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# On $\alpha_r\gamma_s(k)$-perfect graphs

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**Abstract**

For some integer $k \geqslant 0$ and two graph parameters $\pi$ and $\tau$, a graph $G$ is called $\pi\tau(k)$-*perfect*, if $\pi(H) - \tau(H) \leqslant k$ for every induced subgraph $H$ of $G$. For $r \geqslant 1$ let $\alpha_r$ and $\gamma_r$ denote the $r$-(distance)-independence and $r$-(distance)-domination number, respectively. In (J. Graph Theory 32 (1999) 303–310), I. Zverovich gave an ingenious complete characterization of $\alpha_1\gamma_1(k)$-perfect graphs in terms of forbidden induced subgraphs. In this paper we study $\alpha_r\gamma_s(k)$-perfect graphs for $r, s \geqslant 1$. We prove several properties of minimal $\alpha_r\gamma_s(k)$-imperfect graphs. Generalizing Zverovich's main result in (J. Graph Theory 32 (1999) 303–310), we completely characterize $\alpha_{2r-1}\gamma_r(k)$-perfect graphs for $r \geqslant 1$. Furthermore, we characterize claw-free $\alpha_2\gamma_2(k)$-perfect graphs.



*Keywords:* Domination perfect graphs; Domination; Independence; Distance domination number; Distance independence number

## 1. Introduction

All graphs will be finite, undirected and without loops or multiple edges. We will use the standard graph-theoretical terminology (cf. e.g. [7]). Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The set of isolated vertices of $G$ is denoted by $\mathrm{Iso}(G)$. A clique of $G$ is the vertex set of a complete subgraph of $G$. The subgraph of $G$ induced by a set of vertices $V' \subseteq V(G)$ is denoted by $G[V']$. If $H$ is an induced subgraph of $G$, then we write $H \subseteq_{\mathrm{ind}} G$. If $v \in V(G)$, then $G-v=G[V(G)\setminus\{v\}]$. A graph is *claw-free*, if it does not contain the star with three endvertices as an induced subgraph. Let $P_r$ denote the path of order $r \geqslant 1$. We say that *the graph $H$ arises from $G$ by attaching a path $P_r$ to a vertex $v \in V(G)$*, if $V(H) \setminus V(G) = \{v_2, v_3, \ldots, v_r\}$

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and $E(H) \setminus E(G) = \{vv_2, v_2v_3, \ldots, v_{r-1}v_r\}$. The distance of two vertices $u, v \in V(G)$ is denoted by $\mathrm{dist}_G(u, v)$. For $V' \subseteq V(G)$ let $\mathrm{dist}_G(u, V') = \min\{\mathrm{dist}_G(u, v) | v \in V'\}$.

Let $r \geqslant 1$. For $u \in V(G)$ let $N_G^r(u) = \{v \in V(G) | v \neq u, \mathrm{dist}_G(u, v) \leqslant r\}$. The neighbourhood and degree of $u \in V(G)$ are denoted by $N_G(u) = N_G^1(u)$ and $d_G(u) = |N_G(u)|$, respectively. A set $I \subseteq V(G)$ is an *r-independent set* of $G$, if $\mathrm{dist}_G(u, v) \geqslant r + 1$ for all $u, v \in I$. The *r-independence number* $\alpha_r(G)$ of $G$ is the maximum cardinality of an $r$-independent set of $G$. An *$\alpha_r$-set* of $G$ is an $r$-independent set of cardinality $\alpha_r(G)$.

A set $D \subseteq V(G)$ is an *r-dominating set* of $G$, if for each vertex $u \in V(G) \setminus D$ there is some vertex $v \in D$ such that $\mathrm{dist}_G(u, v) \leqslant r$. The *r-domination number* $\gamma_r(G)$ of $G$ is the minimum cardinality of an $r$-dominating set of $G$. A *$\gamma_r$-set* of $G$ is an $r$-dominating set of cardinality $\gamma_r(G)$. For $u \in D \subseteq V(G)$ let

$$PN_G^r(u, D) = (N_G^r(u) \cup \{u\}) \setminus \bigcup_{v \in D \setminus \{u\}} N_G^r(v).$$

In [15], I. Zverovich proposed the following definition of classes of '*perfect*' graphs. Let $k \geqslant 0$ and $\pi$ and $\tau$ be two graph parameters. A graph $G$ is called *$\pi\tau(k)$-perfect*, if $\pi(H) - \tau(H) \leqslant k$ for all $H \subseteq_{\mathrm{ind}} G$. A graph $G$ is a *minimal $\pi\tau(k)$-imperfect graph*, if $\pi(G) - \tau(G) > k$ but $\pi(H) - \tau(H) \leqslant k$ for each $H \subseteq_{\mathrm{ind}} G$ with $H \neq G$. There is an extensive literature about $\pi\tau(0)$-perfect graphs for appropriate choices of $\pi$ and $\tau$. Well-studied examples of $\pi\tau(0)$-perfect graphs involve the notions of independence and domination (cf. e.g. [1,2,5,6,8–14,16,17]).

The main goal in the study of $\pi\tau(k)$-perfect graphs is a characterization in terms of a minimal list of forbidden induced subgraphs. In many cases such characterizations are either trivial (as for $\alpha_1\gamma_1(0)$-perfect graphs) or hard to find (cf. e.g. [16]).

In [15], I. Zverovich was able to give a surprising and ingenious characterization of $\alpha_1\gamma_1(k)$-perfect graphs. We will generalize his result by studying $\alpha_r\gamma_s$-perfect graphs for $r, s \geqslant 1$. If $r + 1 > 2s$, then $|N_G^s(u) \cap I| \leqslant 1$ for each $u \in V(G)$ and every $r$-independent set $I$. Hence $\alpha_r(G) \leqslant \gamma_s(G)$ and all graphs are $\alpha_r\gamma_s(k)$-perfect.

Many choices of the parameters $r$, $s$ and $k$ lead to minimal $\alpha_r\gamma_s(k)$-imperfect graphs that do not have a simple structure. Therefore, in order to obtain results as elegant as in [15] we have to find the '*good*' choices for $r$, $s$ and $k$. Two natural candidates for the generalization of $\alpha_1\gamma_1(k)$-perfect graphs are the classes of $\alpha_{2r-1}\gamma_r(k)$-perfect graphs and $\alpha_r\gamma_r(k)$-perfect graphs.

In Section 2 we prove several properties of minimal $\alpha_r\gamma_s(k)$-imperfect graphs. Generalizing Zverovich's main result in [15], we completely characterize $\alpha_{2r-1}\gamma_r(k)$-perfect graphs for $r \geqslant 1$ in Section 3. Furthermore, in Section 4, we characterize claw-free $\alpha_2\gamma_2(k)$-perfect graphs. The reader that is interested in further results on $\pi\tau(k)$-perfect graphs for $k \geqslant 1$ is referred to [3,4].

## 2. Properties of minimal $\alpha_r\gamma_s(k)$-imperfect graphs

**Lemma 2.1.** *Let $G$ be a minimal $\alpha_r\gamma_s(k)$-imperfect graph for $r, s \geqslant 1$, $r + 1 \leqslant 2s$ and $k \geqslant 0$. Let $I$ be an $\alpha_r$-set and let $D$ be a $\gamma_s$-set of $G$. Then $|PN_G^s(v, D) \cap I| \geqslant 2$ for all $v \in D$. Furthermore, if $r \geqslant s$, then $D \cap I = \emptyset$.*

**Proof.** If $|PN_G^s(v,D) \cap I| \leqslant 1$ for some $v \in D$, then let $H = G[V(G) \setminus PN_G^s(v,D)] \neq G$. Since $I \setminus PN_G^s(v,D) \subseteq V(H)$, $\alpha_r(H) \geqslant \alpha_r(G) - 1$. Since $D \setminus \{v\}$ is an $s$-dominating set of $H$, $\gamma_s(H) \leqslant \gamma_s(G) - 1$. Hence $\alpha_r(H) - \gamma_s(H) \geqslant \alpha_r(G) - \gamma_s(G) \geqslant k + 1$, which is a contradiction. We obtain $|PN_G^s(v,D) \cap I| \geqslant 2$ for all $v \in D$.

Now let $r \geqslant s$. We assume that there is some $u \in D \cap I$. Let $v \in PN_G^s(u,D) \cap I$ with $v \neq u$. We have $r + 1 \leqslant \text{dist}_G(u,v) \leqslant s$, which is a contradiction. □

We will now consider certain paths which allow us to study the structure of the minimal imperfect graphs. For further reference we give the following definition.

**Definition 2.2.** Let $G$ be a minimal $\alpha_r\gamma_s(k)$-imperfect graph for $r, s \geqslant 1$, $r + 1 \leqslant 2s$ and $k \geqslant 0$. Let $I$ be an $\alpha_r$-set and let $D$ be a $\gamma_s$-set of $G$.

For each $u \in I$ let $\text{dom}(u) \in D$ be such that $\text{dist}_G(u, \text{dom}(u)) = \text{dist}_G(u, D)$ and let $P(u)$ be a shortest path from $u$ to $\text{dom}(u)$. (Note that the choice of $\text{dom}(u)$ and $P(u)$ may not be unique.)

**Lemma 2.3.** *Let $r \geqslant s \geqslant 1$ with $r + 1 \leqslant 2s$ and $k \geqslant 0$. Let $G$, $I$, $D$,* dom$(u)$ *and $P(u)$ for $u \in I$ be as in Definition 2.2. Then*

(i) $V(G) = \bigcup_{u \in I} V(P(u))$.
(ii) *If $N_G(u) \cap V(P(v)) \neq \emptyset$ for $u, v \in I$ with $u \neq v$, then $r = s$, $N_G(u) \cap V(P(v)) = \{\text{dom}(v)\}$ and $\text{dist}_G(v, \text{dom}(v)) = s$.*
(iii) *If $d_G(u) \geqslant 2$ for some $u \in I$, then $r = s$ and $N_G(u) \subseteq D$.*
(iv) $\alpha_r(G) - \gamma_s(G) = k + 1$.
(v) $\text{dist}_G(u, I \setminus \{u\}) = r + 1$ *for $u \in I$.*

**Proof.** (i): Let $H = G[\bigcup_{u \in I} V(P(u))]$. Since $I \subseteq V(H)$, $\alpha_r(H) \geqslant \alpha_r(G)$. Since $V(P(u)) \subseteq N_G^s(\text{dom}(u)) \cup \{\text{dom}(u)\}$ for $u \in I$, $\gamma_s(H) \leqslant \gamma_s(G)$. Hence $\alpha_r(H) - \gamma_s(H) \geqslant k + 1$. Since $G$ is minimal $\alpha_r\gamma_s(k)$-imperfect, $G = H$.

(ii): Let $N_G(u) \cap V(P(v)) \neq \emptyset$ for $u, v \in I$ with $u \neq v$. We have $r + 1 \leqslant \text{dist}_G(u,v) \leqslant \text{dist}_G(\text{dom}(v), v) + 1 \leqslant s + 1 \leqslant r + 1$. Hence we have equality throughout this chain of inequalities which implies all desired properties.

(iii): If $d_G(u) \geqslant 2$ for some $u \in I$, then there is some $v \in I$ with $v \neq u$ such that $N_G(u) \cap V(P(v)) \neq \emptyset$. By (ii), $r = s$ and $N_G(u) \cap V(P(v)) = \{\text{dom}(v)\}$. Hence $1 \leqslant \text{dist}_G(u, \text{dom}(u)) \leqslant \text{dist}_G(u, \text{dom}(v)) = 1$. This implies that $N_G(u) \subseteq D$.

(iv): If $\alpha_r(G) - \gamma_s(G) \geqslant k + 2$, then let $u \in PN_G^s(v) \cap I$ for some $v \in D$. By (iii), $d_G(u) = 1$. Let $H = G - u$. Clearly, $\alpha_r(H) \geqslant \alpha_r(G) - 1$ and $\gamma_s(H) \leqslant \gamma_s(G)$. Hence $\alpha_r(H) - \gamma_s(H) \geqslant k + 1$, which is a contradiction.

(v): If $d_G(u) = 1$ for some $u \in I$ and $\text{dist}_G(u, I \setminus \{u\}) \geqslant r + 2$, then let $v \in N_G(u)$ and $H = G - u$. Clearly, $\alpha_r(H) \geqslant |(I \setminus \{u\}) \cup \{v\}| = \alpha_r(G)$ and $\gamma_s(H) \leqslant \gamma_s(G)$. We obtain the same contradiction as above. Hence, by (iii), we can assume that $d_G(u) \geqslant 2$, $r = s$ and $N_G(u) \subseteq D$. If $v \in PN_G^s(\text{dom}(u), D) \cap I$ with $u \neq v$, then $r + 1 \leqslant \text{dist}_G(u,v) \leqslant \text{dist}_G(\text{dom}(u), v) + 1 \leqslant s + 1 = r + 1$. This implies that $\text{dist}_G(u, I \setminus \{u\}) = r + 1$. □

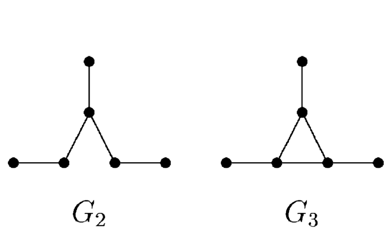

Fig. 1.

Before we come to the next section, we derive some corollaries of the above properties.

**Corollary 2.4.** *Let $G$ be a minimal $\alpha_r\gamma_s(k)$-imperfect graph for $r \geqslant s \geqslant 1$ with $r+1 \leqslant 2s$ and $k \geqslant 0$. Then*

(i) $\alpha_r(G) \geqslant 2\gamma_s(G)$.
(ii) $\gamma_s(G) \leqslant k+1$.
(iii) $|V(G)| \leqslant (k+1)(2r+1)$, *i.e. there are only finitely many non-isomorphic minimal $\alpha_r\gamma_s(k)$-imperfect graphs.*

**Proof.** (i): This immediately follows from Lemma 2.1.

(ii): By Lemma 2.3(iv), $k+1 = \alpha_r(G) - \gamma_s(G) \geqslant 2\gamma_s(G) - \gamma_s(G) = \gamma_s(G)$.

(iii): By Lemma 2.3(i) and (iv), we have $|V(G)| \leqslant s \cdot \alpha_r(G) + \gamma_s(G) = s \cdot (k+1+\gamma_s(G)) + \gamma_s(G) \leqslant s \cdot (k+1+k+1) + k+1 = (k+1)(2s+1)$. □

By Corollary 2.4, there are only finitely many non-isomorphic minimal $\alpha_r\gamma_s(k)$-imperfect graphs and all have order at most $(k+1)(2s+1)$. Hence for any fixed $r$, $s$ and $k$ as in Corollary 2.4, it is a '*finite*' problem to find all minimal $\alpha_r\gamma_s(k)$-imperfect graphs. Once these finitely many graphs have been found, the characterization of the $\alpha_r\gamma_s(k)$-perfect graphs immediately follows. It is e.g. straightforward to check that the three graphs in Fig. 1 are all $\alpha_2\gamma_2(1)$-imperfect graphs. Therefore a graph is $\alpha_2\gamma_2(1)$-perfect graphs if and only if it does not contain $G_1$, $G_2$ or $G_3$ as an induced subgraph.

## 3. $\alpha_{2r-1}\gamma_r(k)$-perfect graphs

Let $r \geqslant 1$. Let $H$ be a graph and let $I_H$ be a maximal independent set of $H$. Let $D = V(H) \backslash (I_H \setminus \mathrm{Iso}(H))$. The graph $f_r(H, I_H)$ arises from $H$ by attaching exactly two paths $P_{r+1}$ to each vertex in $D$ and exactly one path $P_r$ to each vertex in $V(H) \backslash D$. See Fig. 2 for an example.

Since every $(2r-1)$-independent set $I$ of $f_r(H, I_H)$ contains at most one vertex from each of the attached paths, $\alpha_{2r-1}(f_r(H, I_H)) \leqslant 2|D| + |I_H \backslash \mathrm{Iso}(H)|$. On the other hand, for $r=1$ the set $[I_H \backslash \mathrm{Iso}(H)] \cup [V(f_1(H, I_H)) \backslash V(H)]$ and for $r \geqslant 2$ the set of all

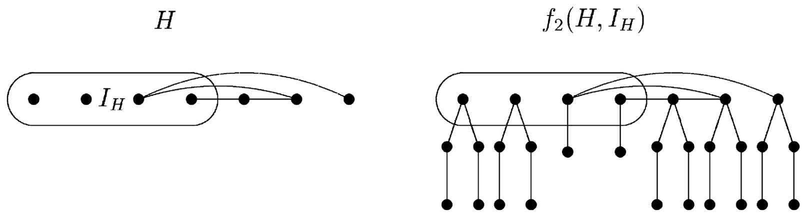


Fig. 2.

endvertices of $f_r(H,I_H)$ is a $(2r-1)$-independent set of cardinality $2|D|+|I_H\backslash \mathrm{Iso}(H)|$. Hence $\alpha_{2r-1}(f_r(H,I_H)) = 2|D| + |I_H\backslash \mathrm{Iso}(H)|$.

Since two paths $P_{r+1}$ have been attached to each vertex in $D$, every $\gamma_r$-set of $f_r(H,I_H)$ contains the set $D$. On the other hand $D$ is a $\gamma_r$-set of $f_r(H,I_H)$. Hence $\gamma_r(f_r(H,I_H)) = |D|$. Together, we obtain

$$\alpha_{2r-1}(f_r(H,I_H)) - \gamma_r(f_r(H,I_H)) = |D| + |I_H\backslash \mathrm{Iso}(H)| = |V(H)|.$$

For $k \geqslant 0$ let

$$\mathscr{F}_r(k) = \{f_r(H,I) | I \text{ is a maximal independent set of } H \text{ and } |V(H)| = k+1\}.$$

The following result generalizes Zverovich's main result from [15]. Our proof works along the lines of [15] dealing with several additional complications.

**Theorem 3.1.** *Let $r \geqslant 1$ and $k \geqslant 0$. A graph is $\alpha_{2r-1}\gamma_r(k)$-perfect if and only if it contains no graph in $\mathscr{F}_r(k)$ as an induced subgraph.*

**Proof.** We will first prove that every minimal $\alpha_{2r-1}\gamma_r(k)$-imperfect graph belongs to $\mathscr{F}_r(k)$. Then we prove that every graph in $\mathscr{F}_r(k)$ is also a minimal $\alpha_{2r-1}\gamma_r(k)$-imperfect graph. These two facts establish the desired result.

Let $G$ be a minimal $\alpha_{2r-1}\gamma_r(k)$-imperfect graph. Let $I$ be an $\alpha_{2r-1}$-set and let $D$ be a $\gamma_r$-set of $G$. Note that $G$ has the properties stated in Lemmas 2.1 and 2.3. Hence $D \cap I = \emptyset$ and $|PN_G^r(v,D) \cap I| \geqslant 2$ for every $v \in D$. For $u \in I$ let $\mathrm{dom}(u)$ and $P(u)$ be as in Definition 2.2.

Let $u \in I$. Let $u' \in PN_G^r(\mathrm{dom}(u),D) \cap I$ with $u' \neq u$. We have $(2r-1)+1 = 2r \leqslant \mathrm{dist}_G(u,u') \leqslant \mathrm{dist}_G(u,\mathrm{dom}(u)) + \mathrm{dist}_G(\mathrm{dom}(u),u') \leqslant r+r = 2r$. Hence, $\mathrm{dist}_G(u,\mathrm{dom}(u)) = r$ for all $u \in I$ and $\mathrm{dist}_G(u,u') = 2r$ for $u,u' \in I$ with $u \neq u'$ and $\mathrm{dom}(u) = \mathrm{dom}(u')$.

For $u \in I$ let $\mathrm{dom}'(u)$ be the unique neighbour of $\mathrm{dom}(u)$ in $V(P(u))$. Since $I$ is $(2r-1)$-independent, we obtain $N_G(v) \subseteq V(P(u))$ for all $u \in I$ and $v \in V(P(u))\backslash \{\mathrm{dom}'(u),\mathrm{dom}(u)\}$. Furthermore, $N_G(\mathrm{dom}'(u))\backslash V(P(u)) \subseteq D$ for all $u \in I$.

Let the set $X$ contain two vertices of $PN_G^r(v,D) \cap I$ for each $v \in D$. Let $H = G[D \cup \{\mathrm{dom}'(u) | u \in I\backslash X\}]$ and $I_H = \mathrm{Iso}(H) \cup \{\mathrm{dom}'(u) | u \in I\backslash X\}$. The set $I_H$ is an independent set of $H$ and the set $D$ is a dominating set of $H$.

If $I_H$ is not a maximal independent set of $H$, then there is a vertex $v \in D\backslash \mathrm{Iso}(H)$ such that $I_H \cup \{v\}$ is an independent set of $H$. Since $v$ has no neighbour in $\{\mathrm{dom}'(u) | u \in I\backslash X\}$, we have $|PN^r_G(v,D) \cap I| = 2$ and thus $PN^r_G(v,D) \cap I \subseteq X$. Since $v \notin \mathrm{Iso}(H)$, there is a vertex $w \in D$ such that $v \in N_H(w)$. Let $\{u_1, u_2\} = PN^r_G(v,D) \cap I$ and let $G' = G[V(G) \setminus [(V(P(u_2) \setminus \{v\}) \cup \{u_1\}]]$. Let $u'_1$ be the unique neighbour of $u_1$ in $G$. It is easy to see that the set $(I \setminus \{u_1, u_2\}) \cup \{u'_1\} \subseteq V(G')$ is a $(2r-1)$-independent set of $G'$. Hence $\alpha_{2r-1}(G') \geqslant \alpha_{2r-1}(G) - 1$. Since $\mathrm{dist}_{G'}(w, u'_1) \leqslant r$, the set $D \setminus \{v\}$ is an $r$-dominating set of $G'$ and hence $\gamma_r(G') \leqslant \gamma_r(G) - 1$. We obtain the contradiction $\alpha_{2r-1}(G') - \gamma_r(G') \geqslant k + 1$. Hence $I_H$ is a maximal independent set of $H$ and $G = f_r(H, I_H)$. Furthermore, $|V(H)| = \alpha_{2r-1}(G) - \gamma_r(G) = k + 1$, i.e. $G \in \mathscr{F}_r(k)$.

Now, let $G = f_r(H, I_H) \in \mathscr{F}_r(k)$. Let $D = V(H) \setminus (I_H \setminus \mathrm{Iso}(H))$. For $r = 1$ let $I = [I_H \setminus \mathrm{Iso}(H)] \cup [V(f_1(H, I_H)) \setminus V(H)]$ and for $r \geqslant 2$ let $I$ be the set of all end-vertices of $G$. $D$ is a $\gamma_r$-set and $I$ is a $\alpha_{2r-1}$-set of $G$. Let $\mathrm{dom}(u)$ and $P(u)$ for $u \in I$ be as in Definition 2.2. Let $G' \subseteq_{\mathrm{ind}} G$ be a minimal $\alpha_{2r-1}\gamma_r(k)$-imperfect graph. We have to prove that $G' = G$. Let

$$D_1 = \{v \in D | PN^r_G(v,D) \cap I \cap V(G') \neq \emptyset\},$$

$$D_2 = \{v \in D | PN^r_G(v,D) \cap I \cap V(G') = \emptyset\} \cap V(G'),$$

$$D_3 = \{v \in D | PN^r_G(v,D) \cap I \cap V(G') = \emptyset\} \setminus V(G'),$$

$$I_1 = \bigcup_{v \in D_1} (PN^r_G(v,D) \cap I \cap V(G')),$$

$$I'_1 = \bigcup_{v \in D_1} (PN^r_G(v,D) \cap I) \setminus V(G'),$$

$$I_2 = \bigcup_{v \in D_2} (PN^r_G(v,D) \cap I),$$

$$I_3 = \bigcup_{v \in D_3} (PN^r_G(v,D) \cap I),$$

$$I_4 = \left[ I \Bigg\backslash \bigcup_{v \in D} PN^r_G(v,D) \right] \cap V(G')$$

and

$$I_5 = \left[ I \Bigg\backslash \bigcup_{v \in D} PN^r_G(v,D) \right] \Bigg\backslash V(G').$$

Let $d_\nu = |D_\nu|$ for $\nu = 1, 2, 3$. Let $i_\nu = |I_\nu|$ for $\nu = 1, 2, \ldots, 5$ and $i'_1 = |I'_1|$. By the construction of $f_r(H, I_H)$, we have $2d_1 \leqslant i_1 + i'_1$, $2d_2 \leqslant i_2$ and $2d_3 \leqslant i_3$. By definition, $D = D_1 \cup D_2 \cup D_3$ and $(I_2 \cup I_3) \cap V(G') = \emptyset$.

Let $u \in I$ and $P(u) : u = u_1 u_2 \ldots u_{l-1} u_l = \mathrm{dom}(u)$. By Lemma 2.1, $G'$ has no component that is isomorphic to a path $P_\nu$ for $\nu < 2r + 1$. Thus for $1 \leqslant i \leqslant l - 2$, $u_i \in V(G')$ implies $u_{i+1} \in V(G')$. Furthermore, if $N^r_G(u) \cap D \cap V(G') = \emptyset$, then $V(P(u)) \cap V(G') = \emptyset$.

This implies that $\bigcup_{u\in I_3} V(P(u))\cap V(G') = \emptyset$. Furthermore, for each $u\in(I_4\cup I_5)$ such that $V(P(u))\cap V(G')\neq\emptyset$ we have $N_G^r(u)\cap(D_1\cup D_2)\neq\emptyset$.

Let $I'$ be an $\alpha_{2r-1}$-set of $G'$. It is easy to see that we can assume without loss of generality that $(I_1\cup I_4)\subseteq I'$. This implies $I'\cap\bigcup_{u\in I_1\cup I_1'\cup I_4} V(P(u)) = I_1\cup I_4$.

For each vertex $v\in D_2$, we have $|(I'\backslash I_1\cup I_4)\cap N_G^r(v)|\leqslant 1$. Since $\mathrm{dist}_G(u,D_2)\leqslant r$ for all $u\in I'\backslash I_1\cup I_4$, this implies that $|I'\backslash(I_1\cup I_4)|\leqslant|D_2|$ and $\alpha_{2r-1}(G')=|I'|\leqslant i_1+i_4+d_2$.

Let $D'$ be an $r$-dominating set of $G'$. We can assume without loss of generality that $D_1\subseteq D'$ and thus $\gamma_r(G')\geqslant d_1$. This implies that

$$\begin{aligned}
k+1 &\leqslant \alpha_{2r-1}(G')-\gamma_r(G')\\
&\leqslant i_1+i_4+d_2-d_1\\
&= (i_1+i_1'+i_2+i_3+i_4+i_5)-(d_1+d_2+d_3)\\
&\quad +2d_2-i_2+2d_3-i_3-i_1'-d_3-i_5\\
&\leqslant |I|-|D|-i_1'-d_3-i_5\\
&\leqslant |I|-|D|\\
&\leqslant k+1.
\end{aligned}$$

We deduce that $i_1'=d_3=i_5=0$, $i_2=2d_2$, $i_3=2d_3=0$, $\alpha_{2r-1}(G')=i_1+i_4+d_2$ and $D_1$ is a $\gamma_r$-set of $G'$.

We assume that $d_2\geqslant 1$. Since $\alpha_{2r-1}(G')>i_1+i_4$, there is some $u\in I'\cap\bigcup_{w\in I_2} V(P(w))$. Let $v\in I_2$ be such that $u\in V(P(v))$. Since $\{u\}\cup I_1\cup I_4\subseteq I'$ and $D_1$ is a $\gamma_r$-set of $G'$, we have $N_G(\mathrm{dom}(v))\cap\bigcup_{w\in I_1\cup I_4} V(P(w))\subseteq D_1\cup D_2$ and $N_G(\mathrm{dom}(v))\cap(D_1\cup D_2)\neq\emptyset$. Hence $\mathrm{dom}(v)\notin\mathrm{Iso}(H)$ and thus $\mathrm{dom}(v)\in V(H)\backslash I_H$.

If $I_H\backslash\mathrm{Iso}(H)\not\subseteq\bigcup_{w\in I_1\cup I_4} V(P(w))$, then there is some $x\in I_H\backslash\mathrm{Iso}(H)$ such that $N_G(x)\cap D=\{y\}$ for some $y\in D_2$. Now the construction of $f_r(H,I_H)$ implies that $i_2\geqslant 2d_2+1$, which is a contradiction. Hence $I_H\backslash\mathrm{Iso}(H)\subseteq\bigcup_{w\in I_1\cup I_4} V(P(w))$ and $\{\mathrm{dom}(v)\}\cup I_H$ is an independent set of $H$, which is a contradiction to the choice of $I_H$. Hence $d_2=0$ and thus $G'=G$. This completes the proof. □

**Corollary 3.2** (I. Zverovich [15]). *Let $k\geqslant 0$. A graph is $\alpha_1\gamma_1(k)$-perfect if and only if it does not contain a graph in $\mathscr{F}_1(k)$ as an induced subgraph.*

## 4. Claw-free $\alpha_2\gamma_2(k)$-perfect graphs

For $l\geqslant 2$ let $G(l)$ consist of a clique of cardinality $l$, an independent set of cardinality $l$ and a perfect matching between these two sets. For $l_1,l_2\geqslant 2$ let $G(l_1,l_2)$ be the graph with vertex set $\{v\}\cup V(G(l_1))\cup V(G(l_2))$ that arises by joining the vertex $v$ to the non-endvertices in $V(G(l_1))\cup V(G(l_2))$. See Fig. 3 for an example.

For $k\geqslant 0$ a graph $G$ belongs to the class $\mathscr{G}(k)$ if and only if $G$ is the disjoint union of graphs $G(l_1),G(l_2),\ldots,G(l_i),G(l_{i+1},l_{i+2}),G(l_{i+3},l_{i+4}),\ldots,G(l_{i+(2j-1)},l_{i+2j})$

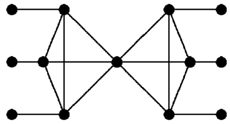

Fig. 3. $G(3,3)$.

such that

$$k+1=\sum_{v=1}^{i}(l_v-1)+\sum_{v=1}^{j}(l_{i+(2v-1)}+l_{i+2v}-1).$$

**Theorem 4.1.** *A claw-free graph G is* $\alpha_2\gamma_2(k)$*-perfect for* $k\geqslant 0$ *if and only if it contains no graph in* $\mathscr{G}(k)$ *as an induced subgraph.*

**Proof.** It is easy to check that $\alpha_2(G(l))-\gamma_2(G(l))=l-1$ and $\alpha_2(G(l_1,l_2))-\gamma_2(G(l_1,l_2))=l_1+l_2-1$. This implies that no $\alpha_2\gamma_2(k)$-perfect graph contains a graph in $\mathscr{G}(k)$ as an induced subgraph.

For the converse let $G$ be a minimal $\alpha_2\gamma_2(k)$-imperfect graph. Let $I$, $D$, $\mathrm{dom}(u)$ and $P(u)$ for $u\in I$ be as in Definition 2.2. The graph $G$ satisfies the properties given in Lemmas 2.1 and 2.3.

Since $I$ is an $\alpha_2$-set of $G$, at most one neighbour of a vertex in $D$ belongs to $I$. Furthermore, $V(P(u))\cap V(P(v))\neq\emptyset$ for $u,v\in I$ implies that $\mathrm{dom}(u)=\mathrm{dom}(v)$, $V(P(u))\cap V(P(v))=\{\mathrm{dom}(u)\}$ and $\max\{\mathrm{dist}_G(u,\mathrm{dom}(u)),\mathrm{dist}_G(v,\mathrm{dom}(u))\}=2$. For $v\in D$ let $I(v)=\{u\in I|\mathrm{dom}(u)=v\}$ and

$$S(v)=(N_G(v)\cap I(v))\cup\left[\bigcup_{u\in I(v)}V(P(u))\backslash(D\cup I)\right]\subseteq N_G(v).$$

Note that $|S(v)|\geqslant 2$.

If $S(v)$ is a clique for some $v\in D$, then let $H=G-v$. Since $I\subseteq V(H)$, $\alpha_2(H)\geqslant\alpha_2(G)$. If $w\in S(v)$, then $(D\backslash\{v\})\cup\{w\}$ is a 2-dominating set of $G$ and thus $\gamma_2(H)\leqslant\gamma_2(G)$. We obtain $\alpha_2(H)-\gamma_2(H)\geqslant k+1$, which is a contradiction.

If $N_G(v)\cap I(v)\neq\emptyset$, then $|N_G(v)\cap I(v)|=1$ and the unique vertex in $N_G(v)\cap I(v)$ has no neighbour in $S(v)$. Since $G$ is claw-free, this implies that $S(v)$ is the union of two cliques one of which consists of the unique vertex in $N_G(v)\cap I(v)$.

If $N_G(v)\cap I(v)=\emptyset$ and there are vertices $x,y,z\in S(v)$ such that $xy,xz\in E(G)$, $yz\notin E(G)$ and $u$ is the unique neighbour of $x$ in $I$, then $G[\{x,u,y,z\}]$ is a claw, which is a contradiction. Hence, also in this case, $S(v)$ is the union of two cliques.

For $i=1,2$ let $v_i\in D$ be such that $S(v_i)=C_i\cup C_i'$ where $C_i$ and $C_i'$ are cliques. If $C_2\subseteq N_G(v_1)$ and $w\in C_2'$, then let $H=G-v_2$. Since $I\subseteq V(H)$, $\alpha_2(H)\geqslant\alpha_2(G)$. Since $(D\backslash\{v_2\})\cup\{w\}$ is a 2-dominating set of $H$, $\gamma_2(H)\leqslant\gamma_2(G)$. We obtain $\alpha_2(H)-\gamma_2(H)\geqslant k+1$, which is a contradiction. Hence, by symmetry, $C_2\nsubseteq N_G(v_1)$, $C_2'\nsubseteq N_G(v_1)$, $C_1\nsubseteq N_G(v_2)$ and $C_1'\nsubseteq N_G(v_2)$.

If $w_2 \in C_2$ is adjacent to $v_1$, then $|C_2| \geqslant 2$. Hence $w_2 \notin I(v_2)$ and there is a vertex $u_2 \in I(v_2)$ such that $N_G(u_2) = \{w_2\}$. Let $w_3 \in C_2 \setminus \{w_2\}$. If $w_3 \notin N_G(v_1)$, then $G[\{v_1, w_2, w_3, u_2\}]$ is a claw. Hence $C_2 \subseteq N_G(v_1)$, which is a contradiction. By symmetry, $v_1$ has no neighbour in $C_2 \cup C_2'$ and $v_2$ has no neighbour in $C_1 \cup C_1'$.

If there are vertices $w_i \in C_i$ for $i = 1, 2$ such that $w_1$ and $w_2$ are adjacent, then $w_1, w_2 \notin I$ and for $i = 1, 2$ there are vertices $u_i \in I(v_i)$ such that $N_G(u_i) = \{w_i\}$. Since $w_2 \notin N_G(v_1)$, $G[\{v_1, w_1, w_2, u_1\}]$ is a claw, which is a contradiction.

If $v_1$ and $v_2$ are adjacent, then for $x_1 \in C_1$ and $x_1' \in C_1'$ the graph $G[\{v_1, v_2, x_1, x_1'\}]$ is a claw, which is a contradiction.

This implies that no edge joins a vertex in $\{v_1\} \cup S(v_1) \cup I(v_1)$ to a vertex in $\{v_2\} \cup S(v_2) \cup I(v_2)$. Hence for each $v \in D$ the set $\{v\} \cup S(v) \cup I(v)$ is the vertex set of a connected component of $G$. If $S(v) = C \cup C'$ for some $v \in D$, $C = \{w\}$ and $w \notin I$, then there is some $u \in I(v)$ such that $N_G(u) = \{w\}$. The graph $H = G - u$ satisfies $\alpha_2(H) - \gamma_2(H) \geqslant k + 1$, which is a contradiction. This finally implies that every component of $G$ is isomorphic either to a graph $G(l)$ for $l \geqslant 2$ or a graph $G(l_1, l_2)$ for $l_1, l_2 \geqslant 1$ which implies that $G \in \mathcal{G}(k)$ and the proof is complete. □

It is easy to see that a graph is $\alpha_r\gamma_r(0)$-perfect if and only if it does not contain $P_{r+2}$ as an induced subgraph. For general $k$ though, we believe that there is no concise description of the minimal $\alpha_r\gamma_r(k)$-imperfect graphs.

## References


[1] B. Bollobás, E.J. Cockayne, Graph-theoretic parameters concerning domination, independence, and irredundance, J. Graph Theory 3 (1979) 241–249.

[2] E.J. Cockayne, O. Favaron, C. Payan, A.G. Thomason, Contributions to the theory of domination, independence and irredundance in graphs, Discrete Math. 33 (1981) 249–258.

[3] L. Dohmen, D. Rautenbach, L. Volkmann, A characterization of $\Gamma\alpha(k)$-perfect graphs, Discrete Math. 224 (2000) 265–271.

[4] L. Dohmen, D. Rautenbach, L. Volkmann, $i\gamma(1)$-perfect graphs, Discrete Math. 234 (2001) 133–138.

[5] O. Favaron, Stability, domination and irredundance in a graph, J. Graph Theory 10 (1986) 429–438.

[6] G. Gutin, V.E. Zverovich, Upper domination and upper irredundance perfect graphs, Discrete Math. 190 (1998) 95–105.

[7] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, NY, 1998.

[8] M.A. Henning, Irredundance perfect graphs, Discrete Math. 142 (1995) 107–120.

[9] M.S. Jacobson, K. Peters, Chordal graphs and upper irredundance, upper domination and independence, Discrete Math. 86 (1990) 59–69.

[10] M.S. Jacobson, K. Peters, A note on graphs which have upper irredundance equal to independence, Discrete Appl. Math. 44 (1993) 91–97.

[11] J. Puech, Irredundance perfection and $P_6$-free graphs, J. Graph Theory 29 (1998) 239–255.

[12] D. Rautenbach, V.E. Zverovich, Perfect graphs of strong and independent strong domination, Discrete Math. 226 (2001) 297–311.

[13] J. Topp, Domination, independence and irredundance in graphs, Dissertationes Math., Vol. 342, 1995, 99pp.

[14] L. Volkmann, V.E. Zverovich, A proof of Favaron's conjecture and a disproof of Henning's conjecture on irredundance perfect graphs, The 5th Twente Workshop on Graphs and Combinatorial Optimization, Enschede, May 1997, pp. 215–217.

[15] I.E. Zverovich, $k$-Bounded classes of dominant-independent perfect graphs, J. Graph Theory 32 (1999) 303–310.
[16] I.E. Zverovich, V.E. Zverovich, An induced subgraph characterization of domination perfect graphs, J. Graph Theory 20 (1995) 375–395.
[17] I.E. Zverovich, V.E. Zverovich, An semi-induced subgraph characterization of upper domination perfect graphs, J. Graph Theory 31 (1999) 139–149.