\mathcal{F}$ upper semi-continuous is $\mathscr{T}^\sharp$ and (i2) the coarsest topology on E which makes all the elements of $\mathcal{G}$ lower semi-continuous is $\mathscr{T}^\flat$. From that it follows, by convexity of E, that (i) the initial topology for $\mathcal{R}$ is $\mathscr{T}$. We are also going to prove that (ii) $x \leqslant y$ iff for every $r \in \mathcal{R}$, $r(x) \leqslant r(y)$.

Let $\mathcal{U}$ be the coarsest topology on E which makes all the elements of $\mathcal{F}$ upper semi-continuous. $\mathcal{U}$ admits as subbase the sets of the form $f_n^{-1}((a, 1])$ with $a \in [0, 1]$ which are open and increasing thus all the sets of $\mathcal{U}$ are open and increasing, that is, $\mathcal{U} \subset \mathscr{T}^\sharp$.

For the converse, let V be open increasing and let $x \in V$. There is some $\epsilon > 0$ such that $P(x, \epsilon) \subset V$, and there is some $c_i \in D(x, \epsilon/2)$. The p-ball $P(c_i, \epsilon/2)$ contains x because $p(c_i, x) \leqslant d(c_i, x) = d(x, c_i) < \epsilon/2$, and $P(c_i, \epsilon/2) \subset P(x, \epsilon) \subset V$ because if $z \in P(c_i, \epsilon/2)$ we have $p(x, z) \leqslant p(x, c_i) + p(c_i, z) \leqslant d(x, c_i) + p(c_i, z) < \epsilon/2 + \epsilon/2 = \epsilon$ thus $z \in P(x, \epsilon)$. These observations imply that the function f_i is such that $x \in f_i^{-1}((1 - \epsilon/2, 1]) \subset V$ which proves that $\mathcal{U}$ is as fine as $\mathscr{T}^\sharp$ and hence equal to it. Actually, it proves more, namely that the sets of the form $f_n^{-1}((a, 1])$ with $a \in [0, 1]$ form a base for $\mathcal{U}$ and hence for $\mathscr{T}^\sharp$. The proof of (i2) is analogous.

As for (ii), if $x \leqslant y$ we get $r(x) \leqslant r(y)$ because all the elements of $\mathcal{R}$ are isotone. If $x \nleqslant y$ then $x \in E \backslash d(y)$ which is open increasing thus, by the just proved result, there is some $f_j \in \mathcal{F}$ and $c \in [0, 1]$ such that $x \in f_j^{-1}((c, 1]) \subset E \backslash d(y)$ which implies $f_j(x) > c > f_j(y)$, thus setting $r = f_j$, $r(x) \nleqslant r(y)$.

The collection $\mathcal{R}$ separates points because if $r(x) = r(y)$ for all $r \in \mathcal{R}$, then, by the just proved result, $x \leqslant y$ and $y \leqslant x$ which implies by the order assumption, $x = y$. By the embedding theorem [15, Theor. 8.12], the map $\rho : E \to H$ whose components are $r_{2i} = f_i$, $r_{2i+1} = g_i$, that is the functions of $\mathcal{R}$, is an embedding, and an order embedding because of (ii).

Let us prove that $\rho(E)$ is not only a subspace but in fact an order subspace of H. For simplicity we shall identify E with $\rho(E)$ thus we shall omit the order homeomorphism between the two spaces. Let V be an open increasing subset of E, we have to find an open increasing subset $V' \subset H$, such that $V' \cap E = V$. For every $x \in V$ there is some $r_{2i} \in \mathcal{F} \subset \mathcal{R}$, $r_{2i} = f_i$, and $c \geqslant 0$ such that $x \in f_i^{-1}((c, 1]) \subset V$. The open set V'_x on H given by the product of all intervals [0,1] but for the (2i)-th term which is $(c, 1]$, is open increasing and such that $V'_x \cap E = f_i^{-1}((c, 1]) \subset V$. Defined $V' = \bigcup_{x \in V} V'_x$ we have $V' \cap E = V$, which is the thesis. The proof in the decreasing case is analogous. □

Remark 3.11. In the ordered case Theorem 3.8 follows also from Theorem 3.10. Indeed, suppose that $(E, \mathscr{T}, \leqslant)$ is a compact quasi-pseudo-metrized ordered space. By the order assumption, $\mathscr{T}$ is Hausdorff and d is a metric. By compactness and metrizability $(E, \mathscr{T})$ is separable, thus $(E, \mathscr{T}, \leqslant)$ is a separable quasi-pseudo-metrizable space which can be regarded as a subset of the ordered Hilbert cube. Since it is compact it is an order subspace [31, Prop. 2.6] and hence a strictly quasi-pseudo-metrizable space.

4. Conclusions

In the framework of topological preordered spaces we have proved that the family of the second countable completely regularly preordered spaces coincides with the family of separable quasi-pseudo-metrizable spaces (Theor. 2.5). The theorem is optimal as there are counterexamples if the latter family is narrowed to the strictly quasi-pseudo-metrizable spaces or the former family is enlarged to include the second countable regularly preordered spaces. We have also shown that in the ordered case the second countable completely regularly ordered spaces are, essentially, subspaces of the ordered Hilbert cube (Theor. 3.7). The difference with the strict case comes from the fact that with the strict condition the subspace can be chosen to be an order subspace (Theor. 3.10).

It remains open the problem of establishing conditions which, starting from second countability and the assumption that E is a completely regularly preordered space could allow one to prove that E is strictly quasi-pseudo-metrizable. We have shown that a compactness condition would be enough (Theor. 3.8) but this assumption is quite strong for applications.

Another direction for further investigation is that of the generalization of the Nagata–Smirnov–Bing metrization theorems to the topological preordered space case. Unfortunately, it seems that several arguments cannot be generalized and analogous quasi-pseudo-metrization results could not hold or could be much harder to prove.

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Appendix A. Quasi-uniformities adapted to uniformities and preorders

A classical problem [5] asks to establish, given a uniformity $\mathcal{O}$ on a preordered space $(E, \leqslant)$, if there is some quasi-uniformity $\mathcal{U}$ such that $\mathcal{U}^* = \mathcal{O}$ and $\bigcap \mathcal{U} = G(\leqslant)$. In this appendix we provide a result which is connected to this problem as well as with the problem of quasi-pseudo-metrization of a topological preordered space.

The canonical quasi-uniformity $\mathcal{R}$ on $\mathbb{R}$ is that generated by the quasi-pseudo-metric $m(x, y) = \max(0, x - y)$. The dual quasi-uniformity $\mathcal{R}^{-1}$ is generated by the quasi-pseudo-metric $n(x, y) = \max(0, y - x)$ and $\mathcal{R}^*$ is generated by the metric $m+n = |x-y|$. Given a family of functions on a topological space E with values in $\mathbb{R}$ one can induce both a weak uniformity or a weak quasi-uniformity on E depending as to whether one endow $\mathbb{R}$ with $\mathcal{R}^*$ or $\mathcal{R}$.

Theorem A.1. *Let $(E, \leqslant)$ be a preordered space, let $\mathcal{O}$ be a uniformity on E, and let $\mathcal{F}$ be a family of uniformly continuous functions with value in $\mathbb{R}$ with the properties*

(i) *$\mathscr{O}$ coincides with the weak uniformity generated by the set of functions $\mathcal{F}$,*
(ii) *$x \leqslant y$ if and only if for every $f \in \mathcal{F}$, $f(x) \leqslant f(y)$,*

then the weak quasi-uniformity $\mathcal{U}$ generated by $\mathcal{F}$ is such that $\mathcal{U}^* = \mathcal{O}$, $\bigcap \mathcal{U} = G(\leqslant)$, and the functions in $\mathcal{F}$ become quasi-uniformly continuous with respect to $\mathcal{U}$.

If $\mathcal{F}$ is countable then $\mathcal{U}$ admits a countable base, thus $(E, \mathcal{U})$ is quasi-pseudo-metrizable (see Theorem 2.6). If $\mathcal{O}$ and $\mathcal{F}$ satisfy (i) and (ii), $\mathcal{O}$ admits a countable base, and $\mathscr{T}(\mathcal{O})$ is second countable (equivalently, $\mathcal{O}$ is induced by a pseudo-metric which makes E a separable pseudo-metric space) then there is a subfamily $\mathcal{F}' \subset \mathcal{F}$ which is countable and satisfies (i) and (ii).

Proof. The weak uniformity $\mathcal{O}$ generated by $\mathcal{F}$ admits a subbase made of subsets of $E \times E$ of the form $(f \times f)^{-1}R$ where $f \in \mathcal{F}$ and $R \in \mathcal{R}^*$ (i.e. a base is made by the finite intersections of sets of that form). For each $R \in \mathcal{R}^*$ there are $U, V \in \mathcal{R}$ such that $U \cap V^{-1} \subset R$ (note that $U \cap V^{-1} \in \mathcal{R}^*$ by definition of the latter family) thus $\mathcal{O}$ admits a subbase made of subsets of $E \times E$ of the form $(f \times f)^{-1}(U \cap V^{-1}) = [(f \times f)^{-1}U] \cap [(f \times f)^{-1}V^{-1}] = [(f \times f)^{-1}U] \cap [(f \times f)^{-1}V]^{-1}$.

The weak quasi-uniformity $\mathcal{U}$ generated by $\mathcal{F}$ admits a subbase made of subsets of $E \times E$ of the form $(f \times f)^{-1}U$ where $f \in \mathcal{F}$ and $U \in \mathcal{R}$ (i.e. a base is made by the finite intersections of sets of that form). A subbase for the quasi-uniformity $\mathcal{U}^*$ is then given by subsets of $E \times E$ of the form $[(f \times f)^{-1}U] \cap [(f \times f)^{-1}V]^{-1}$ for $U, V \in \mathcal{R}$. We conclude that $\mathcal{U}^* = \mathcal{O}$. Finally,

$$\bigcap \mathcal{U} = \bigcap_{f \in \mathcal{F}} \bigcap_{U \in \mathcal{R}} (f \times f)^{-1}U = \bigcap_{f \in \mathcal{F}} (f \times f)^{-1} \bigcap_{U \in \mathcal{R}} U = \bigcap_{f \in \mathcal{F}} (f \times f)^{-1} G(\leqslant_{\mathbb{R}}) = \bigcap_{f \in \mathcal{F}} \{(x, y): f(x) \leqslant f(y)\} = G(\leqslant).$$

The functions in $\mathcal{F}$ are quasi-uniformly continuous with respect to $\mathcal{U}$ by definition of weak quasi-uniformity.

If $\mathcal{F}$ is countable there is a subbase for $\mathcal{U}$ given by $(f_i \times f_i)^{-1}U_m$, where $f_i \in \mathcal{F}$ and $U_m = \{(x, y) \in \mathbb{R} \times \mathbb{R}: x - y < 1/m\}$. Since the subbase is countable the base obtained from all the possible finite intersections is also countable.

If $\mathcal{O}$ admits a countable base then it comes from a pseudo-metric d (e.g. [36, Theor. 13, Chap. 6]). For a topological pseudo-metrizable space second countability is equivalent to separability [36, Theor. 11, Chap. 4] thus (i) $\mathcal{O}$ admits a countable base and $\mathscr{T}$ is second countable, is equivalent to (ii) $\mathcal{O}$ comes from a pseudo-metric d such that (E, d) is a separable pseudo-metric space.

Suppose $\mathcal{O}$ has a countable base O_n and that $\mathscr{T}(\mathcal{O})$ is second countable, then for each n we can find some integers $k_n, m \geqslant 1$, and some functions $f_1^{(n)}, f_2^{(n)}, \ldots, f_{k_n}^{(n)} \in \mathcal{F}$, such that $\bigcap_{i=1}^{k_n} (f_i^{(n)} \times f_i^{(n)})^{-1} R_m \subset O_n$, where $R_m = \{(x, y) \in \mathbb{R} \times \mathbb{R}: |x - y| < 1/m\}$. Consider the family $\mathcal{F}'$ which includes the functions $f_i^{(n)}$ so selected plus another countable family of uniformly continuous functions which we shall define in a moment. We have that the weak uniformity it generates is still $\mathcal{O}$.

Since $\mathscr{T}(\mathcal{O})$ is second countable, the product topology $\mathscr{T} \times \mathscr{T}$ on $E \times E$ is second countable. Since the functions belonging to $\mathcal{F}$ are continuous and $G(\leqslant_{\mathbb{R}})$ is closed, each set $(f \times f)^{-1}G(\leqslant_{\mathbb{R}})$ for $f \in \mathcal{F}$, is closed in the product topology of $E \times E$. By second countability of $E \times E$, the intersection $G(\leqslant) = \bigcap_{f \in \mathcal{F}} (f \times f)^{-1} G(\leqslant_{\mathbb{R}})$ can be reduced to the intersection of a countable number of terms and we include the corresponding elements of $\mathcal{F}$ into $\mathcal{F}'$. The family $\mathcal{F}'$ is then countable and satisfies (i) and (ii). □

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