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Identities induced by Riordan arrays

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ABSTRACT

Historically, there exist two versions of the Riordan array concept. The older one (better known as *recursive matrix*) consists of bi-infinite matrices $(d_{n,k})_{n,k\in\mathbb{Z}}$ ($k > n$ implies $d_{n,k} = 0$), deals with formal Laurent series and has been mainly used to study algebraic properties of such matrices. The more recent version consists of infinite, lower triangular arrays $(d_{n,k})_{n,k\in\mathbb{N}}$ ($k > n$ implies $d_{n,k} = 0$), deals with formal power series and has been used to study combinatorial problems. Here we show that every Riordan array induces two characteristic combinatorial sums in three parameters $n, k, m \in \mathbb{Z}$. These parameters can be specialized and generate an indefinite number of other combinatorial identities which are valid in the bi-infinite realm of recursive matrices.

1. Introduction

The concept of a (proper) Riordan array was introduced [20,21] as a generalization of the Pascal triangle. A Riordan array is defined as a pair of formal power series $D = \mathcal{R}(d(t),\ h(t))$ with $d(0) \neq 0$, $h(0) = 0$, $h'(0) \neq 0$. The usual way to represent the Riordan array $\mathcal{R}(d(t),\ h(t))$ is by means of an infinite matrix $(d_{n,k})$, $n, k \in \mathbb{N}$, its generic element being:

$$d_{n,k} = [t^n]d(t)h(t)^k. \tag{1.1}$$

As shown in Section 4, the Pascal triangle is just the case:

$$P = \mathcal{R}\left(\frac{1}{1-t},\ \frac{t}{1-t}\right).$$

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Many properties of Riordan arrays have been studied in the literature, in particular their connection with combinatorial sums. Actually, if $(s_n)_{n\in\mathbb{N}}$ is any sequence having $s(t) = \sum_{k=0}^{\infty} s_k t^k$ as its generating function, it is possible to prove that:

$$\sum_{k=0}^{n} d_{n,k} s_k = [t^n] d(t) s(h(t))$$

thus reducing the sum to the extraction of a coefficient from a formal power series. For alternative approaches see, e.g., [5,17].

Formally, Riordan arrays are a formulation of the 1-*Umbral Calculus*, as defined by Roman in [19], although their name was only coined later in 1991 by Shapiro et al. [20]. Because of that, the history of Riordan arrays is the same as the history of Umbral calculus, going back to Blissard, Bell, Schur, Jabotinsky and others; the literature about Riordan arrays is still growing. Two important paper related to the present work are Rogers [18] and Barnabei et al. [1]. In particular, Rogers [18], with the aim of generalizing the Pascal triangle, introduced *renewal arrays* (*a-posteriori* a special case of Riordan arrays) and observed that every element $d_{n+1,k+1}$ (not belonging to row or column 0) could be expressed as a linear combination of the elements in the preceding row, i.e.:

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots = \sum_{j=0}^{\infty} a_j d_{n,k+j}. \tag{1.2}$$

The sum is actually finite, the sequence $A = (a_k)_{k\in\mathbb{N}}$ is fixed, has $a_0 \neq 0$ and is called the *A-sequence* of the Riordan array (see also [6,21]); $A(t)$ denotes its generating function. The idea of A-sequence has been later generalized to the concept of A-matrix in [13].

In 1982, Barnabei et al. [1] introduced the concept of recursive matrices, which are bi-infinite Riordan arrays extending (1.1) to all indices $n, k \in \mathbb{Z}$. Recursive matrices have mainly been used for dealing with algebraic properties of the corresponding arrays. We can cite the following papers appearing in the literature and following a similar approach: [2–4,9–12,22,23,25,26]. Instead, Riordan arrays have been used as an approach to counting problems (see, e.g., [14–16]), especially for combinatorial sums and inversion (see, e.g., [21]).

We can summarize the main contributions of the present paper into two points:

(1) We extend the traditional row and column construction of Riordan arrays to recursive matrices. These constructions emphasize the role of the A-sequence in the theory and applications of recursive matrices, an aspect somewhat wasted in the traditional approach to these arrays.
(2) We show that a Riordan array defines two combinatorial identities, representing a horizontal (or row) and a vertical (or column) property of the array (Theorems 3.1 and 3.2). These sums depend on three parameters and therefore they can be specialized into many combinatorial sums with two or also only one parameter. Some of these combinatorial identities are known in the literature (e.g., are present in Gould's collection [5]), but others are new and involve many combinatorial quantities.

Sections 4–6 provide three meaningful examples. In Section 4 we deal with the Pascal triangle and prove that the two corresponding induced identities are essentially related to Vandermonde convolution. In Section 5 we pass to the much more complex example of the triangle derived from the Catalan sequence; in this case the induced identities are new in the literature. Finally, in Section 7, we propose, as a third example, a recursive matrix related to central binomial coefficients.

2. Riordan arrays

We recall the main properties of Riordan arrays which will be used in this paper. The product of two Riordan arrays is defined by:

$$D * E = \mathcal{R}(d_1(t), \quad h_1(t)) * \mathcal{R}(d_2(t), \quad h_2(t)) = \mathcal{R}(d_1(t) d_2(h_1(t)), \quad h_2(h_1(t)));$$

it corresponds to the usual row-by-column product of two (infinite) matrices. The Riordan array $I = \mathcal{R}(1,\ t)$ acts as the identity and the inverse of $D = \mathcal{R}(d(t),\ h(t))$ is the Riordan array:

$$D^* = (d^*_{n,k}) = \mathcal{R}(d^*(t),\ h^*(t)) = \mathcal{R}\left(\frac{1}{d(\overline{h}(t))},\ \overline{h}(t)\right),$$

where $\overline{h}(t)$ is the compositional inverse of $h(t)$, that is the power series such that $h(\overline{h}(t)) = \overline{h}(h(t)) = t$. Since the product $D * D^* = D^* * D$ equals the identity $I = \mathcal{R}(1,\ t)$, every Riordan array induces the two-parameters *basic identity*:

$$\sum_{j=k}^{n} d_{n,j} d^*_{j,k} = \delta_{n,k}, \tag{2.1}$$

where δ is the Kronecker delta.

In general, a superscripted asterisk denotes quantities related to the inverse Riordan arrays; overlining denotes compositional inversion. We observe that $h^*(t) = \overline{h}(t)$, but $d^*(t) \neq \overline{d}(t)$. The set $\mathcal{R}$ of all Riordan arrays is a group with the product defined above. Important subgroups are (see [8]):

$\mathcal{R}_A$	Appell or Toeplitz subgroup	$\mathcal{R}(d(t),\ t)$,
$\mathcal{R}_L$	Lagrange or associated subgroup	$\mathcal{R}(1,\ h(t))$,
$\mathcal{R}_D$	co-Lagrange or Derivative subgroup	$\mathcal{R}(h'(t),\ h(t))$,
$\mathcal{R}_N$	Renewal or Rogers subgroup	$\mathcal{R}(d(t),\ td(t))$,
$\mathcal{R}_H$	Hitting-time subgroup	$\mathcal{R}(th'(t)/h(t),\ h(t))$.

An immediate, but important observation is that every Riordan array can be seen as the product of a Toeplitz by a Lagrange array:

$$\mathcal{R}(d(t), h(t)) = \mathcal{R}(d(t),\ t) * \mathcal{R}(1,\ h(t)). \tag{2.2}$$

The following Theorem 2.2 allows us to compute the generic element of the inverse array of $D = \mathcal{R}(d(t),\ h(t))$ by using the functions $d(t)$ and $h(t)$. It is based on the Lagrange Inversion Formula, which we give in the two following forms and whose proof can be found in the literature (see [7,24]):

Theorem 2.1 (LIF). *Let $h(t)$ be a formal power series with $h(0) = 0$, $h'(0) \neq 0$ and let $\overline{h}(t)$ be its compositional inverse; then we have:*

$$[t^n]\overline{h}(t)^k = \overline{h}^{(k)}_n = [t^{n-k}]h'(t)\left(\frac{t}{h(t)}\right)^{n+1} = \frac{k}{n}[t^{n-k}]\left(\frac{t}{h(t)}\right)^{n}.$$

Besides, let $w = w(t)$ be the solution of the functional equation $w = t\phi(w)$ and let $F(t)$ be any formal power series such that $F(0) \neq 0$; then we have:

$$[t^n]F(w(t)) = \frac{1}{n}[t^{n-1}]F'(t)\phi(t)^n = [t^n]F(t)\phi(t)^{n-1}(\phi(t) - t\phi'(t)).$$

Theorem 2.2. *Given the Riordan array $D = (d_{n,k}) = (d(t),\ h(t))$, the generic element of its inverse is given by:*

$$d^*_{n,k} = [t^{-k-1}]\frac{h'(t)}{d(t)h(t)^{n+1}} = [t^{n-k}]\frac{h'(t)}{d(t)(h(t)/t)^{n+1}} \tag{2.3}$$

or, equivalently:

$$d^*_{n,k} = \frac{1}{n}[t^{n-k}]\left(\frac{k}{d(t)} - \frac{td'(t)}{d(t)^2}\right)\left(\frac{t}{h(t)}\right)^n. \tag{2.4}$$

Proof. The two formulas in Theorem 2.1 prove these results. □

For what concerns the A-sequence, we can state the following theorem, the proof of which can be found in [6,13]:

Theorem 2.3. *An infinite lower triangular array $D = (d_{n,k})_{n,k\in\mathbb{N}}$ is a Riordan array if and only if a sequence $A = (a_0 \neq 0, a_1, a_2, \ldots)$ exists such that for every $n, k \in \mathbb{N}$ relation (1.2) holds true. Besides, the A-sequence is uniquely determined by the function $h(t)$, and vice versa, by the formulas:*

$$h(t) = tA(h(t)) \quad \text{and} \quad A(t) = \left[\frac{h(y)}{y} \;\middle|\; t = h(y)\right] = \left[\frac{t}{y} \;\middle|\; t = h(y)\right]. \tag{2.5}$$

The identity $h(t) = tA(h(t))$ will be called the *basic relation* (not to be confused with the basic identity (2.1)) and from it the following result follows immediately:

Theorem 2.4. *Let $D = \mathcal{R}(d(t),\ h(t))$ any Riordan array, and D^* its inverse; then we have:*

$$A(t) = \frac{t}{\overline{h}(t)} \quad \text{and} \quad A^*(t) = \frac{t}{h(t)}.$$

Proof. The first relation is obtained by setting $t \mapsto \overline{h}(t)$ in the basic relation. The second relation follows from the fact that the h-function of D^* is $\overline{h}(t)$. □

3. Recursive matrices

If $D = \mathcal{R}(d(t),\ h(t))$ is a Riordan array, the corresponding infinite lower triangular matrix is defined by the relation (1.1). Note that if $k > n$ then $d_{n,k} = 0$. This corresponds to the fact that $d(t)h(t)^k$ is the generating function of the elements in column k, the order of which is k according to the conditions imposed on $d(t)$ and $h(t)$. Many Riordan arrays have a combinatorial meaning and their elements count specific characteristics of combinatorial objects. For example, if P is the Pascal triangle, $P_{n,k}$ counts the number of subsets with k elements of a set with n elements. From this point of view, the usual restriction $0 \leqslant k \leqslant n$ makes sense. However, from an algebraic standpoint, $d(t)h(t)^k$ is a *formal Laurent series* when k is negative, and k is its order. Therefore, we can extract the coefficient $[t^n]d(t)h(t)^k$ for any integers n and k thus obtaining the *recursive matrix* $(d_{n,k})_{n,k\in\mathbb{Z}}$. If we consider this bi-infinite triangle $(d_{n,k})$, it contains the Riordan array located at the places (n, k) for $n \geqslant 0$, $k \geqslant 0$. As an example, in Table 1 we show the recursive matrix corresponding to the Catalan Riordan array:

$$C = \mathcal{R}\left(\frac{1-\sqrt{1-4t}}{2t}, \frac{1-\sqrt{1-4t}}{2}\right).$$

In this paper, the recursive matrix derived by the Riordan array $D = \mathcal{R}(d(t),\ h(t))$ will be denoted by $D = \mathcal{X}(d(t),\ h(t))$ without ambiguity.

Combinatorially, this extension is not particularly meaningful, at least at a first sight. For example, row sums, alternate row sums and weighted row sums, all having a specific combinatorial meaning, loose their sense, since rows and columns are all infinite. However, as we will see, recursive matrices are the natural structure for proving some identities induced by Riordan arrays, which are the main topic of this paper.

Table 1
The Catalan recursive matrix.

$n\backslash k$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
-6	1	0	0	0	0	0	0	0	0	0	0	0	0
-5	-5	1	0	0	0	0	0	0	0	0	0	0	0
-4	5	-4	1	0	0	0	0	0	0	0	0	0	0
-3	0	2	-3	1	0	0	0	0	0	0	0	0	0
-2	0	0	0	-2	1	0	0	0	0	0	0	0	0
-1	-1	-1	-1	-1	-1	1	0	0	0	0	0	0	0
0	-5	-4	-3	-2	-1	0	1	0	0	0	0	0	0
1	-20	-14	-9	-5	-2	0	1	1	0	0	0	0	0
2	-75	-48	-28	-14	-5	0	2	2	1	0	0	0	0
3	-275	-165	-90	-42	-14	0	5	5	3	1	0	0	0
4	-1001	-572	-297	-132	-42	0	14	14	9	4	1	0	0
5	-3640	-2002	-1001	-429	-132	0	42	42	28	14	5	1	0
6	-13260	-7072	-3432	-1430	-429	0	132	132	90	48	20	6	1

First of all, it is necessary to point out that the introduction of recursive matrices simply extends the properties of Riordan arrays. In particular, the product of two Riordan arrays becomes the product of the corresponding recursive matrices; we explicitly observe that, in the corresponding recursive matrices, every element is obtained by a finite sum. The identity of this product is $\mathcal{X}(1,\ t)$, the matrix which is everywhere 0, except on the main diagonal, which is all composed by 1's. Finally, the inverse is computed as for Riordan arrays. In this spirit, we can immediately generalize Theorem 2.3, which is a classical result in the theory of Riordan arrays; moreover, in Theorem 3.1 we obtain a particular identity, called the *row* or *horizontal identity* of the recursive matrix. The generating function $A(t)$ of the A-sequence is an invertible formal power series since $a_0 \neq 0$. Therefore, we can also consider its powers $A(t)^m$, where $m \in \mathbb{Z}$, and we will use the notation $a_j^{(m)}$ for the coefficient $[t^j]A(t)^m$. The case $m = -1$ has already been studied in the literature of Riordan arrays and the coefficients of $A(t)^{-1}(= B(t))$ are known as the B-sequence of the array (see [13, p. 309]).

Theorem 3.1. *In every recursive matrix $\mathcal{X}(d(t),\ h(t))$ the following identity holds true for every integer $k, n, m \in \mathbb{Z}$:*

$$d_{n+m,k+m} = \sum_{j=0}^{n-k} a_j^{(m)} d_{n,k+j}. \tag{3.1}$$

Proof. We proceed as in the proof of Theorem 2.3 in [6]. Let $m \in \mathbb{Z}$ and define the recursive matrix $\mathcal{X}(W(t),\ M(t))$ (with $W(t) = \sum_{k=0}^{\infty} w_k t^k$) by the relation:

$$\mathcal{X}(d(t),\ h(t)) * \mathcal{X}(W(t),\ M(t)) = \mathcal{X}(d(t)h(t)^m/t^m,\ h(t)).$$

Obviously, $M(t) = t$ and $W(h(t)) = h(t)^m/t^m$. The generic element of the product on the left hand side is:

$$\sum_j d_{n,j} w_{k-j} = \sum_j d_{n,k+j} w_j.$$

The right hand side gives:

$$[t^n]\frac{d(t)h(t)^m}{t^m}h(t)^k = [t^{n+m}]d(t)h(t)^{k+m} = d_{n+m,k+m}.$$

In Theorem 2.3 it was proved that $A(h(t)) = h(t)/t$ or $h(t)^m/t^m = A(h(t))^m$; this implies $W(h(t)) = A(h(t))^m$ that is $W(t) = A(t)^m$. Finally, we have $w_j = a_j^{(m)}$ and the proof is complete. □

The power series $h(t)$ cannot be inverted with respect to the Cauchy product, since $h(0) = 0$; however, we also have $h'(0) \neq 0$, so that $h(t)/t$ is invertible. Therefore, we will write $h_{j+m}^{(m)}$ for the coefficient $[t^{j+m}]h(t)^m = [t^j](h(t)/t)^m$, for every $m \in \mathbb{Z}$. Depending on the function $h(t)$, we have a *vertical* or *column identity*:

Theorem 3.2. *The following identity holds true for every recursive matrix and for every $m \in \mathbb{Z}$:*

$$d_{n+m,k+m} = \sum_{j=0}^{n-k} h_{j+m}^{(m)} d_{n-j,k}. \tag{3.2}$$

Proof. The proof is almost immediate:

$$d_{n+m,k+m} = [t^{n+m}]d(t)h(t)^{k+m} = [t^n]d(t)h(t)^k \left(\frac{h(t)}{t}\right)^m = \sum_{j=0}^{n} h_{j+m}^{(m)} d_{n-j,k}. \quad \square$$

As pointed out by an anonymous referee, the proof of Theorems 3.1 and 3.2 can be simplified by using the factorization (2.2). However, the method hides the role of the A-sequence, which is fundamental in our approach.

Finally, we will also use $a_j^{*(m)}$ and $h_{j+m}^{*(m)} = \overline{h}_{j+m}^{(m)}$ for the analogous quantities relative to the inverse Riordan array. The following theorem greatly simplifies the computation of the coefficients for the horizontal and vertical identities:

Theorem 3.3. *Let D= $\mathcal{X}(d(t), h(t))$ be any recursive matrix with $A(t)$ being its A-sequence, and let $a_j^{(m)}$, $a_j^{*(m)}$, $h_{j+m}^{(m)}$ and $\overline{h}_{j+m}^{(m)}$ be as above; then we have:*

$$a_j^{(m)} = \overline{h}_{j-m}^{(-m)} \qquad \text{and} \qquad a_j^{*(m)} = h_{j-m}^{(-m)}.$$

Proof. Consider the basic relation $h(t) = tA(h(t))$ and perform the substitution $t \mapsto \overline{h}(t)$; we find $t = \overline{h}(t)A(t)$ or $A(t) = t/\overline{h}(t)$, and this proves the two identities, when we also consider the inverse Riordan array D^*. $\square$

The following shifted version of the horizontal and vertical identities can be useful:

$$d_{n,k} = \sum_{j=0}^{n-k} a_j^{(m)} d_{n-m,k-m+j}, \quad d_{n,k} = \sum_{j=0}^{n-k} h_{j+m}^{(m)} d_{n-m-j,k-m}. \tag{3.3}$$

4. The Pascal triangle

As a very simple example, let us consider the Pascal triangle, the infinite, lower triangular array which generated the concept of a Riordan array. It is very elementary and the identities we find are known. We start by giving the definition of the triangle as a Riordan array:

$$P = \mathcal{R}\left(\frac{1}{1-t}, \frac{t}{1-t}\right).$$

The general term is easily shown to be the appropriate binomial coefficient:

$$P_{n,k} = [t^n]\frac{1}{1-t} \cdot \frac{t^k}{(1-t)^k} = [t^{n-k}](1-t)^{-k-1} = \binom{-k-1}{n-k}(-1)^{n-k}.$$

Table 2
The Pascal recursive matrix.

$n\backslash k$	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
−6	1	0	0	0	0	0	0	0	0	0	0	0	0
−5	−5	1	0	0	0	0	0	0	0	0	0	0	0
−4	10	−4	1	0	0	0	0	0	0	0	0	0	0
−3	−10	6	−3	1	0	0	0	0	0	0	0	0	0
−2	5	−4	3	−2	1	0	0	0	0	0	0	0	0
−1	−1	1	−1	1	−1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0
1	0	0	0	0	0	0	1	1	0	0	0	0	0
2	0	0	0	0	0	0	1	2	1	0	0	0	0
3	0	0	0	0	0	0	1	3	3	1	0	0	0
4	0	0	0	0	0	0	1	4	6	4	1	0	0
5	0	0	0	0	0	0	1	5	10	10	5	1	0
6	0	0	0	0	0	0	1	6	15	20	15	6	1

When $0 \leqslant k \leqslant n$, this is $\binom{n}{k}$; for $k < 0$ and $k \leqslant n$ this formula can be interpreted as a definition of the same binomial coefficient. The central part of the triangle is given in Table 2.

We now solve the functional equation $h(t) = tA(h(t))$, that is we solve $y = t/(1-t)$ in $t = t(y)$ and use the relation $A(y) = y/t(y)$ to find the A-sequence. We have:

$$A(t) = 1 + t \qquad A = (1, 1, 0, 0, 0, \ldots).$$

In order to find the inverse array, we invert the relation $y = t/(1-t)$ and find easily $\overline{h}(t) = t/(1+t)$. We now substitute this expression in $d^*(t) = d(\overline{h}(t))^{-1}$ and eventually get:

$$P^* = \mathcal{R}\left(\frac{1}{1+t}, \frac{t}{1+t}\right).$$

The general term is obtained as before, producing the basic identity (2.1):

$$P^*_{n,k} = (-1)^{n-k}\binom{n}{k} \qquad \sum_{j=k}^{n}\binom{n}{j}(-1)^{j-k}\binom{j}{k} = \delta_{n,k}.$$

The A-sequence is $A^*(t) = 1 - t$ or $A^* = (1, -1, 0, 0, 0, \ldots)$. The horizontal and vertical identities are easily found:

Theorem 4.1. *For every $n, k, m \in \mathbb{Z}$, $k \leqslant n$, we have:*

$$\binom{n+m}{k+m} = \sum_{j=0}^{n-k}\binom{m}{j}\binom{n}{k+j} = \sum_{j=0}^{n-k}\binom{m}{j}\binom{n}{n-k-j}$$

$$\binom{n+m}{k+m} = \sum_{j=0}^{n}\binom{m+j-1}{j}\binom{n-j}{k}.$$

Proof. From the previous expressions we have:

$$a_j^{(m)} = \binom{m}{j} \qquad a_j^{*(m)} = (-1)^j\binom{m}{j}$$

$$h^m_{m+j} = \binom{m+j-1}{j} \qquad \overline{h}^m_{m+j} = (-1)^j\binom{m+j-1}{j},$$

which agree with Theorem 3.3. The two identities now follow immediately. □

The first identity is equivalent to Vandermonde convolution, the second is less common. The corresponding identities relative to P^* are at all similar and we do not give them explicitly.

5. The Catalan triangle

The Catalan triangle is characterized by the function $h(t) = tC(t)$, where:

$$C(t) = \frac{1-\sqrt{1-4t}}{2t} = 1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + 429t^7 + \cdots .$$

is the generating function of the Catalan numbers.

One of the simplest way to introduce the Catalan triangle is to consider the following path problem. Let us define a *Catalan path* as an underdiagonal path starting at the origin and composed by steps $(1, s)$, where $s = 0, 1, 2, 3, \ldots$; the infinite, lower triangular array $(C_{n,k})$ counts the number of paths from the origin to the point $(n, n-k)$. If we fix the number $C_{n+1,k+1}$, it is formed by the contribution of all the paths counted by $C_{n,k+j}$, with $j = 0, 1, 2, \ldots$, that is:

$$C_{n+1,k+1} = C_{n,k} + C_{n,k+1} + \cdots + C_{n,n};$$

obviously, for the elements in column 0 we have:

$$C_{n+1,0} = C_{n,0} + C_{n,1} + \cdots + C_{n,n}.$$

This gives us the A-sequence of the triangle and proves that it is a Riordan array:

$$A = (1, 1, 1, 1, \ldots) \qquad \text{or} \qquad A(t) = \frac{1}{1-t}.$$

Theorem 5.1. *The Catalan triangle is the renewal array:*

$$C = \mathcal{R}\left(\frac{1-\sqrt{1-4t}}{2t}, \frac{1-\sqrt{1-4t}}{2}\right).$$

Proof. By the formula connecting the A-sequence and the function $h(t)$ we have:

$$h(t) = \frac{t}{1-h(t)} \qquad \text{or} \qquad h(t) = \frac{1 \pm \sqrt{1-4t}}{2};$$

since we should have $h(0) = 0$, the minus sign is correct. For what concerns the function $d(t)$, we already observed that every element in column 0 should equal the corresponding element in column 1 (since they are the sum of the same elements), except for $d_{0,0} = 1$ but $d_{0,1} = 0$. In terms of generating function, we have:

$$d(t) - 1 = d(t)h(t) \qquad \text{or} \qquad d(t) = \frac{1-\sqrt{1-4t}}{2t}$$

and this shows that C is a renewal array. □

It is now convenient to find the inverse triangle:

Theorem 5.2. *The inverse of the Catalan triangle is:*

$$C^* = \mathcal{R}(1-t,\ t-t^2)$$

and therefore its generic element is:

$$C^*_{n,k} = (-1)^{n-k}\binom{k+1}{n-k}.$$

Proof. We begin by solving in $t = t(y)$ the functional equation $y = h(t)$:

$$y = \frac{1-\sqrt{1-4t}}{2} \quad \text{or} \quad \sqrt{1-4t} = 1-2t \quad \text{or} \quad t = y - y^2.$$

This result gives $\overline{h}(t) = t(1-t)$. Since the inverse of a renewal array is also a renewal array, we have $d^*(t) = 1-t$. Finally, the generic element is:

$$C^*_{n,k} = [t^{n-k}](1-t)(1-t)^k = (-1)^{n-k}\binom{k+1}{n-k}$$

as desired. □

The two recursive matrices C and C^* are given in Tables 1 and 3. It is easy to find the A-sequence of C^*:

$$A^*(t) = \frac{1+\sqrt{1-4t}}{2} = 1 - \frac{1-\sqrt{1-4t}}{2} = 1 - tC(t);$$

$$A^* = (1, -1, -1, -2, -5, -14, -42, \ldots)$$

We are now in a position to find an explicit expression for the generic element of C.

Theorem 5.3. *The general term of the array C and the basic identity (2.1) read:*

$$C_{n,k} = \frac{k+1}{n+1}\binom{2n-k}{n-k}, \quad \sum_{j=k}^{n} \frac{j+1}{n+1}\binom{2n-j}{n-j}(-1)^{j-k}\binom{k+1}{j-k} = \delta_{n,k}.$$

Proof. The proof makes use of the formula for the explicit form of the element $C^*_{n,k}$, that is the generic element of the inverse Riordan array as given by Theorem 2.2. In this case we apply the formula to $d^*(t) = 1-t$ and $\overline{h}(t) = t - t^2$, as shown in the previous theorem. In other words, we see the Catalan triangle as the inverse of the inverse array:

$$\begin{aligned} C_{n,k} &= [t^{n-k}]\frac{1-2t}{(1-t)(1-t)^{n+1}} = [t^{n-k}]\left(\frac{1}{(1-t)^{n+1}} - \frac{t}{(1-t)^{n+2}}\right) \\ &= \binom{2n-k}{n-k} - \binom{2n-k}{n-k-1} \end{aligned}$$

from which the desired formula follows immediately. □

We observe explicitly that in this proof the application of the Lagrange Inversion Formula is hidden in the general theorem to find the inverse of a Riordan array. We can specialize the basic identity in many ways, but let us only consider the case $n \mapsto 2n$ and $k \mapsto n$ (central coefficients):

$$\sum_{j=n}^{2n} \frac{j+1}{2n+1}\binom{4n-j}{2n-j}(-1)^{j-n}\binom{n+1}{j-n} = \delta_{n,0}.$$

Table 3
The inverse Catalan recursive matrix.

$n\backslash k$	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
−6	1	0	0	0	0	0	0	0	0	0	0	0	0
−5	5	1	0	0	0	0	0	0	0	0	0	0	0
−4	15	4	1	0	0	0	0	0	0	0	0	0	0
−3	35	10	3	1	0	0	0	0	0	0	0	0	0
−2	70	20	6	2	1	0	0	0	0	0	0	0	0
−1	126	35	10	3	1	1	0	0	0	0	0	0	0
0	210	56	15	4	1	0	1	0	0	0	0	0	0
1	330	84	21	5	1	0	−1	1	0	0	0	0	0
2	495	120	28	6	1	0	0	−2	1	0	0	0	0
3	715	165	36	7	1	0	0	1	−3	1	0	0	0
4	1001	220	45	8	1	0	0	0	3	−4	1	0	0
5	1365	286	55	9	1	0	0	0	−1	6	−5	1	0
6	1820	364	66	10	1	0	0	0	0	−4	10	−6	1

Let us now observe that:

$$h_{j+m}^{(m)} = [t^j]\left(\frac{1-\sqrt{1-4t}}{2t}\right)^m = [t^j]C(t)^m$$

and the following theorem becomes important:

Theorem 5.4. *The coefficients of the powers of $C(t)$ have the following form:*

$$C_j^{(m)} = h_{j+m}^m = \begin{cases} [t^j]C(t)^m = \dfrac{m}{m+2j}\dbinom{m+2j}{j} & \text{for } j \neq -m/2 \\ C_j^{(-2j)} = (-1)^j 2 & \text{for } j = -m/2 \end{cases}$$

Proof. We observe that $C_{n,k} = [t^{n-k}]C(t)^{k+1}$ and therefore it is enough to set $j = n-k$ and $m = k+1$, or $k = m-1$ and $n = j+m-1$. Consequently:

$$[t^j]C(t)^m = \frac{m}{j+m}\binom{2j+m-1}{j} = \frac{m}{m+2j}\binom{m+2j}{j}.$$

Since $j \geqslant 0$, for $m > 0$ the two formulas coincide. For $m < 0$ the first formula cannot be computed for $j = -m$, and the second for $j = -m/2$. If, as usual, we use the second formula, we can compute its degenerate case by means of the first formula:

$$C_j^{(-2j)} = [t^j]C(t)^{-2j} = \frac{-2j}{-j}\binom{-1}{j} = 2\binom{1+j-1}{j}(-1)^j = 2(-1)^j$$

as desired. □

Let us observe:

$$a_j^{(m)} = [t^j]A(t)^m = [t^j](1-t)^{-m} = \binom{-m}{j}(-1)^j = \binom{m+j-1}{j}.$$

We can therefore prove the following result:

Theorem 5.5. *The horizontal and vertical identities relative to the Catalan triangle are the identities with three parameters $m, n, k \in \mathbb{Z},\ n \geqslant k$:*

$$\sum_{j=0}^{n-k} \frac{m}{m+2j}\binom{m+2j}{j}\frac{k+1}{n-j+1}\binom{2n-2j-k}{n-j-k} = \frac{k+m+1}{n+m+1}\binom{2n+m-k}{n-k}$$

$$\sum_{j=0}^{n-k} \binom{m+j-1}{j}\frac{k+j+1}{n+1}\binom{2n-j-k}{n-j-k} = \frac{k+m+1}{n+m+1}\binom{2n+m-k}{n-k}.$$

Proof. We apply Theorems 3.1 and 3.2, taking into account, for negative, even values of m, the observation of the previous theorem. $\square$

We can now specialize these identities. For example, by changing $n \mapsto n, m \mapsto n, k \mapsto 0$ we get:

$$\sum_{j=0}^{n} \frac{n}{n+2j}\binom{n+2j}{j}\frac{1}{n-j+1}\binom{2n-2j}{n-j} = \sum_{j=0}^{n}\frac{j+1}{n+1}\binom{n+j-1}{j}\binom{2n-j}{n-j} = \frac{n+1}{2n+1}\binom{3n}{n}.$$

Analogously, by performing the change of variables $n \mapsto 2n, m \mapsto n, k \mapsto n$, we find:

$$\begin{aligned}\sum_{j=0}^{n} \frac{n}{n+2j}\binom{n+2j}{j}\frac{n+1}{2n-j+1}\binom{3n-2j}{n-j} &= \sum_{j=0}^{n}\frac{n+j+1}{2n+1}\binom{n+j-1}{j}\binom{3n-j}{n-j}\\ &= \frac{2n+1}{3n+1}\binom{4n}{n}.\end{aligned}$$

and so on. When we set $m = -k$, column 0 is involved; a two parameters identity is:

$$\begin{aligned}\sum_{j=0}^{n-k} \frac{k}{k-2j}\binom{2j-k}{j}\frac{k+1}{n-j+1}\binom{2n-2j-k}{n-j-k} &= \sum_{j=0}^{n-k}\frac{k+j+1}{n+1}\binom{-k+j-1}{j}\binom{2n-k-j}{n-k-j}\\ &= \frac{1}{n-k+1}\binom{2n-2k}{n-k} = C_{n-k};\end{aligned}$$

some attention should be paid when k is even, because $k-2j$ can annihilate. In that case, the value of the first two factors is $2(-1)^j$ as we observed before. We conclude with the following identity, corresponding to $m = 1$ in the horizontal identity (A-identity):

$$\sum_{j=0}^{n-k} \frac{k+j+1}{n+1}\binom{2n-k-j}{n-k-j} = \frac{k+2}{n+2}\binom{2n-k+1}{n-k};$$

by setting $k \mapsto n$ and $n \mapsto 2n$ we get:

$$\sum_{j=0}^{n} \frac{n+j+1}{2n+1}\binom{3n-j}{n-j} = \frac{n+2}{2n+2}\binom{3n+1}{n};$$

by setting $k \mapsto 0$:

$$\sum_{j=0}^{n} \frac{j+1}{n+1}\binom{2n-j}{n-j} = \frac{2}{n+2}\binom{2n+1}{n} = C_{n+1}.$$

For what concerns the inverse array, $\overline{h}_{j+m}^{(m)}$ can be computed by using Theorem 3.3 or directly:

$$\overline{h}_{j+m}^{(m)} = [t^j]\frac{(t-t^2)^m}{t^m} = [t^j](1-t)^m = (-1)^j\binom{m}{j}.$$

Theorem 3.3 allows us to find $a_j^{*(m)}$:

Theorem 5.6. *For the inverse of the Catalan triangle we have:*

$$a_j^{*(m)} = \begin{cases} (-1)^j \dfrac{m}{j}\dbinom{m-j-1}{j-1} & \text{for } j \neq 0 \\ 1 & \text{for } j = 0. \end{cases}$$

Proof. We have:

$$a_j^{*(m)} = h_{j-m}^{(-m)} = \frac{-m}{-m+2j}\binom{-m+2j}{j} = -\frac{m}{j}\binom{-m+2j-1}{j-1} = (-1)^j\frac{m}{j}\binom{m-j-1}{j-1}.$$

When $j = 0$ this expression does not work; however, we have:

$$a_j^{*(m)} = (-1)^j \frac{m}{m-j}\binom{m-j}{j},$$

(another interesting expression), and if we set $j = 0$ we find immediately $a_0^{*(m)} = 1$ as desired. $\square$

The desired identities are:

Theorem 5.7. *The horizontal and vertical identities related to C^* are:*

$$\sum_{j=1}^{n-k} \frac{m}{j}\binom{m-j-1}{j-1}\binom{k+j+1}{n-k-j} = \binom{k+m+1}{n-k} - \binom{k+1}{n-k}$$

where we isolated the term with $j = 0$, and

$$\sum_{j=0}^{n-k}\binom{m}{j}\binom{k+1}{n-k-j} = \binom{k+m+1}{n-k},$$

this last identity being another version of the Vandermonde convolution.

6. The central binomial triangle

The central binomial triangle is defined by its A-sequence, which is $A(t) = 1 + 2t + t^2 = (1+t)^2$, and by its 0th column, composed by the central binomial coefficients:

$$d(t) = \frac{1}{\sqrt{1-4t}} = 1 + 2t + 6t^2 + 20t^3 + 70t^4 + 252t^5 + 924t^6 + \cdots .$$

By solving the functional equation $h(t) = tA(h(t))$ we find:

$$h(t) = \frac{1-2t-\sqrt{1-4t}}{2t} = t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + 429t^7 + \cdots$$

Table 4
The central binomial recursive matrix.

$n\backslash k$	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
−6	1	0	0	0	0	0	0	0	0	0	0	0	0
−5	−10	1	0	0	0	0	0	0	0	0	0	0	0
−4	36	−8	1	0	0	0	0	0	0	0	0	0	0
−3	−56	21	−6	1	0	0	0	0	0	0	0	0	0
−2	35	−20	10	−4	1	0	0	0	0	0	0	0	0
−1	−6	5	−4	3	−2	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0
1	0	0	0	0	0	1	2	1	0	0	0	0	0
2	0	0	0	0	1	4	6	4	1	0	0	0	0
3	0	0	0	1	6	15	20	15	6	1	0	0	0
4	0	0	1	8	28	56	70	56	28	8	1	0	0
5	0	1	10	45	120	210	252	210	120	45	10	1	0
6	1	12	66	220	495	792	924	792	495	220	66	12	1

showing a relation with the Catalan triangle; actually, we have $h(t) = C(t) - 1$. The existence of the A-sequence assures that the triangle is a Riordan array. In conclusion, we have:

$$\mathbf{B} = \mathcal{R}\left(\frac{1}{\sqrt{1-4t}}, \ \frac{1-2t-\sqrt{1-4t}}{2t}\right);$$

the recursive matrix is shown in Table 4.

First of all, we determine the inverse array $\mathbf{B}^*$

Theorem 6.1. *The inverse Riordan array of* $\mathbf{B}$ *is:*

$$\mathbf{B}^* = \mathcal{R}\left(\frac{1-t}{1+t}, \ \frac{t}{(1+t)^2}\right) \quad \text{with} \quad \mathbf{B}^*_{n,k} = (-1)^{n-k}\frac{2n}{n+k}\binom{n+k}{n-k}.$$

Proof. We begin by finding the compositional inverse of $h(t)$; if we set $y = h(t)$ we have $2ty = 1 - 2t - \sqrt{1-4t}$, and by solving in $t = t(y)$ we find:

$$\overline{h}(t) = \frac{t}{(1+t)^2}$$

after an obvious change of variables. By substitution we also find $d^*(t)$:

$$d^*(t) = \frac{1}{d(\overline{h}(t))} = \sqrt{1 - \frac{4t}{(1+t)^2}} = \frac{1-t}{1+t}.$$

Finally, we have:

$$d^*_{n,k} = [t^n]\frac{1-t}{1+t}\frac{t^k}{(1+t)^{2k}} = [t^{n-k}](1-t)(1+t)^{-2k-1} = (-1)^{n-k}\frac{2n}{n+k}\binom{n+k}{n-k}$$

as desired. □

The A-sequence is obtained from $h(t)$:

$$A^*(t) = \frac{1-2t+\sqrt{1-4t}}{2} = 1-t-tC(t) = 1-2t-t^2-2t^3-5t^4-14t^5-42t^6-132t^7+\cdots.$$

Table 5
The central binomial inverse recursive matrix.

$n\backslash k$	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
−6	1	0	0	0	0	0	0	0	0	0	0	0	0
−5	10	1	0	0	0	0	0	0	0	0	0	0	0
−4	44	8	1	0	0	0	0	0	0	0	0	0	0
−3	110	27	6	1	0	0	0	0	0	0	0	0	0
−2	165	48	14	4	1	0	0	0	0	0	0	0	0
−1	132	42	14	5	2	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0
1	−132	−42	−14	−5	−2	−1	−2	1	0	0	0	0	0
2	−165	−48	−14	−4	−1	0	2	−4	1	0	0	0	0
3	−110	−27	−6	−1	0	0	−2	9	−6	1	0	0	0
4	−44	−8	−1	0	0	0	2	−16	20	−8	1	0	0
5	−10	−1	0	0	0	0	−2	25	−50	35	−10	1	0
6	−1	0	0	0	0	0	2	−36	105	−112	54	−12	1

By Theorem 3.2 in [13], we have the identity:

$$\mathbf{B}^*_{n+1,k+1} = \mathbf{B}^*_{n,k} - 2\mathbf{B}^*_{n,k+1} + \mathbf{B}^*_{n-1,k+1}$$

which can be easily checked in Table 5

At this point we can eventually compute the explicit value of $\mathbf{B}_{n,k}$:

Theorem 6.2. *The central binomial triangle elements have the following form and generate the basic identity (2.1):*

$$\mathbf{B}_{n,k} = \binom{2n}{n-k}, \quad \sum_{j=k}^{n} \binom{2n}{n-j} (-1)^{k-j} \frac{2j}{j+k} \binom{j+k}{j-k} = \delta_{n,k}.$$

Proof. First, we observe that $\overline{h}'(t) = (1-t)/(1+t)^3$ and so we can apply Theorem 2.1:

$$\mathbf{B}_{n,k} = [t^{n-k}] \frac{1-t}{(1+t)^3} \frac{1+t}{1-t} (1+t)^{2n+2} = [t^{n-k}](1+t)^{2n} = \binom{2n}{n-k}$$

as can be easily checked against the values in Table 4. $\square$

The relevant quantities for the horizontal and vertical identities can now be computed:

Theorem 6.3. *We have the following values for the central binomial triangle:*

$$h^{(m)}_{j+m} = \begin{cases} \dfrac{m}{j+m}\dbinom{2m+2j}{j} & \text{for } j \neq -m \\ (-1)^j & \text{for } j = -m \end{cases} \qquad \text{and} \qquad a^{(m)}_j = \binom{2m}{j}.$$

Proof. The coefficients $h^{(m)}_{j+m}$ are computed as in the previous theorem, setting $d(t) = 1$. Obviously, for $j = -m$ the expression is singular, but we can eliminate $j+m$ from the denominator and eventually get $(-1)^j$. For what concerns the A-sequence, we have $a^{(m)}_j = [t^j](1+t)^{2m}$, which immediately gives the desired value. $\square$

Consequently, we have our identities:

Theorem 6.4. *The horizontal and vertical identities relative to the central binomial triangle are:*

$$\sum_{j=0}^{n-k}\binom{2n}{n-k-j}\binom{2m}{j}=\binom{2n+2m}{n-k},$$

another occurrence of Vandermonde convolution, and:

$$\sum_{j=0}^{n-k}\binom{2n-2j}{n-k-j}\frac{m}{j+m}\binom{2m+2j}{j}=\binom{2n+2m}{n-k}.$$

Proof. Immediate. $\square$

We could specialize the three parameters in many ways, but we limit ourselves to the following identity; by setting $k\mapsto 0$ and $m\mapsto n/2$, we obtain:

$$\sum_{j=0}^{n}\binom{2n-2j}{n-j}\frac{n}{n+2j}\binom{n+2j}{j}=\binom{3n}{n}\qquad n\geqslant 1.$$

For what concerns the inverse array (see Table 5), we use Theorem 3.3 to compute the appropriate coefficients:

Theorem 6.5. *For the inverse of the central binomial triangle we have:*

$$\overline{h}_{j+m}^{(m)}=(-1)^j\binom{2m+j-1}{j}\quad\text{and}\quad a_j^{*(m)}=\begin{cases}\dfrac{m}{m-j}\dbinom{2j-2m}{j} & \text{for } j\neq m\\ 2(-1)^j & \text{for } j=m.\end{cases}.$$

Proof. We find:

$$\overline{h}_{j+m}^{(m)}=a_j^{(m)}=\binom{-2m}{j}=(-1)^j\binom{2m+j-1}{j}.$$

Analogously:

$$a_j^{*(m)}=h_{j-m}^{(-m)}=\frac{m}{m-j}\binom{2j-2m}{j},$$

and leave the singular case $j=m$ as an exercise to the reader. $\square$

From these considerations we have:

Theorem 6.6. *The horizontal and vertical identities for the inverse central binomial triangle are:*

$$\sum_{j=0}^{n-k}\frac{m}{m-j}\binom{2j-2m}{j}(-1)^j\frac{2n}{n+k+j}\binom{n+k+j}{n-k-j}=\frac{2n+2m}{n+k+2m}\binom{n+k+2m}{n-k}.$$

$$\sum_{j=0}^{n-k}\binom{2m+j-1}{j}\frac{2n-2j}{n+k-j}\binom{n+k-j}{n-k-j}=\frac{2n+2m}{n+k+2m}\binom{n+k+2m}{n-k}.$$

Proof. Immediate. $\square$

We can specialize these identities in many ways. For example, if we substitute $n \mapsto n$, $k \mapsto 0$ and $m \mapsto n$ in the horizontal identity, and isolate the term with $j = n$, we obtain the following identity:

$$\sum_{j=0}^{n-1}(-1)^j \frac{2n^2}{n^2-j^2}\binom{2j-2n}{j}\binom{n+j}{n-j} = \frac{4}{3}\binom{3n}{n} - 2 \qquad n > 0.$$

An example of a different nature is obtained from the horizontal identity when $m = 1$, corresponding to the so called A-identity. This is:

$$\mathbf{B}^*_{n+1,k+1} = \mathbf{B}^*_{n,k} - \mathbf{B}^*_{n,k+1} - C_0\mathbf{B}^*_{n,k+1} - C_1\mathbf{B}^*_{n,k+2} - \cdots ,$$

where, as before, C_k is the kth Catalan number. This expression can be rewritten as:

$$\mathbf{B}^*_{n+1,k+1} - \mathbf{B}^*_{n,k} + \mathbf{B}^*_{n,k+2} = -\sum_{j=0}^{n-k} C_j\mathbf{B}^*_{n,j+1}.$$

The three constant terms simplify:

$$\frac{2n+2}{n+k+2}\binom{n+k+2}{n-k} - \frac{2n}{n+k}\binom{n+k}{n-k} - \frac{2n}{n+k+1}\binom{n+k+1}{n-k-1} = \frac{2n-1}{2k+1}\binom{n+k-1}{n-k-1},$$

and we conclude with the identity:

$$\sum_{j=0}^{n-k}\frac{(-1)^j}{j+1}\binom{2j}{j}\frac{2n}{n+k+j+1}\binom{n+k+j+1}{n-k-j-1} = \frac{2n-1}{2k+1}\binom{n+k-1}{n-k-1}$$

which is rather remarkable.

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