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ELSEVIER
Finite Fields and Their Applications 10 (2004) 306–314
FINITE FIELDS AND THEIR APPLICATIONS
http://www.elsevier.com/locate/ffa

On the size of complete caps in $PG(3, 2^h)$

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Received 25 September 2002; revised 28 August 2003

Communicated by J.W.P. Hirschfeld

Abstract

Let $m_2'(3,q)$ be the largest value of k $(k < q^2+1)$ for which there exists a complete k-cap in $PG(3,q)$, q even. In this paper, the known upper bound on $m_2'(3,q)$ is improved. We also improve a number of intervals, for k, for which there does not exist a complete k-cap in $PG(3,q)$, q even.

Keywords: Caps; Projective geometry

1. Introduction

A k-cap K in $PG(n,q)$ is a set of k points, no three of which are collinear. A point r of $PG(n,q)$ extends a k-cap K to a $(k+1)$-cap if and only if $K \cup \{r\}$ is a $(k+1)$-cap. A k-cap is complete if and only if it is not contained in a $(k+1)$-cap. A k-cap of $PG(2,q)$ is also called a k-arc of $PG(2,q)$.

Let $m_2(n,q)$ be the maximum value of k for which there exists a (complete) k-cap in $PG(n,q)$. The exact value of $m_2(n,q)$ is known in the following cases

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[1] Supported by the Flemish Institute for the Promotion of Scientific and Technological Research in Industry (IWT), Grant No. IWT/SB/991011/Ferret.

1071-5797/$ - see front matter
doi:10.1016/j.ffa.2003.08.005

(see [12, Theorem 4.1]):

$$m_2(n,2) = 2^n,$$

$$m_2(2,q) = \begin{cases} q+1, & q \text{ odd}, \\ q+2, & q \text{ even}, \end{cases}$$

$$m_2(3,q) = q^2+1 \quad (q>2),$$

$$m_2(4,3) = 20,$$

$$m_2(4,4) = 41,$$

$$m_2(5,3) = 56.$$

The second largest value of k for which there exists a complete k-cap in $\mathrm{PG}(n,q)$ is denoted by $m_2'(n,q)$. This value $m_2'(n,q)$ is only known in some small projective planes [12, Table 2.4], for $n=2, q=2^{2h}$ $(h>1)$ and for $(n,q) = (3,3),(3,4),(3,5),(3,7),(4,3),(4,4),(5,3)$ and $(n,q)=(n,2)$. Namely, $m_2'(2,2^{2h}) = 2^{2h}-2^h+1$ $(h>1)$ [4,10,15], $m_2'(3,3)=8$ [9], $m_2'(3,4)=14$ [13,14], $m_2'(3,5)=20$ [1], $m_2'(3,7)=32$ [8], $m_2'(4,3)=19$ [17], $m_2'(4,4)=40$ [7], $m_2'(5,3)=48$ [2] and $m_2'(n,2)=2^{n-1}+2^{n-3}$, $n\geqslant 3$ [6].

With respect to the other values of $m_2(n,q)$ and $m_2'(n,q)$, only upper bounds are known. We refer to [12] for a list of these upper bounds.

The following theorem gives disjoint intervals for k for which there does not exist a complete k-cap in $\mathrm{PG}(3,q)$, q even.

Theorem 1.1 (Storme and Szőnyi [16]). *There is no complete k-cap K in* $\mathrm{PG}(3,q)$, *q even, $q\geqslant 64$, with*

$$k \in \left[q^2-(a-1)q+a\sqrt{q}+2-a+\binom{a}{2},\, q^2-(a-2)q-a^2\sqrt{q}\right]$$

and with a an integer satisfying

$$2 \leqslant a \leqslant \frac{-2\sqrt{q}+3+\sqrt{16q\sqrt{q}+12q-44\sqrt{q}-7}}{4\sqrt{q}+2}.$$

We will improve the preceding theorem to the following one.

Theorem 1.2. *There is no complete k-kap K in* $\mathrm{PG}(3,q)$, *q even, $q\geqslant 16$, with*

$$k \in [q^2-(c-1)q+(2c^3+c^2-7c+6)/2,\, q^2-(c-2)q-2c^2+5c-2]$$

and with c an integer satisfying $2\leqslant c\leqslant \sqrt[4]{q}$.

There exist complete (q^2+1)-caps in $\mathrm{PG}(3,2^h)$, $h\geqslant 1$, and, by a result of Chao [5], the size of the second largest cap is bounded by $m_2'(3,q=2^h)\leqslant q^2-q+5$, for $q\geqslant 8$ [5].

We improve this upper bound on $m_2'(3,q=2^h)$ to $m_2'(3,q=2^h)\leqslant q^2-q+2$, for $q\geqslant 16$. This improvement is obtained by combining the method of Storme and Szőnyi [16] with the method of Chao [5].

2. Preliminaries

Theorem 2.1. *Let K be a k-arc in $\mathrm{PG}(2,q)$, q even, $q>2$, for which $q+1\geqslant k>q-\sqrt{q}+1$. Then K can be uniquely extended to a $(q+2)$-arc of $\mathrm{PG}(2,q)$.*

Proof. See [11, p. 233, 18]. □

Definition 2.2. A tangent L to a k-cap K in $\mathrm{PG}(n,q)$ is a line which has exactly one point in common with K. We call planes, respectively lines, intersecting K in i points, i-planes, respectively i-lines.

Remark 2.3. Let K be a k-cap in $\mathrm{PG}(n,q)$. Then each point p of K belongs to exactly $t=q^{n-1}+q^{n-2}+\cdots+q+2-k$ tangents.

Theorem 2.4. *Let K be a complete k-cap in $\mathrm{PG}(n,q)$, q even, and let $t=q^{n-1}+q^{n-2}+\cdots+q+2-k$.*

Then each point p of $\mathrm{PG}(n,q)\backslash K$ belongs to $\sigma_1(p)\leqslant t$ tangents to K.

Proof. See [13,14, Lemma 2.3, Lemma 27.4.2]. □

3. Non-existence intervals for complete k-caps in PG$(3,2^h)$

Suppose there exists a complete k-kap K in $\mathrm{PG}(3,q)$, q even, $q\geqslant 16$, with

$$k\in[q^2-(c-1)q+(2c^3+c^2-7c+6)/2, q^2-(c-2)q-2c^2+5c-2]$$

and with c an integer satisfying $2\leqslant c\leqslant \sqrt[4]{q}$.

Lemma 3.1 (cf. Chao [5]). *A plane intersects K in at most c^2-1 points or in at least $q+3-c^2$ points; we will call these planes, respectively,* small *planes and big planes.*

Proof. Let π be an x-plane (cf. [5]). Let S be the set of pairs (r,L), where $r\in\pi\backslash K$ and L is a tangent through r. By Theorem 2.4, we have $|S|\leqslant t(q^2+q+1-x)=t(t+k-1-x)$. We have $t(k-x)$ tangents to K intersecting π in exactly one point of $\pi\backslash K$. In π we have $x(q+2-x)$ tangents to K. Hence $|S| = t(k-x) + x(q+2-x)q$. Solving this inequality yields the result. $\square$

Remark 3.2. Since we impose the condition $c^4 \leqslant q$ on c, large plane intersections are always contained in $(q+2)$-caps of these planes (Theorem 2.1).

As in [16], we will count the cardinality of U, being the set of triples $(L, \alpha, r_{L,\alpha})$, where L is a tangent, α is a big plane through L and $r_{L,\alpha}$ is the unique point on L which extends $K \cap \alpha$ to a larger cap.

Lemma 3.3. $|U| \geqslant (q+2-c)(q^2-(c-2)q-2c^2+5c-2)((c-1)q+2c^2-5c+4)$.

Proof. The k-cap K has $k(q^2+q+2-k) \geqslant t_1 = (q^2-(c-2)q-2c^2+5c-2)(q^2+q+2-(q^2-(c-2)q-2c^2+5c-2))$ tangents.

Let r be the number of small planes through a tangent, then $k-1 \leqslant r(c^2-2)+(q+1-r)q$. So, there are at most $c-1$ small planes through a tangent. Hence there are at least $q+2-c$ big planes through a tangent.

For a fixed tangent L and big plane α, the point $r_{L,\alpha}$ is unique. So $|U| \geqslant (q+2-c)t_1$. $\square$

Lemma 3.4. $|U| \leqslant (c-1)(q+1)(q^3+cq-(2c^3+c^2-7c+4)/2)$.

Proof. There are at most $|PG(3,q)\backslash K| \leqslant q^3+cq-(2c^3+c^2-7c+4)/2$ choices for a point r outside the cap.

If r extends a big plane intersection $\pi \cap K$, then r lies on at least $q+3-c^2$ tangents in π. Hence, if r extends x big plane intersections through r, then r lies on at least $x(q+3-c^2)-\binom{x}{2}$ tangents. Since this number is at most t, we have $x \leqslant c-1$. So $|U| \leqslant (c-1)(q+1)(q^3+cq-(2c^3+c^2-7c+4)/2)$. $\square$

Comparing the bounds of Lemmas 3.3 and 3.4 gives a contradiction; there does not exist a complete k-cap whose size lies in the intervals of Theorem 1.2.

Remark 3.5. When looking for the improved intervals, we described the intervals as $[q^2-(c-1)q+\alpha, q^2-(c-2)q-\beta]$, and looked for the optimal values for α and β. The value α was selected so that a point $r \notin K$ belongs to at most $c-1$ big planes π, in which it extends the intersection $\pi \cap K$ to a larger cap (see proof of Lemma 3.4). Here we already rely on the fact that we know that there are small and big planes (Lemma 3.1). The proof of Lemma 3.3 was first of all done assuming that the upper bound on $|K|$ is equal to $q^2-(c-2)q-\beta$, $\beta > 0$. We then compared the upper bound of Lemma 3.4, using the already selected value of α, with the lower bound of Lemma 3.3. For the selected value of β, the desired contradiction is obtained.

4. A closer look at $m'_2(3,q)$

In this section, we will eliminate the existence of complete k-caps, where k is q^2-q+3, q^2-q+4 or q^2-q+5, in $\mathrm{PG}(3,2^h)$, $h\geqslant 4$. Assume there does exist such a cap K.

Lemma 4.1. *The only small planes are* 0*-planes or* 1*-planes.*

Proof. The reasoning of Lemma 3.1 gives that for $k=q^2-q+3$ we have at most four points in a small plane, and for $k\in\{q^2-q+4, q^2-q+5\}$, we have at most three points in a small plane.

First, we will eliminate 4-planes for a complete (q^2-q+3)-cap K. Assume there exists a 4-plane α. Take a tangent L in α. If L is contained in a $(q+1)$-plane π, then the nucleus n of this plane π lies on at least one tangent $T\neq L$ contained in α. Since $\sigma_1(n)\leqslant t=2q-1$, T cannot be contained in a $(q+1)$-plane. Hence, α always contains a tangent T which is not contained in a $(q+1)$-plane. We have the following planes through T: one 4-plane α, $q-1$ q-planes π_i and one $(q-1)$-plane β. Through the unique point n_i on T which extends $\pi_i\cap K$, we have q tangents contained in π_i, at least one tangent different from T in every other q-plane, and at least one tangent different from T in α. Hence, we have found already t tangents through n_i. Since $\sigma_1(n_i)\leqslant t$, the point n_i lies on the intersection of T and a bisecant in α. There are only three possibilities for such a point, hence we can find indices $i\neq j$ for which $n_i=n_j$, but then we have more than t tangents through n_i; a contradiction.

Now, we will eliminate 2- and 3-planes for $k\in\{q^2-q+4, q^2-q+5\}$. Assume we have a 2-plane α. Take a tangent L in α and let $\{r\}=L\cap K$. Through r go at least $q-1$ tangents in α, hence we have at most $q-1$ tangents left over to divide over the other q planes through L. So we have at least one $(q+1)$-plane π through L. Take the nucleus n of π and let T be the other tangent through n in α. If T is contained in a $(q+1)$-plane, then necessarily n is the nucleus of this plane, and we have more than t tangents through n; a contradiction. So, T lies in at most q-planes, and cannot lie in a 2-plane.

Let α be a 3-plane. Take a tangent T in α, then T is contained in a $(q+1)$-plane π. By the reasoning above, the nucleus of π is on T and on a bisecant in α. Because a point on T cannot be the nucleus of more than one $(q+1)$-plane, this implies that through a tangent T in α, there is exactly one $(q+1)$-plane. Hence, $k=q^2-q+4$ and the planes through T are one 3-plane α, one $(q+1)$-plane π and $q-1$ q-planes π_i. Let n be the nucleus of π. In every plane π_i we must have an even number of tangents through the point n, so besides T, for every q-plane, we have at least one extra tangent through n. But this yields $\sigma_1(n)>t$; a contradiction.

Finally, we will eliminate 2-planes and 3-planes for $k=q^2-q+3$. Assume we have a 2-plane α, and take a tangent L in α. Through L we have at least one $(q+1)$-plane π. Take the tangent $T\neq L$, contained in α, passing through the nucleus n of π. This tangent T cannot be contained in a $(q+1)$-plane, since n would also be the nucleus of this plane and we would have too many tangents through n. Counting incidences with planes through T, we obtain $k-1\leqslant 1+q(q-1)$; a contradiction.

Now assume we have a 3-plane α, and take a tangent L in α. Assume there is no $(q+1)$-plane through L, then all planes $\pi_i \neq \alpha$ through L must be q-planes. Now consider the point n_i on L which extends $K \cap \pi_i$. Through n_i, there are q tangents in π_i, and there is at least one extra tangent for every other q-plane $\pi_j \neq \pi_i$. Hence, we have already found t tangents through n_i, and n_i must be the unique point on L, which lies on the unique bisecant to $(K \cap \alpha)\backslash L$. But we can do this reasoning for every q-plane through L, and we have too many tangents through the point $n_1 = \cdots = n_q$.

Hence, there is a $(q+1)$-plane π through L. We can assume that the nucleus n of $\pi \cap K$ must be the point on L, which lies on a bisecant B to $K \cap \alpha$, or else we are reduced to the preceding situation. Also, π is the only $(q+1)$-plane through L. This yields that there must be at least $q-2$ q-planes through n, all having a tangent different from L through n. Hence $\sigma_1(n) \geqslant q-2+q+1 = t$. Now, consider the planes β_i through B different from the 3-plane. Since such a plane β_i intersects π in a tangent, β_i is not a $(q+2)$-plane, and then β_i has to be a $(q+1)$-plane.

Since n lies on a bisecant B, n lies on exactly one tangent in every plane β_i. This gives $\sigma_1(n) = q+1$; a contradiction with the preceding paragraph where we showed that $\sigma_1(n) = t$. $\square$

Lemma 4.2. *Every tangent L is contained in a $(q+1)$-plane.*

Proof. Let $\{r\} = L \cap K$. If L is contained in a small plane, the result is trivial. Now suppose that L is only contained in big planes. Hence, there exists a point s on L which extends at least two plane intersections through L.

If $k = q^2 - q + 5$, then the reasoning of Lemma 3.1 shows that the big planes contain at least $q-1$ points of K. Then s extends two $(q-1)$-caps lying in planes π_1 and π_2 through L, and there are no tangents through s outside of π_1 or π_2; since $\sigma_1(s) \leqslant t = 2q-3$. Since the number of tangents through s to a planar q-cap is even, there cannot be a q-plane through L. Hence, the planes through L are 3 $(q+1)$-planes and $q-2$ $(q-1)$-planes.

If $k = q^2 - q + 4$, then also here the big planes contain at least $q-1$ points of K. Then a first possibility is that s extends two $(q-1)$-caps lying in planes π_1 and π_2 through L, and there is at most one tangent through s not lying in π_1 or π_2. Then we have at most one q-plane through L and the planes through L must be 1 q-plane, $2(q+1)$-planes and $q-2$ $(q-1)$-planes. The other possibility is that s extends a $(q-1)$-cap and a q-cap lying in planes through L. This case is similar to the previous one.

If $k = q^2 - q + 3$, then big planes share at least $q-2$ points with K. Then we have at most $t - (q-2+q-2-1) = 4$ q-planes through L, and equality might only occur when s extends two $(q-2)$-caps lying in planes through L. Assume that there are no $(q+1)$-planes through L, then L lies in at most four q-planes. Once the number of q-planes through L is fixed, the number of $(q-1)$- and $(q-2)$-planes also is fixed. Checking all cases, only the possibility remains that the planes through L are 4 q-planes and $q-3$ $(q-1)$-planes. But then s lies on $2q-3$ tangents to the two $(q-1)$-planes and to four extra tangents to the four q-planes; but then $\sigma_1(s) = 2q+1 > t$. $\square$

Lemma 4.3. *All tangents through a nucleus n of a $(q+1)$-plane π lie in this $(q+1)$-plane.*

Proof. Assume there is a tangent M through n not contained in π. Then M lies in a $(q+1)$-plane β. Since n lies on at least two tangents M and $\pi \cap \beta$ in β, n is the nucleus of $\beta \cap K$. But then $\sigma_1(n) > t$; a contradiction.

The final contradiction: If $k = q^2 - q + 4$, then take the nucleus n of a $(q+1)$-plane. By Lemma 4.3, we have $\sigma_1(n) = q+1$; while this number of tangents through n should be even.

If $k \in \{q^2 - q + 3, q^2 - q + 5\}$, then we will first show that the only even plane intersections have size 0 or $q+2$. Suppose we have a plane π which intersects K in a q-arc or $(q-2)$-arc. Then take a tangent L in π, this tangent is contained in a $(q+1)$-plane γ. The number of tangents through the nucleus n of $\gamma \cap K$ to $\pi \cap K$ is even; hence there is a tangent through n not contained in γ; a contradiction (Lemma 4.3).

Now, using the standard counting arguments on the numbers $\sum_i n_i, \sum_i i n_i, \sum_i i(i-1)n_i, \sum_i i(i-1)(i-2)n_i$, we can compute

$$\sum_i n_i(i-1)(i-(q-1))(i-(q+1)),$$

where n_i denotes the number of i-planes.

For $k = q^2 - q + 3$, we obtain

$$-n_0(q^2-1) + 3n_{q+2}(q+1) = -5q^3 + 17q^2 - 22q + 16.$$

For $k = q^2 - q + 5$, we obtain

$$-n_0(q^2-1) + 3n_{q+2}(q+1) = -7q^3 + 39q^2 - 68q + 96.$$

In both cases we get a contradiction, computing these equations modulo $q+1$. □

Acknowledgments

The authors wish to thank the referees for their suggestions in improving the article.

Appendix

The crucial element in the improvement of Theorem 1.1 to Theorem 1.2 was the fact that the technique of Chao [5] (cf. proof of Lemma 3.1) showed that planes of $\mathrm{PG}(3,q)$ intersect large complete caps of $\mathrm{PG}(3,q)$ in either a small number of points or in a large number of points.

A check on the known upper bounds for the sizes of caps in $\mathrm{PG}(n,q)$, q even, $n \geqslant 4$, shows that a similar result is valid for large caps in $\mathrm{PG}(n,q)$, q even. We present these results in Theorem 5.2.

To find an upper bound on the sizes of caps in $\mathrm{PG}(n,q)$, q even, we rely on the upper bound on the size of caps in affine spaces $\mathrm{AG}(n,q)$ of Bierbrauer and Edel.

They proved the following result.

Theorem 5.1 (Bierbrauer and Edel [3]). *Let C_n be the largest size of a cap in $\mathrm{AG}(n,q)$, q even, $q>2$, $n \geqslant 4$, then*

$$C_n \leqslant \frac{q^{n-1} + q^n C_{n-1}}{q^{n-1} + C_{n-1}}.$$

As indicated in [3], starting from $C_3 = q^2$, $C_4 = q^3 - q^2 + q$ follows.

In general, for $4 \leqslant n \leqslant 2q/3$, $q \geqslant 8$, the upper bound

$$C_n \leqslant q^{n-1} - (n-3)q^{n-2} + (n-3)^2 q^{n-3}$$

follows. Adding the upper bound q^{n-2} for the size of a cap in $\mathrm{PG}(n-1,q)$, it follows that

$$m_2(n,q) \leqslant q^{n-1} - (n-4)q^{n-2} + (n-3)^2 q^{n-3}$$

for even $q \geqslant 8$ and $4 \leqslant n \leqslant 2q/3$.

We now present an interval theorem on the sizes of the hyperplane sections of hyperplanes of $\mathrm{PG}(n,q)$, q even, with complete large k-caps.

Theorem 5.2. *In $\mathrm{PG}(n,q)$, q even, $n \geqslant 4$, a complete k-cap K, with*

$$k > q^{n-1} - (c-1)q^{n-2} + q^{n-3} + q^{n-4} + \cdots + q + 2 + c^3 q^{n-3}/2$$

with $c^4 \leqslant 4q$, intersects a hyperplane in less than $c^2 q^{n-3}$ points or in larger than $q^{n-2} + \cdots + q + 2 - c^2 q^{n-3}$ points.

Proof. We repeat the arguments of the proof of Lemma 3.1; see also the case $n = 3$ in [5].

Let $|K| = q^{n-1} - (c-1)q^{n-2} + q^{n-3} + q^{n-4} + \cdots + q + 2 + \varepsilon$, then the number of tangents through a point of K is $t = cq^{n-2} - \varepsilon$.

Let π be a hyperplane intersecting K in x points.

We count the number of ordered pairs (r, L), where $r \in \pi \backslash K$, and where L is a tangent to K through r.

Then this number equals $t(k-x) + x(q^{n-2} + \cdots + q + 2 - x)q$, and $t(q^{n-1} + \cdots + q + 1 - x)$ is an upper bound for this number.

This inequality is equivalent to

$$x^2(q-q^2)+x(q^n+q^2-2q)-c^2q^{2n-3}+c^2q^{2n-4}$$
$$+(c+2\varepsilon c)q^{n-1}-(c+2\varepsilon c)q^{n-2}-(\varepsilon+\varepsilon^2)q+\varepsilon+\varepsilon^2\leqslant 0.$$

For $\varepsilon > c^3q^{n-3}/2$, this implies $x < c^2q^{n-3}$ or $x > q^{n-2}+\cdots+q+2-c^2q^{n-3}$. $\square$

Remark 5.3. Presently, no caps in PG(n,q), q even, $n\geqslant 4$ small, of the size of the upper bounds of Theorems 5.1 and 5.2 are known.

Nevertheless, the result of Theorem 5.2 might be useful in eliminating the existence of caps of these sizes.

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