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Journal of Algebra 298 (2006) 248–272
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Generic fiber rings of mixed power series/polynomial rings

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Received 15 February 2005

Available online 23 September 2005

Communicated by Luchezar L. Avramov

Abstract

Let K be a field, m and n positive integers, and $X = \{x_1, \ldots, x_n\}$, and $Y = \{y_1, \ldots, y_m\}$ sets of independent variables over K. Let A be the localized polynomial ring $K[X]_{(X)}$. We prove that every prime ideal P in $\widehat{A} = K[\![X]\!]$ that is maximal with respect to $P \cap A = (0)$ has height $n-1$. We consider the mixed power series/polynomial rings $B := K[\![X]\!][Y]_{(X,Y)}$ and $C := K[Y]_{(Y)}[\![X]\!]$. For each prime ideal P of $\widehat{B} = \widehat{C}$ that is maximal with respect to either $P \cap B = (0)$ or $P \cap C = (0)$, we prove that P has height $n+m-2$. We also prove each prime ideal P of $K[\![X, Y]\!]$ that is maximal with respect to $P \cap K[\![X]\!] = (0)$ is of height either m or $n+m-2$.

1. Introduction and background

Let $(R, \mathbf{m})$ be a Noetherian local integral domain and let $\widehat{R}$ denote the $\mathbf{m}$-adic completion of R. The *generic formal fiber ring* of R is the localization $(R \setminus (0))^{-1}\widehat{R}$ of $\widehat{R}$. The *formal fibers* of R are the fibers of the morphism $\operatorname{Spec} \widehat{R} \to \operatorname{Spec} R$; for a prime ideal P of R, the formal fiber over P is $\operatorname{Spec}((R_P/PR_P) \otimes_R \widehat{R})$. The formal fibers encode im-

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.07.015

portant information about the structure of R. For example, the local ring R is excellent provided it is universally catenary and has geometrically regular formal fibers [2, (7.8.3), p. 214].

Let $R \hookrightarrow S$ be an injective homomorphism of commutative rings. If R is an integral domain, the *generic fiber ring* of the map $R \hookrightarrow S$ is the localization $(R \setminus (0))^{-1}S$ of S. In this article we study generic fiber rings for "mixed" polynomial and power series rings over a field. More precisely, for K a field, m and n positive integers, and $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ sets of variables over K, we consider the local rings $A := K[X]_{(X)}$, $B := K[\![X]\!][Y]_{(X,Y)}$ and $C := K[Y]_{(Y)}[\![X]\!]$, as well as their completions $\widehat{A} = K[\![X]\!]$ and $\widehat{B} = \widehat{C} = K[\![X, Y]\!]$. Notice that there is a canonical inclusion map $B \hookrightarrow C$.

We have the following local embeddings.

$$A := K[X]_{(X)} \hookrightarrow \widehat{A} := K[\![X]\!], \quad \widehat{A} \hookrightarrow \widehat{B} = \widehat{C} = K[\![X, Y]\!] \quad \text{and}$$
$$B := K[\![X]\!][Y]_{(X,Y)} \hookrightarrow C := K[Y]_{(Y)}[\![X]\!] \hookrightarrow \widehat{B} = \widehat{C} = K[\![X]\!][\![Y]\!].$$

Matsumura proves in [7] that the generic formal fiber ring of A has dimension $n - 1 = \dim A - 1$, and the generic formal fiber rings of B and C have dimension $n + m - 2 = \dim B - 2 = \dim C - 2$. However he does not address the question of whether all maximal ideals of the generic formal fiber rings for A, B and C have the same height. If the field K is countable, it follows from [3, Proposition 4.10, p. 36] that all maximal ideals of the generic formal fiber ring of A have the same height.

In answer to a question raised by Matsumura in [7], Rotthaus in [10] establishes the following result. Let n be a positive integer. Then there exist excellent regular local rings R such that $\dim R = n$ and such that the generic formal fiber ring of R has dimension t, where the value of t may be taken to be any integer between 0 and $\dim R - 1$. It is also shown in [10, Corollary 3.2] that there exists an excellent regular local domain having the property that its generic formal fiber ring contains maximal ideals of different heights.

Let $\widehat{T}$ be a complete Noetherian local ring and let $\mathcal{C}$ be a finite set of incomparable prime ideals of $\widehat{T}$. Charters and Loepp in [1] (see also [6, Theorem 17]) determine necessary and sufficient conditions for $\widehat{T}$ to be the completion of a Noetherian local domain T such that the generic formal fiber of T has as maximal elements precisely the prime ideals in $\mathcal{C}$. If $\widehat{T}$ is of characteristic zero, Charters and Loepp give necessary and sufficient conditions to obtain such a domain T that is excellent. The finite set $\mathcal{C}$ may be chosen to contain prime ideals of different heights. This provides many examples where the generic formal fiber ring contains maximal ideals of different heights.

Our main results may be summarized as follows.

1.1. Theorem. *With the above notation, we prove that all maximal ideals of the generic formal fiber rings of A, B and C have the same height. In particular, we prove*:

(1) *If P is a prime ideal of $\widehat{A}$ maximal with respect to $P \cap A = (0)$, then* $\operatorname{ht} P = n - 1$.
(2) *If P is a prime ideal of $\widehat{B}$ maximal with respect to $P \cap B = (0)$, then* $\operatorname{ht} P = n + m - 2$.
(3) *If P is a prime ideal of $\widehat{C}$ maximal with respect to $P \cap C = (0)$, then* $\operatorname{ht} P = n + m - 2$.
(4) *In addition, there are at most two possible values for the height of a maximal ideal of the generic fiber ring $(\widehat{A} \setminus (0))^{-1}\widehat{C}$ of the inclusion map $\widehat{A} \hookrightarrow \widehat{C}$.*

(a) *If $n \geqslant 2$ and P is a prime ideal of $\widehat{C}$ maximal with respect to $P \cap \widehat{A} = (0)$, then either* $\operatorname{ht} P = n + m - 2$ *or* $\operatorname{ht} P = m$.
(b) *If $n = 1$, then all maximal ideals of the generic fiber ring $(\widehat{A} \setminus (0))^{-1}\widehat{C}$ have height m.*

We were motivated to consider generic fiber rings for the embeddings displayed above because of questions related to [4,5] and ultimately because of the following question posed by Melvin Hochster.

1.2. Question. Let R be a complete local domain. Can one describe or somehow classify the local maps of R to a complete local domain S such that $U^{-1}S$ is a field, where $U = R \setminus (0)$, i.e., such that the generic fiber of $R \hookrightarrow S$ is trivial?

Hochster remarks that if, for example, R is equal characteristic zero, one obtains such extensions by starting with

$$R = K[\![x_1, \dots, x_n]\!] \hookrightarrow T = L[\![x_1, \dots, x_n, y_1, \dots, y_m]\!] \to T/P = S, \qquad (1.2.1)$$

where K is a subfield of L, the x_i, y_j are formal indeterminates, and P is a prime ideal of T maximal with respect to being disjoint from the image of $R \setminus \{0\}$. Of course, such prime ideals P correspond to the maximal ideals of the generic fiber $(R \setminus (0))^{-1}T$.

In Theorem 7.2, we answer Question 1.2 in the special case where the extension arises from the embedding in (1.2.1) with the field $L = K$. We prove in this case that the dimension of the extension ring S must be either 2 or n.

In [5] we study extensions of integral domains $R \overset{\varphi}{\hookrightarrow} S$ such that, for every nonzero $Q \in \operatorname{Spec} S$, we have $Q \cap R \neq (0)$. Such extensions are called *trivial generic fiber extensions* or *TGF extensions* in [5]. One obtains such an extension by considering a composition $R \hookrightarrow T \to T/P = S$, where T is an extension ring of R and $P \in \operatorname{Spec} T$ is maximal with respect to $P \cap R = (0)$. Thus the generic fiber ring and so also Theorem 1.1 give information regarding TGF extensions in the case where the smaller ring is a mixed polynomial/power series ring.

In addition, Theorem 1.1 is useful in the study of (1.2.1), because the map in (1.2.1) factors through:

$$R = K[\![x_1, \dots, x_n]\!] \hookrightarrow K[\![x_1, \dots, x_n]\!][y_1, \dots, y_m] \hookrightarrow T = L[\![x_1, \dots, x_n, y_1, \dots, y_n]\!].$$

Section 2 contains implications of Weierstrass' Preparation Theorem to the prime ideals of power series rings. We first prove a technical proposition regarding a change of variables that provides a "nice" generating set for a given prime ideal P of a power series ring; then in Theorem 2.3 we prove that, in certain circumstances, a larger prime ideal can be found with the same contraction as P to a certain subring. In Sections 3 and 4, we prove parts (2) and (3) of Theorem 1.1 stated above. In Section 5 we use a result of Valabrega for the two-dimensional case. We then apply this result in Section 6 to prove part (1) of Theorem 1.1, and in Section 7 we prove part (4).

2. Variations on a theme of Weierstrass

In this section, we apply the Weierstrass Preparation Theorem [12, Theorem 5, p. 139, and Corollary 1, p. 145] to examine the structure of a given prime ideal P in the power series ring $\widehat{A} = K[\![X]\!]$, where $X = \{x_1, \dots, x_n\}$ is a set of n variables over the field K. Here $A = K[X]_{(X)}$ is the localized polynomial ring in these variables. Our procedure is to make a change of variables that yields a regular sequence in P of a nice form.

2.1. Notation. By a *change of variables*, we mean a finite sequence of 'polynomial' change of variables of the type described below, where $X = \{x_1, \dots, x_n\}$ is a set of n variables over the field K. For example, with $e_i, f_i \in \mathbb{N}$, consider

$$x_1 \mapsto x_1 + x_n^{e_1} = z_1, \qquad x_2 \mapsto x_2 + x_n^{e_2} = z_2, \qquad \dots,$$
$$x_{n-1} \mapsto x_{n-1} + x_n^{e_{n-1}} = z_{n-1}, \qquad x_n \mapsto x_n = z_n,$$

followed by

$$z_1 \mapsto z_1 = t_1, \qquad z_2 \mapsto z_2 + z_1^{f_2} = t_2, \qquad \dots,$$
$$z_{n-1} \mapsto z_{n-1} + z_1^{f_{n-1}} = t_{n-1}, \qquad x_n \mapsto z_n + z_1^{f_n} = t_n.$$

Thus a change of variables defines an automorphism of $\widehat{A}$ that restricts to an automorphism of A.

We also consider a change of variables for subrings of A and $\widehat{A}$. For example, if $A_1 = K[x_2, \dots, x_n] \subseteq A$ and $S = K[\![x_2, \dots, x_n]\!] \subseteq \widehat{A}$, then by a *change of variables inside* A_1 and S, we mean a finite sequence of automorphisms of A and $\widehat{A}$ of the type described above on $x_2, \dots, x_n$ that leave the variable x_1 fixed. In this case we obtain an automorphism of $\widehat{A}$ that restricts to an automorphism on each of S, A and A_1.

2.2. Proposition. *Let $\widehat{A} := K[\![X]\!] = K[\![x_1, \dots, x_n]\!]$ and let $P \in \operatorname{Spec} \widehat{A}$ with $x_1 \notin P$ and $\operatorname{ht} P = r$, where $1 \leqslant r \leqslant n-1$. There exists a change of variables $x_1 \mapsto z_1 := x_1$ (x_1 is fixed), $x_2 \mapsto z_2, \dots, x_n \mapsto z_n$ and a regular sequence $f_1, \dots, f_r \in P$ so that, upon setting $Z_1 = \{z_1, \dots, z_{n-r}\}$, $Z_2 = \{z_{n-r+1}, \dots, z_n\}$ and $Z = Z_1 \cup Z_2$, we have*

$$\begin{array}{ll} f_1 \in K[\![Z_1]\!][z_{n-r+1}, \dots, z_{n-1}][z_n] & \textit{is monic as a polynomial in } z_n, \\ f_2 \in K[\![Z_1]\!][z_{n-r+1}, \dots, z_{n-2}][z_{n-1}] & \textit{is monic as a polynomial in } z_{n-1}, \textit{ etc.}, \\ \vdots & \\ f_r \in K[\![Z_1]\!][z_{n-r+1}] & \textit{is monic as a polynomial in } z_{n-r+1}. \end{array}$$

In addition:

(1) *P is a minimal prime of the ideal $(f_1, \dots, f_r)\widehat{A}$.*

(2) *The* (Z_2)*-adic completion of* $K[\![Z_1]\!][Z_2]_{(Z)}$ *is identical to the* $(f_1, \ldots, f_r)$*-adic completion and both equal* $\widehat{A} = K[\![X]\!] = K[\![Z]\!]$.
(3) *If* $P_1 := P \cap K[\![Z_1]\!][Z_2]_{(Z)}$*, then* $P_1\widehat{A} = P$*, that is,* P *is extended from* $K[\![Z_1]\!][Z_2]_{(Z)}$.
(4) *The ring extension*:

$$K[\![Z_1]\!] \hookrightarrow K[\![Z_1]\!][Z_2]_{(Z)}/P_1 \cong K[\![Z]\!]/P$$

is finite (*and integral*).

Proof. Since $\widehat{A}$ is a unique factorization domain, there exists a nonzero prime element f in P. The power series f is therefore not a multiple of x_1, and so f must contain a monomial term $x_2^{i_2} \ldots x_n^{i_n}$ with a nonzero coefficient in K. This nonzero coefficient in K may be assumed to be 1. There exists an automorphism $\sigma : \widehat{A} \to \widehat{A}$ defined by the change of variables:

$$x_1 \mapsto x_1, \quad x_2 \mapsto t_2 := x_2 + x_n^{e_2}, \quad \ldots, \quad x_{n-1} \mapsto t_{n-1} := x_{n-1} + x_n^{e_{n-1}}, \quad x_n \mapsto x_n,$$

with $e_2, \ldots, e_{n-1} \in \mathbb{N}$ chosen suitably so that f written as a power series in the variables $x_1, t_2, \ldots, t_{n-1}, x_n$ contains a term $a_n x_n^{s_n}$, where s_n is a positive integer, and $a_n \in K$ is nonzero. We assume that the integer s_n is minimal among all integers i such that a term $a x_n^i$ occurs in f with a nonzero coefficient $a \in K$; we further assume that the coefficient $a_n = 1$. By Weierstrass we have that:

$$f = m\epsilon,$$

where $m \in K[\![x_1, t_2, \ldots, t_{n-1}]\!][x_n]$ is a monic polynomial in x_n of degree s_n and ϵ is a unit in $\widehat{A}$. Since $f \in P$ is a prime element, $m \in P$ is also a prime element. Using Weierstrass again, every element $g \in P$ can be written as:

$$g = mh + q,$$

where $h \in K[\![x_1, t_2, \ldots, t_{n-1}, x_n]\!] = \widehat{A}$ and $q \in K[\![x_1, t_2, \ldots, t_{n-1}]\!][x_n]$ is a polynomial in x_n of degree less than s_n. Note that

$$K[\![x_1, t_2, \ldots, t_{n-1}]\!] \hookrightarrow K[\![x_1, t_2, \ldots, t_{n-1}]\!][x_n]/(m)$$

is an integral (finite) extension. Thus the ring $K[\![x_1, t_2, \ldots, t_{n-1}]\!][x_n]/(m)$ is complete. Moreover, the two ideals $(x_1, t_2, \ldots, t_{n-1}, m) = (x_1, t_2, \ldots, t_{n-1}, x_n^{s_n})$ and $(x_1, t_2, \ldots, t_{n-1}, x_n)$ of $B_0 := K[\![x_1, t_2, \ldots, t_{n-1}]\!][x_n]$ have the same radical. Therefore $\widehat{A}$ is the (m)-adic and the (x_n)-adic completion of B_0 and P is extended from B_0.

This implies the statement for $r = 1$, with $f_1 = m$, $z_n = x_n$, $z_1 = x_1$, $z_2 = t_2, \ldots, z_{n-1} = t_{n-1}$, $Z_1 = \{x_1, t_2, \ldots, t_{n-1}\}$ and $Z_2 = \{z_n\} = \{x_n\}$. In particular, when $r = 1$, P is minimal over $m\widehat{A}$, so $P = m\widehat{A}$.

For $r > 1$ we continue by induction on r. Let $P_0 := P \cap K[\![x_1, t_2, \ldots, t_{n-1}]\!]$. Since $m \notin K[\![x_1, t_2, \ldots, t_{n-1}]\!]$ and P is extended from $B_0 := K[\![x_1, t_2, \ldots, t_{n-1}]\!][x_n]$, then $P \cap B_0$ has height r and $\operatorname{ht} P_0 = r - 1$. Since $x_1 \notin P$, we have $x_1 \notin P_0$, and by the induction hypothesis there is a change of variables $t_2 \mapsto z_2, \dots, t_{n-1} \mapsto z_{n-1}$ of $K[\![x_1, t_2, \dots, t_{n-1}]\!]$ and elements $f_2, \dots, f_r \in P_0$ so that:

$$\begin{aligned} &f_2 \in K[\![x_1, z_2, \dots, z_{n-r}]\!][z_{n-r+1}, \dots, z_{n-2}][z_{n-1}] &&\text{is monic in } z_{n-1},\\ &f_3 \in K[\![x_1, z_2, \dots, z_{n-r}]\!][z_{n-r+1}, \dots, z_{n-3}][z_{n-2}] &&\text{is monic in } z_{n-2}, \text{ etc.},\\ &\quad\vdots\\ &f_r \in K[\![x_1, z_2, \dots, z_{n-r}]\!][z_{n-r+1}] &&\text{is monic in } z_{n-r+1}, \end{aligned}$$

and $f_2, \dots, f_r$ satisfy the assertions of Proposition 2.2 for P_0.

It follows that $m, f_2, \dots, f_r$ is a regular sequence of length r and that P is a minimal prime of the ideal $(m, f_2, \dots, f_r)\widehat{A}$. Set $z_n = x_n$. We now prove that m may be replaced by a polynomial $f_1 \in K[\![x_1, z_2, \dots, z_{n-r}]\!][z_{n-r+1}, \dots, z_n]$. Write

$$m = \sum_{i=0}^{s_n} a_i z_n,$$

where the $a_i \in K[\![x_1, z_2, \dots, z_{n-1}]\!]$. For each $i < s_n$, apply Weierstrass to a_i and f_2 in order to obtain:

$$a_i = f_2 h_i + q_i,$$

where h_i is a power series in $K[\![x_1, z_2, \dots, z_{n-1}]\!]$ and $q_i \in K[\![x_1, z_2, \dots, z_{n-2}]\!][z_{n-1}]$ is a polynomial in z_{n-1}. With $q_{s_n} = 1 = a_{s_n}$, we define

$$m_1 = \sum_{i=0}^{s_n} q_i z_n^i.$$

Now $(m_1, f_2, \dots, f_r)\widehat{A} = (m, f_2, \dots, f_r)\widehat{A}$ and we may replace m by m_1 which is a polynomial in z_{n-1} and z_n. To continue, for each $i < s_n$, write:

$$q_i = \sum_{j,k} b_{ij} z_{n-1}^j \quad \text{with } b_{ij} \in K[\![x_1, z_2, \dots, z_{n-2}]\!].$$

For each b_{ij}, we apply Weierstrass to b_{ij} and f_3 to obtain:

$$b_{ij} = f_3 h_{ij} + q_{ij},$$

where $q_{ij} \in K[\![x_1, z_2, \dots, z_{n-3}]\!][z_{n-2}]$. Set

$$m_2 = \sum_{i,j} q_{ij} z_{n-1}^j z_n^i \in K[\![x_1, z_2, \dots, z_{n-3}]\!][z_{n-2}, z_{n-1}, z_n]$$

with $q_{s_n 0} = 1$. It follows that

$$(m_2, f_2, \dots, f_r)\widehat{A} = (m, f_2, \dots, f_r)\widehat{A}.$$

Continuing this process by applying Weierstrass to the coefficients of $z_{n-2}^k z_{n-1}^j z_n^i$ and f_4, we establish the existence of a polynomial $f_1 \in K[\![Z_1]\!][z_{n-r+1}, \dots, z_n]$ that is monic in z_n so that $(f_1, f_2, \dots, f_r)\widehat{A} = (m, f_2, \dots, f_r)\widehat{A}$. Therefore P is a minimal prime of $(f_1, \dots, f_r)\widehat{A}$.

The extension

$$K[\![Z_1]\!] \to K[\![Z_1]\!][Z_2]/(f_1, \dots, f_r)$$

is integral and finite. Thus the ring $K[\![Z_1]\!][Z_2]/(f_1, \dots, f_r)$ is complete. This implies $\widehat{A} = K[\![x_1, z_2, \dots, z_n]\!]$ is the $(f_1, \dots, f_r)$-adic (and the (Z_2)-adic) completion of $K[\![Z_1]\!][Z_2]_{(Z)}$ and that P is extended from $K[\![Z_1]\!][Z_2]_{(Z)}$. This completes the proof of Proposition 2.2. □

The following theorem is the technical heart of the paper.

2.3. Theorem. *Let K be a field and let y and $X = \{x_1, \dots, x_n\}$ be variables over K. Assume that V is a discrete valuation domain with completion $\widehat{V} = K[\![y]\!]$ and that $K[y] \subseteq V \subseteq K[\![y]\!]$. Also assume that the field $K((y)) = K[\![y]\!][1/y]$ has uncountable transcendence degree over the quotient field $\mathcal{Q}(V)$ of V. Set $R_0 := V[\![X]\!]$ and $R = \widehat{R_0} = K[\![y, X]\!]$. Let $P \in \operatorname{Spec} R$ be such that*

(i) $P \subseteq (X)R$ *(so $y \notin P$), and*
(ii) $\dim(R/P) > 2$.

Then there is a prime ideal $Q \in \operatorname{Spec} R$ such that

(1) $P \subset Q \subset XR$,
(2) $\dim(R/Q) = 2$, *and*
(3) $P \cap R_0 = Q \cap R_0$.

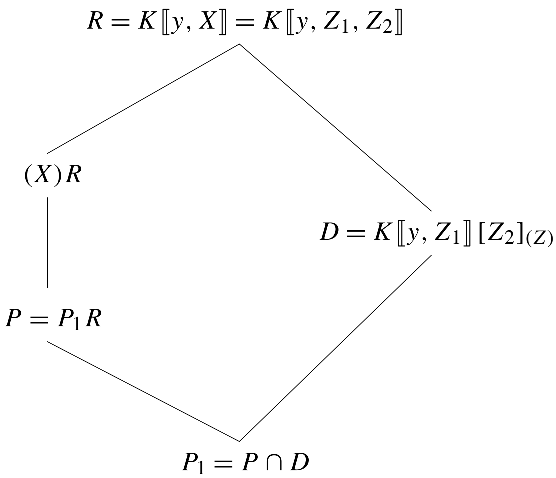
In particular, $P \cap K[\![X]\!] = Q \cap K[\![X]\!]$.

Proof. Assume that P has height r. Since $\dim(R/P) > 2$, we have $0 \leqslant r < n-1$. If $r > 0$, then there exist a transformation $x_1 \mapsto z_1, \dots, x_n \mapsto z_n$ and elements $f_1, \dots, f_r \in P$, by Proposition 2.2, so that the variable y is fixed, and

$$\begin{array}{ll} f_1 \in K[\![y, z_1, \dots, z_{n-r}]\!][z_{n-r+1}, \dots, z_n] & \text{is monic in } z_n, \\ f_2 \in K[\![y, z_1, \dots, z_{n-r}]\!][z_{n-r+1}, \dots, z_{n-1}] & \text{is monic in } z_{n-1}, \text{ etc.,} \\ \vdots & \\ f_r \in K[\![y, z_1, \dots, z_{n-r}]\!][z_{n-r+1}] & \text{is monic in } z_{n-r+1}, \end{array}$$

and the assertions of Proposition 2.2 are satisfied. In particular, P is a minimal prime of $(f_1, \dots, f_r)R$. Let $Z_1 = \{z_1, \dots, z_{n-r}\}$ and $Z_2 = \{z_{n-r+1}, \dots, z_{n-1}, z_n\}$. By Proposition 2.2, if $D := K[\![y, Z_1]\!][Z_2]_{(Z)}$ and $P_1 := P \cap D$, then $P_1 R = P$.

The following diagram shows these rings and ideals.



Note that $f_1, \dots, f_r \in P_1$. Let $g_1, \dots, g_s \in P_1$ be other generators such that $P_1 = (f_1, \dots, f_r, g_1, \dots, g_s)D$. Then $P = P_1 R = (f_1, \dots, f_r, g_1, \dots, g_s)R$. For each $(i) := (i_1, \dots, i_n) \in \mathbb{N}^n$ and j, k with $1 \leqslant j \leqslant r$, $1 \leqslant k \leqslant s$, let $a_{j,(i)}, b_{k,(i)}$ denote the coefficients in $K[\![y]\!]$ of the f_j, g_k, so that

$$f_j = \sum_{(i) \in \mathbb{N}^n} a_{j,(i)} z_1^{i_1} \dots z_n^{i_n}, \qquad g_k = \sum_{(i) \in \mathbb{N}^n} b_{k,(i)} z_1^{i_1} \dots z_n^{i_n} \in K[\![y]\!][\![Z]\!].$$

Define

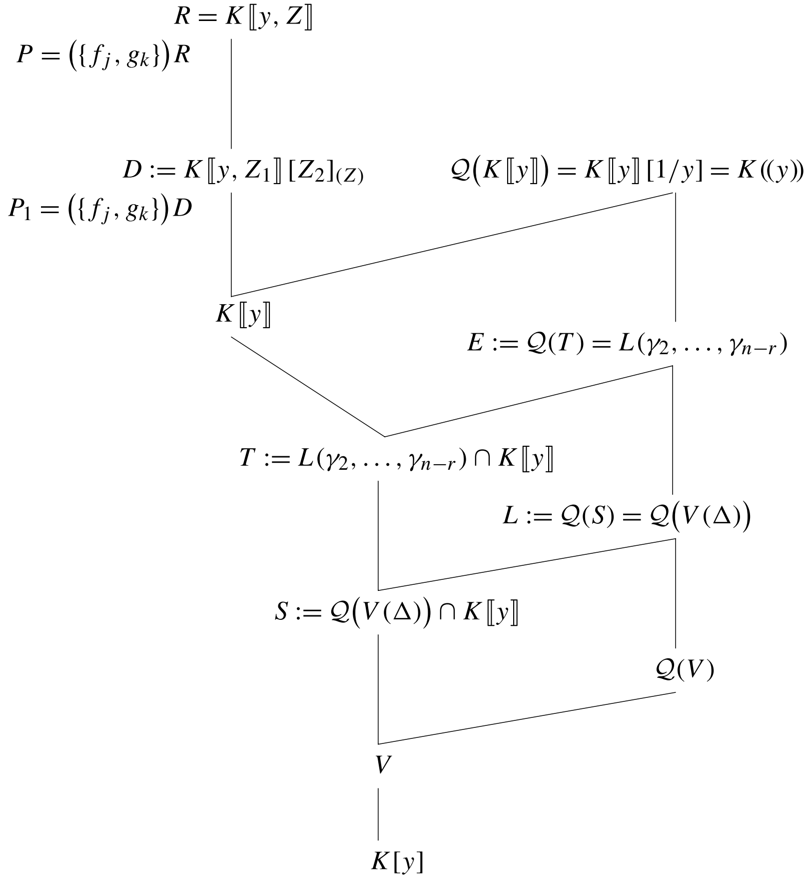
$$\Delta := \begin{cases} \{a_{j,(i)}, b_{k,(i)}\} \subseteq K[\![y]\!], & \text{for } r > 0, \\ \emptyset, & \text{for } r = 0. \end{cases}$$

A key observation here is that in either case the set Δ is countable.

To continue the proof, we consider $S := \mathcal{Q}(V(\Delta)) \cap K[\![y]\!]$, a discrete valuation domain, and its field of quotients $L := \mathcal{Q}(V(\Delta))$. Since Δ is a countable set, the field $K(\!(y)\!)$ is (still) of uncountable transcendence degree over L. Let $\gamma_2, \dots, \gamma_{n-r}$ be elements of $K[\![y]\!]$ that are algebraically independent over L. We define

$$T := L(\gamma_2, \dots, \gamma_{n-r}) \cap K[\![y]\!] \quad \text{and} \quad E := \mathcal{Q}(T) = L(\gamma_2, \dots, \gamma_{n-r}).$$

The diagram below shows the prime ideals P and P_1 and the containments between the relevant rings.



Let $P_2 := P \cap S[\![Z_1]\!][Z_2]_{(Z)}$. Since $f_1, \ldots, f_r, g_1, \ldots, g_s \in S[\![Z_1]\!][Z_2]_{(Z)}$, we have $P_2 R = P$. Since $P \subseteq (x_1, \ldots, x_n)R = (Z)R$, there is a prime ideal $\widetilde{P}$ in $L[\![Z]\!]$ that is minimal over $P_2 L[\![Z]\!]$. Since $L[\![Z]\!]$ is flat over $S[\![Z]\!]$, $\widetilde{P} \cap S[\![Z]\!] = P_2 S[\![Z]\!]$. Note that $L[\![X]\!] = L[\![Z]\!]$ is the $(f_1, \ldots, f_r)$-adic (and the (Z_2)-adic) completion of $L[\![Z_1]\!][Z_2]_{(Z)}$. In particular,

$$L[\![Z_1]\!][Z_2]/(f_1, \ldots, f_r) = L[\![Z_1]\!][\![Z_2]\!]/(f_1, \ldots, f_r)$$

and this also holds with the field L replaced by its extension field E.

Since $L[\![Z]\!]/\widetilde{P}$ is a homomorphic image of $L[\![Z]\!]/(f_1, \ldots, f_r)$, it follows that $L[\![Z]\!]/\widetilde{P}$ is integral (and finite) over $L[\![Z_1]\!]$. This yields the commutative diagram:

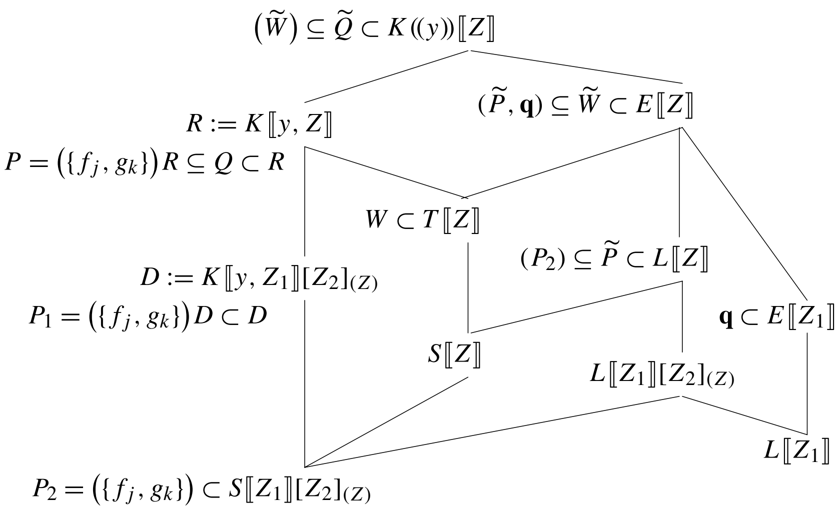
$$\begin{tikzcd}
E[\![Z_1]\!] \arrow[r] & E[\![Z_1]\!][\![Z_2]\!]/\widetilde{P}E[\![Z]\!] \\
L[\![Z_1]\!] \arrow[r] \arrow[u] & L[\![Z_1]\!][\![Z_2]\!]/\widetilde{P} \arrow[u]
\end{tikzcd} \tag{2.3.0}$$

with injective integral (finite) horizontal maps. Recall that E is the subfield of $K((y))$ obtained by adjoining $\gamma_2, \dots, \gamma_{n-r}$ to the field L. Thus the vertical maps of (2.3.0) are faithfully flat.

Let $\mathbf{q} := (z_2 - \gamma_2 z_1, \dots, z_{n-r} - \gamma_{n-r} z_1)E[\![Z_1]\!] \in \operatorname{Spec}(E[\![Z_1]\!])$ and let $\widetilde{W}$ be a minimal prime of the ideal $(\widetilde{P}, \mathbf{q})E[\![Z]\!]$. Since

$$f_1, \dots, f_r, z_2 - \gamma_2 z_1, \dots, z_{n-r} - \gamma_{n-r} z_1$$

is a regular sequence in $T[\![Z]\!]$, the prime ideal $W := \widetilde{W} \cap T[\![Z]\!]$ has height $n-1$. Let $\widetilde{Q}$ be a minimal prime of $\widetilde{W} K((y))[\![Z]\!]$ and let $Q := \widetilde{Q} \cap R$. Then $W = Q \cap T[\![Z]\!]$, $P \subset Q \subset ZR = XR$, and pictorially we have:



Notice that $\mathbf{q}$ is a prime ideal of height $n-r-1$. Also, since $K((y))[\![Z]\!]$ is flat over $K[\![y, Z]\!] = R$, we have $\operatorname{ht} Q = n-1$ and $\dim(R/Q) = 2$. We clearly have $P_2 \subseteq W \cap S[\![Z_1]\!][Z_2]_{(Z)}$.

2.3.1. Claim. $\mathbf{q} \cap L[\![Z_1]\!] = (0)$.

To show this we argue as in [7]. Suppose that

$$h = \sum_{m \in \mathbb{N}} H_m \in \mathbf{q} \cap L[\![z_1, \dots, z_{n-r}]\!],$$

where $H_m \in L[z_1, \dots, z_{n-r}]$ is a homogeneous polynomial of degree m:

$$H_m = \sum_{|(i)|=m} c_{(i)} z_1^{i_1} \dots z_{n-r}^{i_{n-r}},$$

where $(i) := (i_1, \dots, i_{n-r}) \in \mathbb{N}^{n-r}$, $|(i)| := i_1 + \dots + i_{n-r}$ and $c_{(i)} \in L$. Consider the E-algebra homomorphism $\pi : E[\![Z_1]\!] \to E[\![z_1]\!]$ defined by $\pi(z_1) = z_1$ and $\pi(z_i) = \gamma_i z_1$ for $2 \leqslant i \leqslant n-r$. Then $\ker \pi = \mathbf{q}$, and for each $m \in \mathbb{N}$:

$$\pi(H_m) = \pi\left(\sum_{|(i)|=m} c_{(i)} z_1^{i_1} \dots z_{n-r}^{i_{n-r}}\right) = \sum_{|(i)|=m} c_{(i)} \gamma_2^{i_2} \dots \gamma_{n-r}^{i_{n-r}} z_1^m$$

and

$$\pi(h) = \sum_{m \in \mathbb{N}} \pi(H_m) = \sum_{m \in \mathbb{N}} \sum_{|(i)|=m} c_{(i)} \gamma_2^{i_2} \dots \gamma_{n-r}^{i_{n-r}} z_1^m.$$

Since $h \in \mathbf{q}$, $\pi(h) = 0$. Since $\pi(h)$ is a power series in $E[\![z_1]\!]$, each of its coefficients is zero, that is, for each $m \in \mathbb{N}$,

$$\sum_{|(i)|=m} c_{(i)} \gamma_2^{i_2} \dots \gamma_{n-r}^{i_{n-r}} = 0.$$

Since the γ_i are algebraically independent over L, each $c_{(i)} = 0$. Therefore $h = 0$, and so $\mathbf{q} \cap L[\![Z_1]\!] = (0)$. This proves Claim 2.3.1.

Using the commutativity of the displayed diagram (2.3.0) and that the horizontal maps of this diagram are integral extensions, we deduce that $\widetilde{W} \cap E[\![Z_1]\!] = \mathbf{q}$, and $\mathbf{q} \cap L[\![Z_1]\!] = (0)$ implies $\widetilde{W} \cap L[\![Z_1]\!] = (0)$. We conclude that $Q \cap S[\![Z]\!] = P \cap S[\![Z]\!]$ and therefore $Q \cap R_0 = P \cap R_0$. □

We record the following corollary.

2.4. Corollary. *Let K be a field and let $R = K[\![y, X]\!]$, where $X = \{x_1, \dots, x_n\}$ and y are independent variables over K. Assume $P \in \operatorname{Spec} R$ is such that*

(i) $P \subseteq (x_1, \dots, x_n)R$ *and*
(ii) $\dim(R/P) > 2$.

Then there is a prime ideal $Q \in \operatorname{Spec} R$ so that

(1) $P \subset Q \subset (x_1, \dots, x_n)R$,
(2) $\dim(R/Q) = 2$, *and*
(3) $P \cap K[y]_{(y)}[\![X]\!] = Q \cap K[y]_{(y)}[\![X]\!]$.

In particular, $P \cap K[\![x_1, \dots, x_n]\!] = Q \cap K[\![x_1, \dots, x_n]\!]$.

Proof. With notation as in Theorem 2.3, let $V = K[y]_{(y)}$. □

3. Weierstrass implications for the ring $B = K[\![X]\!][Y]_{(X,Y)}$

As before K denotes a field, n and m are positive integers, and $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ denote sets of variables over K. Let $B := K[\![X]\!][Y]_{(X,Y)} = K[\![x_1, \dots, x_n]\!][y_1, \dots, y_m]_{(x_1,\dots,x_n,y_1,\dots,y_m)}$. The completion of B is $\widehat{B} = K[\![X, Y]\!]$.

3.1. Theorem. *With the notation as above, every ideal Q of $\widehat{B} = K[\![X, Y]\!]$ maximal with the property that $Q \cap B = (0)$ is a prime ideal of height $n + m - 2$.*

Proof. Suppose first that Q is such an ideal. Then clearly Q is prime. Matsumura shows in [7, Theorem 3] that the dimension of the generic formal fiber of B is at most $n+m-2$. Therefore $\operatorname{ht} Q \leqslant n + m - 2$.

Now suppose $P \in \operatorname{Spec} \widehat{B}$ is an arbitrary prime ideal of height $r < n + m - 2$ with $P \cap B = (0)$. We construct a prime $Q \in \operatorname{Spec} \widehat{B}$ with $P \subset Q$, $Q \cap B = (0)$, and $\operatorname{ht} Q = n + m - 2$. This will show that all prime ideals maximal in the generic fiber have height $n + m - 2$.

For the construction of Q we consider first the case where $P \not\subseteq X\widehat{B}$. Then there exists a prime element $f \in P$ that contains a term $\theta := y_1^{i_1} \cdots y_m^{i_m}$, where the i_j's are nonnegative integers and at least one of the i_j is positive. Notice that $m \geqslant 2$ for otherwise with $y = y_1$ we have $f \in P$ contains a term y^i. By Weierstrass it follows that $f = g\epsilon$, where $g \in K[\![X]\!][y]$ is a nonzero monic polynomial in y and ϵ is a unit of $\widehat{B}$. But $g \in P$ and $g \in B$ implies $P \cap B \neq (0)$, a contradiction to our assumption that $P \cap B = (0)$.

For convenience we now assume that the last exponent i_m appearing in θ above is positive. We apply a change of variables: $y_m \to t_m := y_m$ and, for $1 \leqslant \ell < m$, let $y_\ell \to t_\ell := y_\ell + t_m^{e_\ell}$, where the e_ℓ are chosen so that f, expressed in the variables $t_1, \dots, t_m$, contains a term t_m^q, for some positive integer q. This change of variables induces an automorphism of B. By Weierstrass $f = g_1 h$, where h is a unit in $\widehat{B}$ and $g_1 \in K[\![X, t_1, \dots, t_{m-1}]\!][t_m]$ is monic in t_m. Set $P_1 = P \cap K[\![X, t_1, \dots, t_{m-1}]\!]$. If $P_1 \subseteq XK[\![X, t_1, \dots, t_{m-1}]\!]$, we stop the procedure and take $s = m - 1$ in what follows. If $P_1 \not\subseteq XK[\![X, t_1, \dots, t_{m-1}]\!]$, then there exists a prime element $\tilde{f} \in P_1$ that contains a term ${t_1}^{j_1} \cdots {t_{m-1}}^{j_{m-1}}$, where the j_k's are nonnegative integers and at least one of the j_k is positive. We then repeat the procedure using the prime ideal P_1. That is, we replace $t_1, \dots, t_{m-1}$ with a change of variables so that a prime element of P_1 contains a term monic in some one of the new variables. After a suitable finite iteration of changes of variables, we obtain an automorphism of $\widehat{B}$ that restricts to an automorphism of B and maps $y_1, \dots, y_m \mapsto z_1, \dots, z_m$. Moreover, there exist a positive integer $s \leqslant m - 1$ and elements $g_1, \dots, g_{m-s} \in P$ such that

$$
\begin{aligned}
&g_1 \in K[\![X, z_1, \dots, z_{m-1}]\!][z_m] && \text{is monic in } z_m,\\
&g_2 \in K[\![X, z_1, \dots, z_{m-2}]\!][z_{m-1}] && \text{is monic in } z_{m-1}, \text{etc.},\\
&\quad\vdots\\
&g_{m-s} \in K[\![X, z_1, \dots, z_s]\!][z_{s+1}] && \text{is monic in } z_{s+1},
\end{aligned}
$$

and such that, for $R_s := K[\![X, z_1, \dots, z_s]\!]$ and $P_s := P \cap R_s$, we have $P_s \subseteq XR_s$.

As in the proof of Proposition 2.2 we replace the regular sequence $g_1, \dots, g_{m-s}$ by a regular sequence $f_1, \dots, f_{m-s}$ so that:

$$\begin{aligned} &f_1 \in R_s[z_{s+1}, \dots, z_m] && \text{is monic in } z_m, \\ &f_2 \in R_s[z_{s+1}, \dots, z_{m-1}] && \text{is monic in } z_{m-1}, \text{ etc.}, \\ &\quad\vdots \\ &f_{m-s} \in R_s[z_{s+1}] && \text{is monic in } z_{s+1}, \end{aligned}$$

and $(g_1, \dots, g_{m-s})\widehat{B} = (f_1, \dots, f_{m-s})\widehat{B}$.

Let $G := K[\![X, z_1, \dots, z_s]\!][z_{s+1}, \dots, z_m] = R_s[z_{s+1}, \dots, z_m]$. By Proposition 2.2, P is extended from G. Let $\mathbf{q} := P \cap G$ and extend $f_1, \dots, f_{m-s}$ to a generating system of $\mathbf{q}$, say, $\mathbf{q} = (f_1, \dots, f_{m-s}, h_1, \dots, h_t)G$. For integers k, ℓ with $1 \leqslant k \leqslant m-s$ and $1 \leqslant \ell \leqslant t$, express the f_k and h_ℓ in G as power series in $\widehat{B} = K[\![z_1]\!][\![z_2, \dots, z_m]\!][\![X]\!]$ with coefficients in $K[\![z_1]\!]$:

$$f_k = \sum a_{k(i)(j)} z_2^{i_2} \dots z_m^{i_m} x_1^{j_1} \dots x_n^{j_n} \quad \text{and} \quad h_\ell = \sum b_{\ell(i)(j)} z_2^{i_2} \dots z_m^{i_m} x_1^{j_1} \dots x_n^{j_n},$$

where $a_{k(i)(j)}, b_{\ell(i)(j)} \in K[\![z_1]\!]$, $(i) = (i_2, \dots, i_m)$ and $(j) = (j_1, \dots, j_n)$. The set $\Delta = \{a_{k(i)(j)}, b_{\ell(i)(j)}\}$ is countable. We define $V := K(z_1, \Delta) \cap K[\![z_1]\!]$. Then V is a discrete valuation domain with completion $K[\![z_1]\!]$ and $K((z_1))$ has uncountable transcendence degree over $\mathcal{Q}(V)$. Let $V_s := V[\![X, z_2, \dots, z_s]\!] \subseteq R_s$. Notice that $R_s = \widehat{V_s}$, the completion of V_s. Also $f_1, \dots, f_{m-s} \in V_s[z_{s+1}, \dots, z_m] \subseteq G$ and $(f_1, \dots, f_{m-s})G \cap R_s = (0)$. Furthermore the extension

$$V_s := V[\![X, z_2, \dots, z_s]\!] \hookrightarrow V_s[z_{s+1}, \dots, z_m]/(f_1, \dots, f_{m-s})$$

is finite. Set $P_0 := P \cap V_s$. Then $P_0 \subseteq XR_s \cap V_s = XV_s$.

Consider the commutative diagram:

$$\begin{array}{ccc} R_s := K[\![X, z_1, \dots, z_s]\!] & \longrightarrow & R_s[\![z_{s+1}, \dots, z_m]\!]/(f_1, \dots, f_{m-s}) \\ \uparrow & & \uparrow \\ V_s := V[\![X, z_2, \dots, z_s]\!] & \longrightarrow & V_s[z_{s+1}, \dots, z_m]/(f_1, \dots, f_{m-s}). \end{array} \tag{3.1.1}$$

The horizontal maps are injective and finite and the vertical maps are completions.

The prime ideal $\bar{\mathbf{q}} := PR_s[\![z_{s+1}, \dots, z_m]\!]/(f_1, \dots, f_{m-s})$ lies over P_s in R_s. By assumption $P_s \subseteq (X)R_s$ and by Theorem 2.3 there is a prime ideal Q_s of R_s such that $P_s \subseteq Q_s \subseteq (X)R_s$, $Q_s \cap V_s = P_s \cap V_s = P_0$, and $\dim(R_s/Q_s) = 2$. There is a prime ideal $\bar{Q}$ in $R_s[\![z_{s+1}, \dots, z_m]\!]/(f_1, \dots, f_{m-s})$ lying over Q_s with $\bar{\mathbf{q}} \subseteq \bar{Q}$ by the "going-up theorem" [8, Theorem 9.4]. Let Q be the preimage in $\widehat{B} = K[\![X, z_1, \dots, z_m]\!]$ of $\bar{Q}$. We show the rings and ideals of Theorem 3.1 below.

$$\widehat{B} = K[\![X, Y]\!] = K[\![X, z_1, \ldots, z_m]\!] = R_s[\![z_{s+1}, \ldots, z_m]\!]$$

$$(\mathbf{q}, Q_s)\widehat{B} \subseteq Q$$

$$P \not\subseteq X\widehat{B}$$

$$G := R_s[z_{s+1}, \ldots, z_m]$$

$$\mathbf{q} := P \cap G$$

$$\mathbf{q} = \bigl(\{f_i, h_j\}\bigr)G$$

$$f_i \notin R_s := K[\![X, z_1, \ldots, z_s]\!]$$

$$P_s \subseteq Q_s \subset R_s$$

$$P_s := P \cap R_s \subseteq XR_s$$

$$V_s := V[\![X, z_2, \ldots, z_s]\!] \qquad \widehat{V} = K[\![z_1]\!]$$

$$P_0 := P \cap V_s$$

$$V := K(z_1, \Delta) \cap K[\![z_1]\!]$$

Then Q has height $n + s - 2 + m - s = n + m - 2$. Moreover, from diagram (3.1.1), it follows that Q and P have the same contraction to $V_s[z_{s+1}, \ldots, z_m]$. This implies that $Q \cap B = (0)$ and completes the proof in the case where $P \not\subseteq X\widehat{B}$.

In the case where $P \subseteq X\widehat{B}$, let $h_1, \ldots, h_t \in \widehat{B}$ be a finite set of generators of P, and as above, let $b_{\ell(i)(j)} \in K[\![z_1]\!]$ be the coefficients of the h_ℓ's. Consider the countable set $\Delta = \{b_{\ell(i)(j)}\}$ and the valuation domain $V := K(z_1, \Delta) \cap K[\![z_1]\!]$. Set $P_0 := P \cap V[\![X, z_2, \ldots, z_m]\!]$. By Theorem 2.3, there exists a prime ideal Q of $\widehat{B} = K[\![X, z_1, \ldots, z_m]\!]$ of height $n + m - 2$ such that $P \subset Q$ and

$$Q \cap V[\![X, z_2, \ldots, z_m]\!] = P \cap V[\![X, z_2, \ldots, z_m]\!] = P_0.$$

Therefore $Q \cap B = (0)$. This completes the proof of Theorem 3.1. $\square$

4. Weierstrass implications for the ring $C = K[Y]_{(Y)}[\![X]\!]$

As before K denotes a field, n and m are positive integers, and $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ denote sets of variables over K. Consider the ring $C = K[y_1, \ldots, y_m]_{(y_1, \ldots, y_m)}[\![x_1, \ldots, x_n]\!] = K[Y]_{(Y)}[\![X]\!]$. Then the completion of C is $\widehat{C} = K[\![Y, X]\!]$.

4.1. Theorem. *With notation as above, let $Q \in \operatorname{Spec} \widehat{C}$ be maximal with the property that $Q \cap C = (0)$. Then $\operatorname{ht} Q = n + m - 2$.*

Proof. Let $B = K[\![X]\!][Y]_{(X,Y)} \subset C$. If $P \in \operatorname{Spec}\widehat{C} = \operatorname{Spec}\widehat{B}$ and $P \cap C = (0)$, then $P \cap B = (0)$, so $\operatorname{ht} P \leqslant n + m - 2$ by Theorem 3.1. Consider a nonzero prime $P \in \operatorname{Spec}\widehat{C}$ with $P \cap C = (0)$ and $\operatorname{ht} P = r < n + m - 2$. If $P \subseteq X\widehat{C}$ then Theorem 2.3 implies the existence of $Q \in \operatorname{Spec}\widehat{C}$ with $\operatorname{ht} Q = n + m - 2$ such that $P \subset Q$ and $Q \cap C = (0)$.

Assume that P is not contained in $X\widehat{C}$ and consider the ideal $J := (P, X)\widehat{C}$. Since C is complete in the XC-adic topology, [9, Lemma 2] implies that if J is primary for the maximal ideal of $\widehat{C}$, then P is extended from C. Since we are assuming $P \cap C = (0)$, J is not primary for the maximal ideal of $\widehat{C}$ and we have $\operatorname{ht} J = n + s < n + m$, where $0 < s < m$. Let $W \in \operatorname{Spec}\widehat{C}$ be a minimal prime of J such that $\operatorname{ht} W = n + s$. Let $W_0 = W \cap K[\![Y]\!]$. Then $W = (W_0, X)\widehat{C}$ and W_0 is a prime ideal of $K[\![Y]\!]$ with $\operatorname{ht} W_0 = s$. By Proposition 2.2 applied to $K[\![Y]\!]$ and the prime ideal $W_0 \in \operatorname{Spec} K[\![Y]\!]$, there exists a change of variables $Y \mapsto Z$ with $y_1 \mapsto z_1, \dots, y_m \mapsto z_m$ and elements $f_1, \dots, f_s \in W_0$ so that with $Z_1 = \{z_1, \dots, z_{m-s}\}$, we have

$$\begin{array}{ll} f_1 \in K[\![Z_1]\!][z_{m-s+1}, \dots, z_m] & \text{is monic in } z_m, \\ f_2 \in K[\![Z_1]\!][z_{m-s+1}, \dots, z_{m-1}] & \text{is monic in } z_{m-1}, \text{ etc.,} \\ \quad\vdots & \\ f_s \in K[\![Z_1]\!][z_{m-s+1}] & \text{is monic in } z_{m-s+1}. \end{array}$$

Now $z_1, \dots, z_{m-s}, f_1, \dots, f_s$ is a regular sequence in $K[\![Z]\!] = K[\![Y]\!]$. Let $T = \{t_{m-s+1}, \dots, t_m\}$ be a set of additional variables and consider the map:

$$\varphi : K[\![Z_1, T]\!] \to K[\![z_1, \dots, z_m]\!]$$

defined by $z_i \mapsto z_i$ for all $1 \leqslant i \leqslant m - s$ and $t_{m-i+1} \mapsto f_i$ for all $1 \leqslant i \leqslant s$. The embedding φ is finite (and free) and so is the extension to power series rings in X:

$$\rho : K[\![Z_1, T]\!][\![X]\!] \to K[\![z_1, \dots, z_m]\!][\![X]\!] = \widehat{C}.$$

Since $W \in \operatorname{Spec}\widehat{C}$ is of height $n + s$, so is its contraction $\rho^{-1}(W) \in \operatorname{Spec} K[\![Z_1, T, X]\!]$. Moreover $\rho^{-1}(W)$ contains $(T, X)K[\![Z_1, T, X]\!]$, a prime ideal of height $n + s$. Therefore $\rho^{-1}(W) = (T, X)K[\![Z_1, T, X]\!]$. By construction, $P \subseteq W$ which yields that $\rho^{-1}(P) \subseteq (T, X)K[\![Z_1, T, X]\!]$.

To complete the proof we construct a suitable base ring related to C. Consider the expressions for the f_i's as power series in $z_2, \dots, z_m$ with coefficients in $K[\![z_1]\!]$:

$$f_j = \sum a_{j(i)} z_2^{i_2} \dots z_m^{i_m},$$

where $(i) := (i_2, \dots, i_m)$, $1 \leqslant j \leqslant s$, $a_{j(i)} \in K[\![z_1]\!]$. Also consider a finite generating system $g_1, \dots, g_q$ for P and expressions for the g_k, where $1 \leqslant k \leqslant q$, as power series in $z_2, \dots, z_m, x_1, \dots, x_n$ with coefficients in $K[\![z_1]\!]$:

$$g_k = \sum b_{k(i)(\ell)} z_2^{i_2} \dots z_m^{i_m} x_1^{\ell_1} \dots x_n^{\ell_n},$$

where $(i) := (i_2, \dots, i_m)$, $(\ell) := (\ell_1, \dots, \ell_n)$, and $b_{k(i)(\ell)} \in K[\![z_1]\!]$. We take the subset $\Delta = \{a_{j(i)}, b_{k(i)(\ell)}\}$ of $K[\![z_1]\!]$ and consider the discrete valuation domain:

$$V := K(z_1, \Delta) \cap K[\![z_1]\!].$$

Since V is countably generated over $K(z_1)$, the field $K(\!(z_1)\!)$ has uncountable transcendence degree over $\mathcal{Q}(V) = K(z_1, \Delta)$. Moreover, by construction the ideal P is extended from $V[\![z_2, \dots, z_m]\!][\![X]\!]$. Consider the embedding:

$$\psi : V[\![z_2, \dots, z_{m-s}, T]\!] \to V[\![z_2, \dots, z_m]\!],$$

which is the restriction of φ above, so that $z_i \mapsto z_i$ for all $2 \leqslant i \leqslant m - s$ and $t_{m-i+1} \mapsto f_i$ for all i with $1 \leqslant i \leqslant s$.

Let σ be the extension of ψ to the power series rings:

$$\sigma : V[\![z_2, \dots, z_{m-s}, T]\!][\![X]\!] \to V[\![z_2, \dots, z_m]\!][\![X]\!]$$

with $\sigma(x_i) = x_i$ for all i with $1 \leqslant i \leqslant n$.

Notice that ρ defined above is the completion $\hat{\sigma}$ of the map σ, that is, the extension of σ to the completions. Consider the commutative diagram:

$$\begin{tikzcd} K[\![Z_1, T]\!][\![X]\!] \arrow[r, "\hat{\sigma}=\rho"] & K[\![Z]\!][\![X]\!] = \widehat{C} \\ V[\![z_2, \dots, z_{m-s}, T]\!][\![X]\!] \arrow[r, "\sigma"] \arrow[u] & V[\![z_2, \dots, z_m]\!][\![X]\!] \arrow[u] \end{tikzcd} \tag{4.1.0}$$

where $\hat{\sigma} = \rho$ is a finite map.

Recall that $\rho^{-1}(W) = (T, X)K[\![Z_1, T, X]\!]$, and so $\rho^{-1}(P) \subseteq (T, X)K[\![Z_1, T, X]\!]$ by Diagram 4.1.0. By Theorem 2.3, there exists a prime ideal Q_0 of the ring $K[\![Z_1, T, X]\!]$ such that $\rho^{-1}(P) \subseteq Q_0$, $\operatorname{ht} Q_0 = n + m - 2$, and

$$Q_0 \cap V[\![z_2, \dots, z_{m-s}, T]\!][\![X]\!] = \rho^{-1}(P) \cap V[\![z_2, \dots, z_{m-s}, T]\!][\![X]\!].$$

By the "going-up theorem" [8, Theorem 9.4], there is a prime ideal $Q \in \operatorname{Spec} \widehat{C}$ that lies over Q_0 and contains P. Moreover, Q also has height $n + m - 2$. The commutativity of diagram (4.1.0) implies that

$$P_1 := P \cap V[\![z_2, \dots, z_{m-s}, T]\!][\![X]\!] \subseteq Q_1 := Q \cap V[\![z_2, \dots, z_{m-s}, T]\!][\![X]\!].$$

Consider the finite homomorphism:

$$\lambda : V[\![z_2, \dots, z_{m-s}]\!][T]_{(Z_1,T)}[\![X]\!] \to V[\![z_2, \dots, z_{m-s}]\!][z_{m-s+1}, \dots, z_m]_{(Z)}[\![X]\!]$$

(determined by $t_i \mapsto f_i$ for $1 \leqslant i \leqslant m$) and the commutative diagram:

$$\begin{tikzcd} V[\![z_2,\dots,z_{m-s}]\!][\![T]\!][\![X]\!] \arrow[r, "\sigma"] & V[\![z_2,\dots,z_m]\!][\![X]\!] \\ V[\![z_2,\dots,z_{m-s}]\!][T]_{(Z_1,T)}[\![X]\!] \arrow[u] \arrow[r, "\lambda"] & V[\![z_2,\dots,z_{m-s}]\!][z_{m-s+1},\dots,z_m]_{(Z)}[\![X]\!]. \arrow[u] \end{tikzcd}$$

Since $Q \cap V[\![z_2,\dots,z_{m-s},T]\!][\![X]\!] = P \cap V[\![z_2,\dots,z_{m-s},T]\!][\![X]\!]$ and since λ is a finite map we conclude that

$$\begin{aligned} &Q_1 \cap V[\![z_2,\dots,z_{m-s}]\!][z_{m-s+1},\dots,z_m]_{(Z)}[\![X]\!] \\ &\quad = P_1 \cap V[\![z_2,\dots,z_m]\!][z_{m-s+1},\dots,z_m]_{(Z)}[\![X]\!]. \end{aligned}$$

Since $C \subseteq V[\![z_2,\dots,z_{m-s}]\!][z_{m-s+1},\dots,z_m]_{(Z)}[\![X]\!]$, we obtain $Q \cap C = P \cap C = (0)$. This completes the proof of Theorem 4.1. $\square$

4.2. Remark. With B and C as in Sections 3 and 4, we have

$$B = K[\![X]\!][Y]_{(X,Y)} \hookrightarrow K[Y]_{(Y)}[\![X]\!] = C \quad \text{and} \quad \widehat{B} = K[\![X,Y]\!] = \widehat{C}.$$

Thus for $P \in \operatorname{Spec} K[\![X,Y]\!]$, if $P \cap C = (0)$, then $P \cap B = (0)$. By Theorems 3.1 and 4.1, each prime of $K[\![X,Y]\!]$ maximal in the generic formal fiber of B or C has height $n+m-2$. Therefore each $P \in \operatorname{Spec} K[\![X,Y]\!]$ maximal with respect to $P \cap C = (0)$ is also maximal with respect to $P \cap B = (0)$. However, if $n+m \geqslant 3$, the generic fiber of $B \hookrightarrow C$ is nonzero [4], so there exist primes of $K[\![X,Y]\!]$ maximal in the generic formal fiber of B that are not in the generic formal fiber of C.

5. Subrings of the power series ring $K[\![z,t]\!]$

In this section we establish properties of certain subrings of the power series ring $K[\![z,t]\!]$ that will be useful in considering the generic formal fiber of localized polynomial rings over the field K.

5.1. Notation. Let K be a field and let z and t be independent variables over K. Consider countably many power series:

$$\alpha_i(z) = \sum_{j=0}^{\infty} a_{ij} z^j \in K[\![z]\!]$$

with coefficients $a_{ik} \in K$. Let s be a positive integer and let $\omega_1, \dots, \omega_s \in K[\![z,t]\!]$ be power series in z and t, say:

$$\omega_i = \sum_{j=0}^{\infty} \beta_{ij} t^j, \quad \text{where} \quad \beta_{ij}(z) = \sum_{k=0}^{\infty} b_{ijk} z^k \in K[\![z]\!] \quad \text{and} \quad b_{ijk} \in K,$$

for each i with $1 \leqslant i \leqslant s$. Consider the subfield $K(z, \{\alpha_i\}, \{\beta_{ij}\})$ of $K((z))$ and the discrete rank-one valuation domain

$$V := K\bigl(z, \{\alpha_i\}, \{\beta_{ij}\}\bigr) \cap K[\![z]\!].$$

The completion of V is $\widehat{V} = K[\![z]\!]$. Assume that $\omega_1, \dots, \omega_r$ are algebraically independent over $\mathcal{Q}(V)(t)$ and that the elements $\omega_{r+1}, \dots, \omega_s$ are algebraic over the field $\mathcal{Q}(V)(t, \{\omega_i\}_{i=1}^r)$. Notice that the set $\{\alpha_i\} \cup \{\beta_{ij}\}$ is countable, and that also the set of coefficients of the α_i and β_{ij}

$$\Delta := \{a_{ij}\} \cup \{b_{ijk}\}$$

is a countable subset of the field K. Let K_0 denote the prime subfield of K and let F denote the algebraic closure in K of the field $K_0(\Delta)$. The field F is countable and the power series $\alpha_i(z)$ and $\beta_{ij}(z)$ are in $F[\![z]\!]$. Consider the subfield $F(z, \{\alpha_i\}, \{\beta_{ij}\})$ of $F((z))$ and the discrete rank-one valuation domain

$$V_0 := F\bigl(z, \{\alpha_i\}, \{\beta_{ij}\}\bigr) \cap F[\![z]\!].$$

The completion of V_0 is $\widehat{V_0} = F[\![z]\!]$. Since $\mathcal{Q}(V_0)(t) \subseteq \mathcal{Q}(V)(t)$, the elements $\omega_1, \dots, \omega_r$ are algebraically independent over the field $\mathcal{Q}(V_0)(t)$.

Consider the subfield $E_0 := \mathcal{Q}(V_0)(t, \omega_1, \dots, \omega_r)$ of $\mathcal{Q}(V_0[\![t]\!])$ and the subfield $E := \mathcal{Q}(V)(t, \omega_1, \dots, \omega_r)$ of $\mathcal{Q}(V[\![t]\!])$. A result of Valabrega [11] implies that the integral domains:

$$D_0 := E_0 \cap V_0[\![t]\!] \quad \text{and} \quad D := E \cap V[\![t]\!] \tag{5.1.1}$$

are two-dimensional regular local rings with completions $\widehat{D_0} = F[\![z,t]\!]$ and $\widehat{D} = K[\![z,t]\!]$, respectively. Moreover, $\mathcal{Q}(D_0) = E_0$ is a countable field.

5.2. Proposition. *Let D_0 be as defined in* 5.1. *Then there exists a power series $\gamma \in zF[\![z]\!]$ such that the prime ideal $(t-\gamma)F[\![z,t]\!] \cap D_0 = (0)$, i.e., $(t-\gamma)F[\![z,t]\!]$ is in the generic formal fiber of D_0.*

Proof. Since D_0 is countable there are only countably many prime ideals in D_0 and since D_0 is Noetherian there are only countably many prime ideals in $\widehat{D_0} = F[\![z,t]\!]$ that lie over a nonzero prime of D_0. There are uncountably many primes in $F[\![z,t]\!]$, which are generated by elements of the form $t - \sigma$ for some $\sigma \in zF[\![z]\!]$. Thus there must exist an element $\gamma \in zF[\![z]\!]$ with $(t-\gamma)F[\![z,t]\!] \cap D_0 = (0)$. $\square$

For $\omega_i = \omega_i(t) = \sum_{j=0}^{\infty} \beta_{ij} t^j$ as in 5.1 and γ an element of $zK[\![z]\!]$, let $\omega_i(\gamma)$ denote the following power series in $K[\![z]\!]$:

$$\omega_i(\gamma) := \sum_{j=0}^{\infty} \beta_{ij} \gamma^j \in K[\![z]\!].$$

5.3. Proposition. *Let D be as defined in* (5.1.1). *For an element $\gamma \in zK[\![z]\!]$ the following conditions are equivalent*:

(i) $(t-\gamma)K[\![z,t]\!] \cap D = (0)$.
(ii) *The elements $\gamma, \omega_1(\gamma), \ldots, \omega_r(\gamma)$ are algebraically independent over $\mathcal{Q}(V)$.*

Proof. (i) $\Rightarrow$ (ii). Assume by way of contradiction that the set $\{\gamma, \omega_1(\gamma), \ldots, \omega_r(\gamma)\}$ is algebraically dependent over $\mathcal{Q}(V)$ and let $d_{(k)} \in V$ be finitely many elements such that

$$\sum_{(k)} d_{(k)} \omega_1(\gamma)^{k_1} \ldots \omega_r(\gamma)^{k_r} \gamma^{k_{r+1}} = 0$$

is a nontrivial equation of algebraic dependence for $\gamma, \omega_1(\gamma), \ldots, \omega_r(\gamma)$, where each $(k) = (k_1 \ldots, k_r, k_{r+1})$ is an $(r+1)$-tuple of nonnegative integers. It follows that

$$\sum_{(k)} d_{(k)} \omega_1^{k_1} \ldots \omega_r^{k_r} t^{k_{r+1}} \in (t-\gamma)K[\![z,t]\!] \cap D = (0).$$

Since $\omega_1, \ldots, \omega_r$ are algebraically independent over $\mathcal{Q}(V)(t)$, we have $d_{(k)} = 0$ for all (k), a contradiction. This completes the proof that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). If $(t-\gamma)K[\![z,t]\!] \cap D \neq (0)$, then there exists a nonzero element

$$\tau = \sum_{(k)} d_{(k)} \omega_1^{k_1} \ldots \omega_r^{k_r} t^{k_{r+1}} \in (t-\gamma)K[\![z,t]\!] \cap V[t, \omega_1, \ldots, \omega_r].$$

But this implies that

$$\tau(\gamma) = \sum_{(k)} d_{(k)} \omega_1(\gamma)^{k_1} \ldots \omega_r(\gamma)^{k_r} \gamma^{k_{r+1}} = 0.$$

Since $\gamma, \omega_1(\gamma), \ldots, \omega_r(\gamma)$ are algebraically independent over $\mathcal{Q}(V)$, it follows that all the coefficients $d_{(k)} = 0$, a contradiction to the assumption that τ is nonzero. $\square$

Let $\gamma \in zF[\![z]\!]$ be as in Proposition 5.2 with $(t-\gamma)F[\![z,t]\!] \cap D_0 = (0)$. Then:

5.4. Proposition. *With notation as above, we also have $(t-\gamma)K[\![z,t]\!] \cap D = (0)$, that is, $(t-\gamma)K[\![z,t]\!]$ is in the generic formal fiber of D.*

Proof. Let $\{t_i\}_{i\in I}$ be a transcendence basis of K over F and let $L := F(\{t_i\}_{i\in I})$. Then K is algebraic over L. Let $\{\alpha_i\}, \{\beta_{ij}\} \subset F[\![z]\!]$ be as in 5.1 and define

$$V_1 = L\big(z, \{\alpha_i\}, \{\beta_{ij}\}\big) \cap L[\![z]\!] \quad \text{and} \quad D_1 = \mathcal{Q}(V_1)(t, \omega_1, \ldots, \omega_r) \cap L[\![z, t]\!].$$

Then V_1 is a discrete rank-one valuation domain with completion $L[\![z]\!]$ and D_1 is a two-dimensional regular local domain with completion $\widehat{D_1} = L[\![z, t]\!]$. Note that $\mathcal{Q}(V)$ and $\mathcal{Q}(D)$ are algebraic over $\mathcal{Q}(V_1)$ and $\mathcal{Q}(D_1)$, respectively. Since $(t-\gamma)K[\![z, t]\!] \cap L[\![z, t]\!] = (t-\gamma)L[\![z, t]\!]$, it suffices to prove that $(t-\gamma)L[\![z, t]\!] \cap D_1 = (0)$. By Proposition 5.3, it suffices to show that $\gamma, \omega_1(\gamma), \ldots, \omega_r(\gamma)$ are algebraically independent over $\mathcal{Q}(V_1)$. The commutative diagram

$$\begin{tikzcd} F[\![z]\!] \arrow[rr, "\{t_i\}\text{ algebraically ind.}"] & & L[\![z]\!] \\ \mathcal{Q}(V_0) \arrow[u] \arrow[rr, "\text{transcendence basis }\{t_i\}"] & & \mathcal{Q}(V_1) \arrow[u] \end{tikzcd}$$

implies that the set $\{\gamma, \omega_1(\gamma), \ldots, \omega_r(\gamma)\} \cup \{t_i\}$ is algebraically independent over $\mathcal{Q}(V_0)$. Therefore $\{\gamma, \omega_1(\gamma), \ldots, \omega_r(\gamma)\}$ is algebraically independent over $\mathcal{Q}(V_1)$, which completes the proof of Proposition 5.4. □

5.5. Remark. We remark that with $\omega_{r+1}, \ldots, \omega_s$ algebraic over $\mathcal{Q}(V)(\omega_1, \ldots, \omega_r)$ as in 5.1, if we define

$$\widetilde{D} := \mathcal{Q}(V)(t, \omega_1, \ldots, \omega_s) \cap V[\![t]\!],$$

then again by Valabrega [11], $\widetilde{D}$ is a two-dimensional regular local domain with completion $K[\![z, t]\!]$. Moreover, $\mathcal{Q}(\widetilde{D})$ is algebraic over $\mathcal{Q}(D)$ and $(t-\gamma)K[\![z, t]\!] \cap D = (0)$ implies that $(t-\gamma)K[\![z, t]\!] \cap \widetilde{D} = (0)$.

6. Weierstrass implications for the localized polynomial ring $A = K[X]_{(X)}$

Let n be a positive integer, let $X = \{x_1, \ldots, x_n\}$ be a set of n variables over a field K, and let $A := K[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)} = K[X]_{(X)}$ denote the localized polynomial ring in these n variables over K. Then the completion of A is $\widehat{A} = K[\![X]\!]$.

6.1. Theorem. *For the localized polynomial ring $A = K[X]_{(X)}$ defined above, if Q is an ideal of $\widehat{A}$ maximal with respect to $Q \cap A = (0)$, then Q is a prime ideal of height $n-1$.*

Proof. Again it is clear that Q as described in the statement is a prime ideal. Also the assertion holds for $n = 1$. Thus we assume $n \geqslant 2$. By Proposition 5.4, there exists a nonzero prime $\mathbf{p}$ in $K[\![x_1, x_2]\!]$ such that $\mathbf{p} \cap K[x_1, x_2]_{(x_1,x_2)} = (0)$. It follows that $\mathbf{p}\widehat{A} \cap A = (0)$. Thus the generic formal fiber of A is nonzero.

Let $P \in \operatorname{Spec} \widehat{A}$ be a nonzero prime ideal with $P \cap A = (0)$ and $\operatorname{ht} P = r < n-1$. We construct $Q \in \operatorname{Spec} \widehat{A}$ of height $n-1$ with $P \subseteq Q$ and $Q \cap A = (0)$. By Proposition 2.2, there exists a change of variables $x_1 \mapsto z_1, \dots, x_n \mapsto z_n$ and polynomials

$$\begin{aligned}
&f_1 \in K[\![z_1, \dots, z_{n-r}]\!][z_{n-r+1}, \dots, z_n] && \text{monic in } z_n,\\
&f_2 \in K[\![z_1, \dots, z_{n-r}]\!][z_{n-r+1}, \dots, z_{n-1}] && \text{monic in } z_{n-1}, \text{etc.},\\
&\quad\vdots\\
&f_r \in K[\![z_1, \dots, z_{n-r}]\!][z_{n-r+1}] && \text{monic in } z_{n-r+1},
\end{aligned}$$

so that P is a minimal prime of $(f_1, \dots, f_r)\widehat{A}$ and P is extended from

$$R := K[\![z_1, \dots, z_{n-r}]\!][z_{n-r+1}, \dots, z_n].$$

Let $P_0 := P \cap R$ and extend $f_1, \dots, f_r$ to a system of generators of P_0, say:

$$P_0 = (f_1, \dots, f_r, g_1, \dots, g_s)R.$$

Using an argument similar to that in the proof of Theorem 2.3, write

$$f_j = \sum_{(i) \in \mathbb{N}^{n-1}} a_{j,(i)} z_2^{i_2} \dots z_n^{i_n} \quad \text{and} \quad g_k = \sum_{(i) \in \mathbb{N}^{n-1}} b_{k,(i)} z_2^{i_2} \dots z_n^{i_n},$$

where $a_{j,(i)}, b_{k,(i)} \in K[\![z_1]\!]$. Let

$$V_0 := K(z_1, a_{j,(i)}, b_{k,(i)}) \cap K[\![z_1]\!].$$

Then V_0 is a discrete rank-one valuation domain with completion $K[\![z_1]\!]$, and $K((z_1))$ has uncountable transcendence degree over the field of fractions $\mathcal{Q}(V_0)$ of V_0. Let $\gamma_3, \dots, \gamma_{n-r} \in K[\![z_1]\!]$ be algebraically independent over $\mathcal{Q}(V_0)$ and define

$$\mathbf{q} := (z_3 - \gamma_3 z_2, z_4 - \gamma_4 z_2, \dots, z_{n-r} - \gamma_{n-r} z_2) K[\![z_1, \dots, z_{n-r}]\!].$$

We see that $\mathbf{q} \cap V_0[\![z_2, \dots, z_{n-r}]\!] = (0)$ by an argument similar to that in [7] and in Claim 2.3.1. Let $R_1 := V_0[\![z_2, \dots, z_{n-r}]\!][z_{n-r+1}, \dots, z_n]$, let $P_1 := P \cap R_1$ and consider the commutative diagram:

$$\begin{tikzcd}
K[\![z_1, \dots, z_{n-r}]\!] \arrow[r] & R/P_0 \\
V_0[\![z_2, \dots, z_{n-r}]\!] \arrow[r] \arrow[u] & R_1/P_1 \arrow[u]
\end{tikzcd}$$

The horizontal maps are injective finite integral extensions. Let W be a minimal prime of $(\mathbf{q}, P)\widehat{A}$. Then $\operatorname{ht} W = n-2$ and $\mathbf{q} \cap V_0[\![z_2, \dots, z_{n-r}]\!] = (0)$ implies that $W \cap R_1 = P_1$.

We have found a prime ideal $W \in \operatorname{Spec} \widehat{A}$ such that $\operatorname{ht} W = n-2$, $W \cap A = (0)$ and $P \subseteq W$. Since $f_1, \dots, f_r \in W$ and since $\widehat{A} = K[\![z_1, \dots, z_n]\!]$ is the $(f_1, \dots, f_r)$-adic completion of $K[\![z_1, \dots, z_{n-r}]\!][z_{n-r+1}, \dots, z_n]$, the prime ideal W is extended from $K[\![z_1, \dots, z_{n-r}]\!][z_{n-r+1}, \dots, z_n]$.

We claim that W is actually extended from $K[\![z_1, z_2]\!][z_3, \dots, z_n]$. To see this let $g \in W \cap K[\![z_1, \dots, z_{n-r}]\!][z_{n-r+1}, \dots, z_n]$ and write:

$$g = \sum_{(i)} a_{(i)} z_{n-r+1}^{i_{n-r+1}} \dots z_n^{i_n} \in K[\![z_1, \dots, z_{n-r}]\!][z_{n-r+1}, \dots, z_n],$$

where the sum is over all $(i) = (i_{n-r}, \dots, i_n)$ and $a_{(i)} \in K[\![z_1, \dots, z_{n-r}]\!]$. For all $a_{(i)}$ by Weierstrass we can write

$$a_{(i)} = (z_{n-r} - \gamma_{n-r} z_2) h_{(i)} + q_{(i)},$$

where $h_{(i)} \in K[\![z_1, \dots, z_{n-r}]\!]$ and $q_{(i)} \in K[\![z_1, \dots, z_{n-r-1}]\!]$. If $n - r > 3$, we write

$$q_{(i)} = (z_{n-r-1} - \gamma_{n-r-1} z_2) h'_{(i)} + q'_{(i)},$$

where $h'_{(i)} \in K[\![z_1, \dots, z_{n-r-1}]\!]$ and $q'_{(i)} \in K[\![z_1, \dots, z_{n-r-2}]\!]$. In this way we replace a generating set for W in $K[\![z_1, \dots, z_{n-r}]\!][z_{n-r+1}, \dots, z_n]$ by a generating set for W in $K[\![z_1, z_2]\!][z_3, \dots, z_n]$.

In particular, we can replace the elements $f_1, \dots, f_r$ by elements:

$$\begin{array}{ll}
h_1 \in K[\![z_1, z_2]\!][z_3, \dots, z_n] & \text{monic in } z_n, \\
h_2 \in K[\![z_1, z_2]\!][z_3, \dots, z_{n-1}] & \text{monic in } z_{n-1}, \text{ etc.}, \\
\quad \vdots & \\
h_r \in K[\![z_1, z_2]\!][z_3 \dots, z_{n-r+1}] & \text{monic in } z_{n-r+1},
\end{array}$$

and set $h_{r+1} = z_3 - \gamma_3 z_2, \dots, h_{n-2} = z_{n-r} - \gamma_{n-r} z_2$, and then extend to a generating set $h_1, \dots, h_{n+s-2}$ for

$$W_0 = W \cap K[\![z_1, z_2]\!][z_3, \dots, z_n]$$

such that $W_0 \widehat{A} = W$. Consider the coefficients in $K[\![z_1]\!]$ of the h_j:

$$h_j = \sum_{(i)} c_{j(i)} z_2^{i_2} \dots z_n^{i_n}$$

with $c_{j(i)} \in K[\![z_1]\!]$. The set $\{c_{j(i)}\}$ is countable. Define

$$V := \mathcal{Q}(V_0)\bigl(\{c_{j(i)}\}\bigr) \cap K[\![z_1]\!].$$

Then V is a rank-one discrete valuation domain that is countably generated over $K[z_1]_{(z_1)}$ and W is extended from $V[\![z_2]\!][z_3,\dots,z_n]$.

We may also write each h_i as a polynomial in $z_3,\dots,z_n$ with coefficients in $V[\![z_2]\!]$:

$$h_i = \sum \omega_{(i)} z_3^{i_3} \dots z_n^{i_n}$$

with $\omega_{(i)} \in V[\![z_2]\!] \subseteq K[\![z_1,z_2]\!]$. By the result of Valabrega [11], the integral domain

$$D := \mathcal{Q}(V)\bigl(z_2, \{\omega_{(i)}\}\bigr) \cap K[\![z_1,z_2]\!]$$

is a two-dimensional regular local domain with completion $\widehat{D} = K[\![z_1,z_2]\!]$. Let $W_1 := W \cap D[z_3,\dots,z_n]$. Then $W_1\widehat{A} = W$. We have shown in Section 5 that there exists a prime element $q \in K[\![z_1,z_2]\!]$ with $qK[\![z_1,z_2]\!] \cap D = (0)$. Consider the finite extension

$$D \to D[z_3,\dots,z_n]/W_1.$$

Let $Q \in \operatorname{Spec}\widehat{A}$ be a minimal prime of $(q,W)\widehat{A}$. Since $\operatorname{ht} W = n-2$ and $q \notin W$, $\operatorname{ht} Q = n-1$. Moreover, $P \subseteq W$ implies $P \subseteq Q$. We claim that

$$Q \cap D[z_3,\dots,z_n] = W_1 \quad \text{and therefore} \quad Q \cap A = (0).$$

To see this consider the commutative diagram:

$$\begin{array}{ccc} K[\![z_1,z_2]\!] & \longrightarrow & K[\![z_1,\dots,z_n]\!]/W \\ \uparrow & & \uparrow \\ D & \longrightarrow & D[z_3,\dots,z_n]/W_1 \end{array}$$

which has injective finite horizontal maps. Since $qK[\![z_1,z_2]\!] \cap D = (0)$, it follows that $Q \cap D[z_3,\dots,z_n] = W_1$. This completes the proof of Theorem 6.1. □

7. Generic fibers of power series ring extensions

In this section we apply the Weierstrass machinery from Section 2 to the generic fiber rings of power series extensions.

7.1. Theorem. *Let $n \geqslant 2$ be an integer and let $y, x_1,\dots,x_n$ be variables over the field K. Let $X = \{x_1,\dots,x_n\}$. Consider the formal power series ring $R_1 = K[\![X]\!]$ and the extension $R_1 \hookrightarrow R_1[\![y]\!] = R$. Let $U = R_1 \setminus (0)$. For $P \in \operatorname{Spec} R$ such that $P \cap U = \emptyset$ we have*:

(1) *If $P \nsubseteq XR$, then $\dim(R/P) = n$ and P is maximal in the generic fiber $U^{-1}R$.*
(2) *If $P \subseteq XR$, then there exists $Q \in \operatorname{Spec} R$ such that $P \subseteq Q$, $\dim(R/Q) = 2$ and Q is maximal in the generic fiber $U^{-1}R$.*

If $n > 2$ for each prime ideal Q maximal in the generic fiber $U^{-1}R$, we have

$$\dim(R/Q) = \begin{cases} n, & \text{and } R_1 \hookrightarrow R/Q \text{ is finite, or} \\ 2, & \text{and } Q \subset XR. \end{cases}$$

Proof. Let $P \in \operatorname{Spec} R$ be such that $P \cap U = \emptyset$ or equivalently $P \cap R_1 = (0)$. Then R_1 embeds in R/P. If $\dim(R/P) \leqslant 1$, then the maximal ideal of R_1 generates an ideal primary for the maximal ideal of R/P. By [8, Theorem 8.4] R/P is finite over R_1, and so $\dim R_1 = \dim(R/P)$, a contradiction. Thus $\dim(R/P) \geqslant 2$.

If $P \not\subseteq XR$, then there exists a prime element $f \in P$ that contains a term y^s for some positive integer s. By Weierstrass, it follows that $f = g\epsilon$, where $g \in K[\![X]\!][y]$ is a nonzero monic polynomial in y and ϵ is a unit of R. We have $fR = gR \subseteq P$ is a prime ideal and $R_1 \hookrightarrow R/gR$ is a finite integral extension. Since $P \cap R_1 = (0)$, we must have $gR = P$.

If $P \subseteq XR$ and $\dim(R/P) > 2$, then Theorem 2.3 implies there exists $Q \in \operatorname{Spec} R$ such that $\dim(R/Q) = 2$, $P \subset Q \subset XR$ and $P \cap R_1 = (0) = Q \cap R_1$, and so P is not maximal in the generic fiber. Thus $Q \in \operatorname{Spec} R$ maximal in the generic fiber of $R_1 \hookrightarrow R$ implies that the dimension of $\dim(R/Q)$ is 2, or equivalently that $\operatorname{ht} Q = n - 1$. □

7.2. Theorem. *Let n and m be positive integers, and let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ be sets of independent variables over the field K. Consider the formal power series rings $R_1 = K[\![X]\!]$ and $R = K[\![X, Y]\!]$ and the extension $R_1 \hookrightarrow R_1[\![Y]\!] = R$. Let $U = R_1 \setminus (0)$. Let $Q \in \operatorname{Spec} R$ be maximal with respect to $Q \cap U = \emptyset$. If $n = 1$, then* $\dim(R/Q) = 1$ *and $R_1 \hookrightarrow R/Q$ is finite.*

If $n \geqslant 2$, there are two possibilities:

(1) *$R_1 \hookrightarrow R/Q$ is finite, in which case* $\dim(R/Q) = \dim R_1 = n$*, or*
(2) $\dim(R/Q) = 2$.

Proof. First assume $n = 1$, and let $x = x_1$. Since Q is maximal with respect to $Q \cap U = \emptyset$, for each $P \in \operatorname{Spec} R$ with $Q \subsetneq P$ we have $P \cap U$ is nonempty and therefore $x \in P$. It follows that $\dim(R/Q) = 1$, for otherwise,

$$Q = \bigcap \{P \mid P \in \operatorname{Spec} R \text{ and } Q \subsetneq P\},$$

which implies $x \in Q$. By [8, Theorem 8.4], $R_1 \hookrightarrow R/Q$ is finite.

It remains to consider the case where $n \geqslant 2$. We proceed by induction on m. Theorem 7.1 yields the assertion for $m = 1$. Suppose $Q \in \operatorname{Spec} R$ is maximal with respect to $Q \cap U = \emptyset$. As in the proof of Theorem 7.1, we have $\dim(R/Q) \geqslant 2$. If $Q \subseteq (X, y_1, \ldots, y_{m-1})R$, then by Theorem 2.3 with $R_0 = K[y_m]_{(y_m)}[\![X, y_1, \ldots, y_{m-1}]\!]$, there exists $Q' \in \operatorname{Spec} R$ with $Q \subseteq Q'$, $\dim(R/Q') = 2$, and $Q \cap R_0 = Q' \cap R_0$. Since $R_1 \subseteq R_0$, we have $Q' \cap U = \emptyset$. Since Q is maximal with respect to $Q \cap U = \emptyset$, we have $Q = Q'$, so $\dim(R/Q) = 2$.

Otherwise, if $Q \not\subseteq (X, y_1, \ldots, y_{m-1})R$, then there exists a prime element $f \in Q$ that contains a term y_m^s for some positive integer s. Let $R_2 = K[\![X, y_1, \ldots, y_{m-1}]\!]$. By Weierstrass, it follows that $f = g\epsilon$, where $g \in R_2[y_m]$ is a nonzero monic polynomial in y_m and

ϵ is a unit of R. We have $fR = gR \subseteq Q$ is a prime ideal and $R_2 \hookrightarrow R/gR$ is a finite integral extension. Thus $R_2/(Q \cap R_2) \hookrightarrow R/Q$ is an integral extension. It follows that $Q \cap R_2$ is maximal in R_2 with respect to being disjoint from U. By induction $\dim(R_2/(Q \cap R_2))$ is either n or 2. Since R/Q is integral over $R_2/(Q \cap R_2)$, $\dim(R/Q)$ is either n or 2. □

7.3. Remark. In the notation of Theorem 1.1, Theorem 7.2 proves the second part of the theorem, since $\dim R = n + m$. Thus if $n = 1$, $\operatorname{ht} Q = m$. If $n \geqslant 2$, the two cases are (i) $\operatorname{ht} Q = m$ and (ii) $\operatorname{ht} Q = n + m - 2$, as in (a) and (b) of Theorem 1.1.

Using the TGF terminology discussed in the introduction, we have the following corollary to Theorem 7.2.

7.4. Corollary. *With the notation of Theorem* 7.2*, assume* $P \in \operatorname{Spec} R$ *is such that* $R_1 \hookrightarrow R/P =: S$ *is a TGF extension. Then* $\dim S = \dim R_1 = n$ *or* $\dim S = 2$.

Acknowledgments

The authors are grateful for the hospitality and cooperation of Michigan State, Nebraska and Purdue, where several work sessions on this research were conducted. Wiegand thanks the National Security Agency for its support.

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