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# Generators, extremals and bases of max cones ☆

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**Abstract**

Max cones are max-algebraic analogs of convex cones. In the present paper we develop a theory of generating sets and extremals of max cones in $\mathbb{R}^n_+$. This theory is based on the observation that extremals are minimal elements of max cones under suitable scalings of vectors. We give new proofs of existing results suitably generalizing, restating and refining them. Of these, it is important that any set of generators may be partitioned into the set of extremals and the set of redundant elements. We include results on properties of open and closed cones, on properties of totally dependent sets and on computational bounds for the problem of finding the (essentially unique) basis of a finitely generated cone.


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## 1. Introduction

By *max algebra* we understand the analog of linear algebra obtained by considering $\mathbb{R}_+$ (the nonnegative reals) with max times operations:

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$$a \oplus b := \max(a, b),$$
$$a \otimes b := ab$$

extended to matrices and vectors. That is, if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\mathbb{R}_+$ and $\alpha \in \mathbb{R}_+$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all $i, j$, $C = A \otimes B$ if $c_{ij} = \bigoplus_k a_{ik} \otimes b_{kj} = \max_k(a_{ik}b_{kj})$ for all $i, j$ and $C = \alpha \otimes A$ if $c_{ij} = \alpha \otimes a_{ij}$ for all $i, j$. There are several essentially equivalent[1] definitions of max algebra. An early paper presenting the above version is [24], another early paper presenting an equivalent version is [7]. For more information on max algebra, its generalizations and applications the reader is referred e.g. to [2,5,8–10,16,28]. See also [20] for recent developments in the area and for further references.

We give a summary of the contents of this paper. In Section 2 we begin by defining (max) cones, extremals, generating sets, independent sets and totally dependent sets. The key observation is Proposition 11 which extends [18, Proposition 2.9]. We deduce in Theorem 14 that extremals are minimal elements of max cones under suitable scalings of vectors. This leads us to a reformulation and new proof of the basic result Theorem 16 which is also easily derived from [26, Proposition 2.5.3]: Every generating set of a max cone can be partitioned into the set of the extremals of a cone and a set of redundant elements. It follows that if a cone has a basis then (under a scaling condition) it consists of the extremals of the cone and hence the cone has a basis unique up to scalar multiples, see Theorem 18 and its corollaries for more detail. In Corollary 22 we then turn to the case when the set of extremals of the cone is empty, in which case every generating set is totally dependent. Discussion of totally dependent sets specifically may be new. Towards the end of this section we consider topological notions. In Corollary 23 we show that (under a natural restriction) every open cone has totally dependent generators and in Proposition 24 we prove an analogue of Minkowski's theorem for closed cones. This result extends a result due to [17] and it also appears as [14, Theorem 3.1] where a different proof is given.

In Section 3 we give two simple versions of an algorithm, based on [9, Theorem 16.2] for finding the (essentially unique) basis of a finitely generated max cone and a MATLAB program which implements one version. We also relate our problem to the classical problem of finding maxima of a set of vectors which is described in [19] and in [21, Section 3], and give the bounds for computational complexity.

We now relate the concepts and techniques of our proofs to those in other publications. Most of our concepts appear in [17,25,26], sometimes under different names. For instance, extremals are called irreducible elements in [25,26], and minimal elements are called efficient points in [17]. Our key Proposition 11 may also be derived from (possibly slightly extended) results found in some of our references. Examples are results in [24] Section 2 in terms of set coverings, see also [7], [9, Theorem 15.6] and [5] Section 2, or the fundamental results of [9] and [10, Section 3] concerning max linear systems. The latter are also found in [8,12] in terms of a projection operator. Further, such results on max linear systems as Propositions 11 and 31 can be extended to the case of functional Galois connections, as it is shown in [1]. The generalizations considered in [1] are useful in many areas including abstract convex analysis, the theory of Hamilton–Jacobi equation and the Monge–Kantorovitch mass transportation problem.

[1] That is, algebraically isomorphic.

Our topic is also related to (and partially stimulated by) the emerging field of tropical geometry which develops basic concepts of max algebra in a different form and with different terminology and applies these to finitely generated structures, see [12,18,3]. In particular, Proposition 11 can also be seen as a minor extension of [18, Proposition 2.9] which is important in the theory of tropical halfspaces. The emphasis of these papers is on geometry, while in this paper it is on algebraic and order theoretic results. Max cones are also studied in [13,14]. The main effort of these papers is to develop the theory of max-plus convex sets and their recession cones. This theory is not present in our paper. In turn, we deal with more general cones and we emphasize the link to set maxima and give a more detailed description of bases and generating sets.

In max algebra as in linear algebra a basis is normally defined as an *indexed set*, that is a *sequence* if the basis is finite or countable, see [26] for a definition in max algebra or [4, p. 10] in linear algebra. Since we wish to show the inclusion of the set of extremals (which do not have a natural order) in every generating set or basis for a cone we define the latter in term of *sets* in Section 2. We thereby exclude the possibility of a repetition of elements in generating sets. But we change our point of view in Section 3 on algorithms for finitely generated cones since we wish to consider the generators as columns of a matrix.

Max cones have much in common with convex cones, see [22] for a general reference. This has been exploited (and generalized) in many papers including those just quoted and e.g. [6,27]. To this end, the basic concepts of this paper and such results as Theorem 16, Proposition 24 and Proposition 25 have their direct analogs in terms of positive linear combinations and in convex analysis. We do not provide details, as convex geometry is also beyond our scope here.

## 2. Generating sets, bases and extremals

We begin with two standard definitions of max algebra.

**Definition 1.** A subset $K$ of $\mathbb{R}^n_+$ is a *max cone* in $\mathbb{R}^n_+$ if it is closed under $\oplus$ and $\otimes$ by nonnegative reals.

**Definition 2.** Let $S \subseteq \mathbb{R}^n_+$. Then $u$ is a *max combination* of $S$ if

$$u = \bigoplus_{x\in S} \lambda_x x, \quad \lambda_x \in \mathbb{R}_+, \tag{1}$$

where only finite number of $\lambda_x \neq 0$. The set of all max combinations will be denoted by span$(S)$. We put span$(\emptyset) = \{0\}$.

Evidently, span$(S)$ is a cone. If span$(S) = K$, we call $S$ a *set of generators* for $K$.

**Definition 3.** An element $u \in K$ is an *extremal* in $K$ if

$$u = v \oplus w, \quad v, w \in K \Longrightarrow u = v \text{ or } u = w. \tag{2}$$

If $u$ is an extremal in $K$ and $\lambda > 0$ then $\lambda u$ is also an extremal in $K$.

**Definition 4.** An element $x \in \mathbb{R}^n_+$ is *scaled* if $\|x\| = 1$.

For most of this section, $\|x\|$ may be any norm in $\mathbb{R}^n$ (they are all equivalent). However, in the end we specialize to the max norm, $\|x\| = \max x_i$, in order to exploit the property that it is max linear on $\mathbb{R}^n_+$. If $S \subseteq \mathbb{R}^n_+$ we may call $S$ scaled to indicate that it consists of scaled elements.

**Definition 5.** Let $S$ be a set of vectors in $\mathbb{R}^n_+$.

1. The set $S$ is *dependent* if, for some $x \in S$, $x$ is a max combination of $S \setminus \{x\}$. Otherwise, $S$ is *independent*.
2. The set $S$ is *totally dependent* if every $x \in S$ is a max combination of $S \setminus \{x\}$.

Thus the empty set of vectors is both independent and totally dependent. Since span(∅) $= \{0\}$, the set $\{0\}$ is totally dependent.

**Definition 6.** Let $K$ be a cone in $\mathbb{R}^n_+$. A set $S$ of vectors in $\mathbb{R}^n_+$ is a *basis* for $K$ if it is an independent set of generators for $K$.

The set of all unit vectors $\{e^p,\ p = 1, \dots, n\}$ defined, as usual, by

$$e^p_j = \begin{cases} 1, & j = p, \\ 0, & j \neq p, \end{cases} \tag{3}$$

is a basis of $\mathbb{R}^n_+$, which is called *standard*.

**Lemma 7.** *Let $S$ be a set of scaled generators for the cone $K$ in $\mathbb{R}^n_+$ and let $u$ be a scaled extremal in $K$. Then $u \in S$.*

**Proof.** Suppose $u$ is given by the max combination (1). Since the number of nonzero $\lambda_x$ is finite, we may use Definition 3 and induction to show that $u = \lambda_x x$ for some $x$. But $u$ and $x$ are both scaled, hence $u = x$ and $u \in S$. □

**Lemma 8.** *The set of scaled extremals of a cone is independent.*

**Proof.** If the set $E$ of scaled extremals is nonempty let $u$ be a scaled extremal in $K$ and apply Lemma 7 to the cone $K_1 := \mathrm{span}(E \backslash \{u\})$. This shows $u \notin K_1$ and the result is proved. □

Below we use subscripts for elements of vectors in $\mathbb{R}^n_+$ and superscripts to label vectors.

**Definition 9.** Let $v \in \mathbb{R}^n_+$. Then the *support* of $v$ is defined by

$$\mathrm{supp}(v) = \{j \in \{1, \dots, n\} : v_j > 0\}.$$

The cardinality of $\mathrm{supp}(u)$ will be written as $|\mathrm{supp}(u)|$.

In order to relate the natural partial order on $\mathbb{R}^n_+$ to results on extremals of cones we introduce a scaling of vectors in $\mathbb{R}^n_+$ for each $j \in \{1, \dots, n\}$ such that for each scaled vector $v_j = 1$.

**Definition 10**
1. Let $u \in \mathbb{R}^n_+$ and suppose $j \in \mathrm{supp}(u)$. Then we define $u(j) = u/u_j$.
2. Let $S \subseteq \mathbb{R}^n_+$. We define $S(j) = \{u(j) : u \in S \text{ and } j \in \mathrm{supp}(u)\}$ for all $j = 1, \dots, n$.
3. Let $S \subseteq \mathbb{R}^n_+$. An element $u \in S$ is called *minimal* in $S$, if $v \leqslant u$ and $v \in S$ implies that $v = u$.
4. Let $K$ be a cone in $\mathbb{R}^n_+$, let $u \in K$, and let $j \in \mathrm{supp}(u)$. We define

   $$D_j(u) = \{v \in K(j) : v \leqslant u(j)\}.$$

Our key observation is the following proposition. It can be viewed as a minor but needed extension of [18, Proposition 2.9], see also the remarks concerning it in our Introduction.

**Key Proposition 11.** *Let $S \subseteq \mathbb{R}^n_+$. Then the following are equivalent*:

1. $u \in \text{span}(S)$.
2. *For each* $j \in \text{supp}(u)$ *there is an* $x^j \in S$ *such that* $j \in \text{supp}(x^j)$ *and* $x^j(j) \in D_j(u)$.

**Proof.** 2. $\Longrightarrow$ 1. If 2. holds, then $u = \bigoplus_{j\in\text{supp}(u)} \lambda_j x^j$ where $\lambda_j = u_j/x^j_j$.

1. $\Longrightarrow$ 2. Conversely if 1. holds, then it follows immediately from (1) that for each $j \in \text{supp}(u)$ there is an $x^j \in S$ with $\lambda_j x^j \leqslant u$ and $(\lambda_j x^j)_j = u_j$. Clearly, $\lambda_j = u_j/x^j_j$ which yields 2. $\square$

The following immediate corollary to Proposition 11 is essentially found as [17, Theorem II.1] and as [12, Proposition 5]. It is analogous to Carathéodory's Theorem.

**Corollary 12.** *Let $S \subseteq \mathbb{R}^n_+$. Then $u \in \text{span}(S)$ if and only if there are k vectors $x^1, \ldots, x^k \in S$, where $k \leqslant |\text{supp}(u)|$, such that $u \in \text{span}\{x^1, \ldots, x^k\}$.*

**Corollary 13.** *Let K be a cone in $\mathbb{R}^n_+$ and let T be a set of generators for K. Let $U \subseteq T$ and let $S = T\backslash U$. Then S generates K if and only if each $u \in T$ satisfies condition* 2. *of Proposition* 11.

**Theorem 14.** *Let K be a cone in $\mathbb{R}^n_+$ generated by S and let $u \in S$, $u \neq 0$. Then the following are equivalent*:

1. *u is an extremal in K.*
2. *For some* $j \in \text{supp}(u)$, $u(j)$ *is minimal in* $K(j)$.
3. *For some* $j \in \text{supp}(u)$, $u(j)$ *is minimal in* $S(j)$.

**Proof.** 1. $\Longrightarrow$ 3. If $|\text{supp}(u)| = 1$ then $u(j)$ is minimal in $S(j)$. So suppose that $|\text{supp}(u)| > 1$ and that $u(j)$ is not minimal in $S(j)$ for any $j \in \text{supp}(u)$. Then for each $j \in \text{supp}(u)$ there exists $x^j \in S(j)$ such that $x^j \leqslant u(j)$, $x^j \neq u(j)$. Therefore $u = \bigoplus_{j\in\text{supp}(u)} u_j x^j$, and $u$ is proportional with none of $x^j$. Hence $u$ is not an extremal in $K$.

3. $\Longrightarrow$ 2. Let $v \in K$ and assume that $j \in \text{supp}(v)$ and $v(j) \leqslant u(j)$. We need to show that $v(j) = u(j)$. By Proposition 11, there is a $w \in S$ such that $w(j) \leqslant v(j)$. Thus $w(j) \leqslant v(j) \leqslant u(j)$ and by 3. it follows that $w(j) = v(j) = u(j)$.

2. $\Longrightarrow$ 1. Let $u(j)$ be minimal in $K(j)$ for some $j \in \text{supp}(u)$ and suppose that $u = v \oplus w$, $v, w \in K$. Then both $v \leqslant u$ and $w \leqslant u$ and either $v_j = u_j$ or $w_j = u_j$, say (without loss of generality) that $v_j = u_j$. Hence $v(j) \leqslant u(j)$ and it follows from 2. that $v(j) = u(j)$. Hence also $v = u$ which proves 1. $\square$

Note that in Theorem 14 we can of course have $S = K$. Also note that Corollary 13 may be combined with Theorem 14 to yield conditions for a set of generators to be redundant.

**Corollary 15.** *Let K be a cone in $\mathbb{R}^n_+$. If $D_j(u)$ has a minimal element for each $u \in K$ and each $j \in \text{supp}(u)$, then K is generated by its extremals.*

**Proof.** Suppose that $x^j$ is a minimal element of $D_j(u)$. Since, for $v \in K(j)$, $v \leqslant x^j$ implies that $v \in D_j(u)$, $x^j$ is also a minimal element of $K(j)$. We now obtain the Corollary by combining Proposition 11 and Theorem 14. □

Essentially, the following fundamental result was proved in [26, Proposition 2.5.3]. We suitably restate it: every set of generators $S$ for a cone $K$ can be partitioned as $E \cup F$, where $E$ is a set of extremals for $K$ and the remainder $F$ is redundant. Our proof is a combination of Proposition 11 and Theorem 14.

**Theorem 16.** *Let $S$ be a set of scaled generators for a cone $K$ in $\mathbb{R}^n_+$ and let $E$ be the set of scaled extremals in $K$. Then*

1. $E \subseteq S$.
2. *Let $F = S\backslash E$. Then for any $u \in F$, the set $S\backslash\{u\}$ is a set of generators for $K$.*

**Proof.** Assertion 1 repeats Lemma 7.

To prove Assertion 2, let $u \in F$. Since $u$ is not an extremal, by Theorem 14 for each $j \in \text{supp}(u)$ there is $z^j \in K$ such that $z^j(j) < u(j)$. Since $K = \text{span}(S)$, by Proposition 11 we also have $y^j \in S$ such that $y^j(j) \leqslant z^j(j) < u(j)$. Evidently $y^j \neq u$, and applying Proposition 11 again, we get that $u$ is a max combination of $\{y^j : j \in \text{supp}(u)\}$, where $y^j \in S$ are different from $u$. Thus in any max combination involving $u$, this vector can be replaced by a max combination of vectors in $S\backslash\{u\}$, and the theorem is proved. □

The following example shows that the set $F$ of Theorem 16 need not be totally dependent.

**Example 17.** Let $K$ be the cone in $R^2_+$ generated by $u^r = [1,\ 1/r]^{\mathrm{T}},\ r = 1, \ldots$ The elements of $K$ scaled with respect to the max norm are $[1,\ a]^{\mathrm{T}}$ with $0 < a \leqslant 1$. Thus $u^1$ is the unique scaled extremal in $K$. But the set $F = \{u^r : r = 2, \ldots\}$ is not totally dependent since $u^2$ is an extremal in span$(F)$ whose scaled elements are $[1,\ a]^{\mathrm{T}}$ with $0 < a \leqslant 1/2$.

The following is a refinement of Theorem 16, and also of [26, Theorem 5].

**Theorem 18.** *Let $E$ be the set of scaled extremals in a max cone $K$. Let $S \subseteq K$ consist of scaled elements. Then the following are equivalent*:

1. *The set $S$ is a minimal set of generators for $K$.*
2. *$S = E$ and $S$ generates $K$.*
3. *The set $S$ is a basis for $K$.*

**Proof.** 1. $\Longrightarrow$ 2. By Theorem 16 we have $S = E \cup F$ where every element of $F$ is redundant in $S$. But since $S$ is a minimal set of generators, we must have $F = \emptyset$. Hence $S = E$.

2. $\Longrightarrow$ 3. The set $E$ is independent and a generating set.

3. $\Longrightarrow$ 1. By independence of $S$ the span of a proper subset of $S$ is strictly contained in span$(S)$. □

Theorem 18 shows that if a cone has a (scaled) basis then it must be its set of (scaled) extremals, hence the basis is essentially unique. We note that a maximal independent set in a cone $K$ need not be a basis for $K$ as is shown by the following example.

**Example 19.** Let $K \subseteq \mathbb{R}^2_+$ consist of all $[x_1,\ x_2]^{\mathrm{T}}$ with $x_1 \geqslant x_2 > 0$. If $1 > a > b > 0$, then $\{[1,\ a]^{\mathrm{T}},\ [1,\ b]^{\mathrm{T}}\}$ is a maximal independent set in $K$ which does not generate $K$.

The following corollary is found e.g. as [12, Proposition 21], [18, Proposition 2.5] and also in [23, Proposition 1], where it is used to obtain uniqueness results for definite max-plus matrices. As a special case of this corollary, the standard basis of $\mathbb{R}^n_+$ is essentially unique.

**Corollary 20.** *If $K$ is a finitely generated cone, then its set of scaled extremals is the unique scaled basis for $K$.*

**Proof.** Since $K$ is finitely generated, there exists a minimal set of generators $S$. By Theorem 18 $S = E$ and $S$ is a basis. □

Note that in the tropical geometry [12,18] *vertices* of a polytope are defined to be the essentially unique generators determined in Corollary 20 and hence vertices correspond to our extremals (and to Wagneur's irreducible elements). Next we obtain some corollaries concerning totally dependent sets.

**Corollary 21.** *If $S$ is a nonempty scaled totally dependent set in $\mathbb{R}^n_+$ then $S$ is infinite.*

**Proof.** Suppose that $S$ is finite and let $K = \mathrm{span}(S)$. By Corollary 20 $K$ contains scaled extremals which, by Theorem 16, must be contained in $S$ given that $K = \mathrm{span}(S)$. But then $S$ is not totally dependent. This contradiction proves the result. □

**Corollary 22.** *Let $K$ be a cone in $\mathbb{R}^n_+$. The following are equivalent:*

1. *There is no extremal in $K$.*
2. *There exists a totally dependent set of generators for $K$.*
3. *Every set of generators for $K$ is totally dependent.*

**Proof.** Since there always exists a set of generators for $K$ (e.g. $K$ itself), each of the Conditions 2 and 3 is equivalent to Condition 1 by Theorem 16. □

We now consider $\mathbb{R}^n_+$ in the topology induced by the Euclidean topology of $\mathbb{R}^n$. That is, a set in $\mathbb{R}^n_+$ will be called *open* if and only if it is the intersection of an open subset of $\mathbb{R}^n$ with $\mathbb{R}^n_+$. A cone $K$ is called open if $K\backslash\{0\}$ is open, and it is called closed if it is closed as a subset of $\mathbb{R}^n_+$, or equivalently of $\mathbb{R}^n$.

**Corollary 23.** *If $K$ is an open cone in $\mathbb{R}^n_+$ that does not contain unit vectors, then every generating set for $K$ is totally dependent.*

**Proof.** It is enough to show that there is no extremal in $K$, for then the result follows by Theorem 16. Let $u \in K$. Since $u$ is not a unit vector, there are at least two indices $k, l \in \mathrm{supp}(u)$. Since $K$

is open, we have $w^p = u - \varepsilon e^p \in K,\ p = k, l$ for sufficiently small $\varepsilon$ and $u = w^k \oplus w^l$. None of $w^p,\ p = k, l$ is equal to $u$, hence $u$ is not an extremal, and the corollary follows. □

An example of an open cone in $\mathbb{R}^n_+$ is furnished by the cone $K$ of all positive vectors in $\mathbb{R}^n_+$. We note that, for this particular case, Corollary 23 was shown in [11]. Another example of an open cone consists of all vectors $[a,\ b]^{\mathrm{T}}$ in $\mathbb{R}^2_+$ with $a > b > 0$. We acknowledge the following proposition, which is analogous to Minkowski's Theorem, to [14, Theorem 3.1]. There the result is proved directly by a minimality argument; here we deduce it from a corollary to Theorem 14 which characterizes extremals of cones that may not be closed. It extends earlier results of [17].

**Proposition 24.** *Let $K$ be a closed cone in $\mathbb{R}^n_+$. Then $K$ is generated by its set of extremals, and any point in $K$ is a max combination of not more than $n$ extremals.*

**Proof.** Let $u \in K$ and let $j \in \mathrm{supp}(u)$. It is easily shown that $D_j(u)$ is compact since $K$ is closed. Hence $D_j(u)$ contains a minimal element $x^j$. The result now follows by Corollary 15 and Corollary 12. □

The max norm is max linear on $\mathbb{R}^n_+$:

$$\|\lambda u \oplus \mu v\| = \lambda\|u\| \oplus \mu\|v\|. \tag{4}$$

This is exploited in the following proposition.

**Proposition 25.** *If $S \subset \mathbb{R}^n_+$ is compact and $0 \notin S$, then the cone $K = \mathrm{span}(S)$ is closed.*

**Proof.** Consider a sequence $u^i \in K$ converging to $v$. Then, by Corollary 12 we have

$$u^i = \bigoplus_{s=1}^{n} \lambda_{is} w^{is},$$

where $w^{is} \in S$ and $\lambda_{is} \in \mathbb{R}_+$. By (4)

$$\|u^i\| = \bigoplus_{s=1}^{n} \lambda_{is}\|w^{is}\|. \tag{5}$$

Since the sequence $u^i$ converges (to $v$), the norms $\|u^i\|$ are bounded from above by some $M_1 > 0$. On the other hand, we have $\|w^{is}\| \geqslant M_2$ for some $M_2 > 0$, since $S$ is closed and does not contain 0. Then by (5) $\lambda_{is}\|w^{is}\| \leqslant M_1$ for all $i$ and $s$, and $\lambda_{is} \leqslant M_1 M_2^{-1}$ for all $i$ and $s$. Thus $\lambda_{is}$ are bounded from above. But $\|w^{is}\|$ are also bounded from above, since $S$ is compact. This implies that there is a subsequence $u^{j(i)}$ such that for all $s = 1, \ldots, n$ the sequences $w^{j(i)s}$ and $\lambda_{j(i)s}$ converge. Denote their limits by $\bar{w}^s$ and $\bar{\lambda}_s$, respectively, then $\bar{w}^s \in S$ and $\bar{\lambda}_s \leqslant M_1 M_2^{-1}$. By continuity of $\oplus$ and $\otimes$ we obtain that

$$v = \bigoplus_{s=1}^{n} \bar{\lambda}_s \bar{w}^s. \tag{6}$$

Thus $v \in K$. □

**Corollary 26.** *If the set of scaled extremals of a max cone $K$ is closed and generates $K$, then $K$ is closed.*

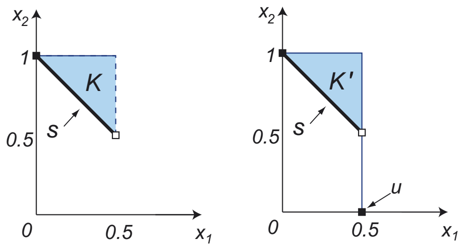


Fig. 1. Max cones of Example 28.

**Corollary 27.** *Any finitely generated max cone $K$ is closed.*

We now give a counterexample to the converses of Corollary 15 and Proposition 24 (part 1), and to the converse of Corollary 26 (part 2).

**Example 28.** 1. In $\mathbb{R}^3_+$ let $S$ consist of all vectors $[x_1,\ x_2,\ 1]^{\mathrm{T}},\ 0 \leqslant x_1 < 1/2$ such that $x_1 + x_2 = 1$ and let $K = \mathrm{span}(S)$. Then the section of $K$ given by $x_3 = 1$ consists of all vectors $[x_1,\ x_2,\ 1]^{\mathrm{T}},\ 0 \leqslant x_1 < 1/2,\ 0 \leqslant x_2 \leqslant 1$ such that $x_1 + x_2 \geqslant 1$. Note that $S$ is the set of extremals of $K$ scaled with respect to the max norm, but $K = \mathrm{span}(S)$ is not closed and for any $u \in K$ there are no minimal elements in $D_1(u)$ and $D_2(u)$.

2. Now let $S' = S \cup \{u\}$, where $u = [1/2,\ 0,\ 1]^{\mathrm{T}}$ and let $K' = \mathrm{span}(S')$. Then the section of $K'$ given by $x_3 = 1$ consists of $K$ together with the line segment whose end points are $u$ and $[1/2,\ 1,\ 1]^{\mathrm{T}}$. Thus $K'$ is closed. The set of scaled extremals of $K'$ is $S'$ which is not closed.

The cross sections of $K$ and $K'$ by $x_3 = 1$ are shown in Fig. 1, together with the generating sets $S$ and $S' = S \cup \{u\}$.

## 3. Algorithmic considerations

As explained in the introduction we redefine our basic concepts for this section which is concerned with finitely generated cones. We also restate a suitable adaptation of Corollary 20.

**Definition 29.** Let $V \in \mathbb{R}^{nk}_+$ and let $V_{\hat{i}}$ be the matrix obtained from $V$ by deleting column $i,\ i = 1, \ldots, k$. Then the cone $K$ *generated* by the columns $v^1, \ldots, v^k$ of $V$ consists of all vectors of form $V \otimes x,\ x \in \mathbb{R}^k_+$. Further, the columns of $V$ form a *basis* for $K$ if, for $i,\ i = 1, \ldots, k$, there is no $x \in \mathbb{R}^{k-1}_+$ such that $V_{\hat{i}} \otimes x = v^i$.

**Proposition 30.** *Let $V \in \mathbb{R}^{nk}_+$. Then there exists a submatrix $U \in \mathbb{R}^{np}_+$, $0 \leqslant p \leqslant k$ whose columns form a basis for the cone generated by the columns of $V$ (and every other basis is of form $UPD$, where $P$ is a permutation matrix and $D$ is a diagonal matrix with nonzero diagonal elements).*

We shall apply the following proposition. Note that all statements in this proposition have been proved in a more general setting in [8]. See also [5], [10, Chapter III] and [24].

**Proposition 31.** *Let $U \in \mathbb{R}^{nk}_+$ with all columns nonzero and let $v \in \mathbb{R}^n_+$. Let $x \in \mathbb{R}^k_+$ be defined by*

$$x_i = \min\{v_j/u^i_j : u^i_j \neq 0, j = 1, \ldots, n\} \tag{7}$$

*for* $i = 1, \ldots, k$. *Then*

$$U \otimes x \leqslant v, \tag{8}$$

$$x = \max\{z \in \mathbb{R}^k_+ : U \otimes z \leqslant v\}, \tag{9}$$

$$U \otimes x = \max\{U \otimes z : z \in \mathbb{R}^k_+, U \otimes z \leqslant v\}. \tag{10}$$

*Further, there exists* $z \in \mathbb{R}^k_+$ *such that* $U \otimes z = v$ *if and only if* $U \otimes x = v$.

**Proof.** Assertion (8) follows from the observation that $U \otimes z \leqslant v$ if and only if $z_i \leqslant v_j/u^i_j$ if $j \in \text{supp}(u^i)$, $i = 1, \ldots, k$. Note that $x \in \mathbb{R}^k_+$ since no column of $U$ is zero. Since $\otimes$ is isotone (that is, $x \leqslant y$ implies $A \otimes x \leqslant A \otimes y$), assertions (9) and (10) follow immediately. For the final statement assume that $U \otimes z = v$ for some $z$. By (8) and (10) we have $v = U \otimes z \leqslant U \otimes x \leqslant v$, and the statement follows. The converse is trivial. □

**Algorithm 32**

*Input*: $V \in \mathbb{R}^{nk}_+$.

*Output*: An $n \times p$ submatrix $U$ of $V$ whose columns form the essentially unique basis for the cone generated by the columns of $V$.

*Step 1*. Initialize $U = V$.

*Step 2*. For each $j = 1, \ldots, k$ if $u^j \neq 0$ set $v = u^j$, and for each $i \neq j$ compute $x_i$ by (7), if $u^i \neq 0$, and set $x_i = 0$ otherwise. If $U_{\hat{j}}x = v$, set $u^j = 0$.

*Step 3*. Delete the zero columns of $U$. The remaining columns of $U$ are the basis we seek.

**Remark 33.** The restriction in Proposition 31 that each column $U \in \mathbb{R}^{nk}_+$ must have a positive element was imposed to avoid definitions for $a/0$, $a > 0$, or $0/0$. The restriction is inessential in the sense that for general $U \in \mathbb{R}^{nk}_+$ we may define $x_i$ by (7) whenever $u^i \neq 0$ and choose $x_i$ arbitrarily in $\mathbb{R}^k_+$ whenever $u^i = 0$. Then all assertions of the Proposition still hold, with exception of (9). It is possible to extend $\mathbb{R}^n_+$ by adding a maximal element $\infty$ so that (9) still holds.

We omit details and present the MATLAB program maxbas that implements Algorithm 32 but employs such an extension. We also give an example with some elements equal to 0. Note that in [9, Theorem 16.2] a related algorithm called *A-test* has been presented. It enables us to identify columns that are dependent on other columns of an $n \times k$ matrix in $O(nk^2)$ time. However, there is no discussion of bases in connection with this method in [9].

```
%the unique max times basis for the max col space of A
%function [B,f] = {\ul maxbas}(A),
%B = the unique max times basis for the max col space of A
%f = indices of columns of B in A
%calls {\ul maxpr},  max multiplication  of matrices
 function  [B,f] = maxbas(A)
 [m,n] = size(A); B = A; t = max(max(A));
 for j  = 1:n
```

```
     v = compl(j,n);
     c = B(:,j); BB = B(:,v); warning('off'),
     e = ones(1,n-1); C = c*e; x = min(C./BB)';
     z= maxpr(BB,x);
     if abs(c-z) < t*eps, B(:,j) = 0; end,
end u = max(B); f = find(u >t*eps); B = B(:,f);
A =
  Columns 1 through 5
     1     9    10     5     9
     2    10    10     0    10
     3    15    14     7     0
     4    20    16     8    12
>> [B,f] = maxbas(A)
B =
     1     5     9
     2     0    10
     3     7     0
     4     8    12
f =
     1     4     5
```

We note that a second form of the algorithm may be based on *set covering* condition (11) below, which appears in [9, Theorem 15.6]. It can also be found in [24] Section 2 and in [5] Section 2 but only in the case when all vectors are positive. See also [1, Theorem 3.5] for an interesting functional generalization of this condition (and more). With $v$ and $U$ as in Step 2, denote by $N_i$ the set $\{j: v(j) \geqslant u^i(j)\}$. By Proposition 11, $v \in \text{span}(u^1, \ldots, u^m)$ if and only if

$$\bigcup_{i=1}^{m} N_i = \text{supp}(v). \tag{11}$$

With $x$ given by (7), we note that

$$N_i = \begin{cases} \{j \in \text{supp}(u^i) : v_j/u^i_j = x_i\} & \text{if } x_i \neq 0, \\ \emptyset, & \text{if } x_i = 0. \end{cases} \tag{12}$$

Thus Step 2 in Algorithm 32 may be replaced by

*Step 2′*: For each $j = 1, \ldots, k$ such that $u^j \neq 0$: set $v = u^j$ and for each $i \neq j$ compute $N_i = \{j: v(j) \geqslant u^i(j)\}$ according to (12), if $u^i \neq 0$, and set $N_i = \emptyset$ otherwise. If $\bigcup_{i \neq j} N_i = \text{supp}(v)$, set $u^j = 0$.

The version with Step 2′ is also well-known. It is implemented in the max-plus toolbox of Scilab, a freely distributed software. See [15, Sect.III-B] for the documentation.

Our algorithms are of complexity $O(nk^2)$.

If $S$ is the set of columns of the matrix $U$, then it follows from Theorem 14 that a basis for the cone generated by $S$ consists of the union of the $n$ sets $M(j)$, $j = 1, \ldots, n$, where $M(j)$ consist of the vectors minimal in $S(j)$. The problem of finding all maxima (or minima) of $k$ vectors in $\mathbb{R}^n$ is considered in [19], and also in [21, Section 4.1.3], where it is dubbed the problem of Erehwon Kings. The computational complexity of methods developed in [19,21] is bounded from above by $O(n^2k(\log_2 k)^{n-2}) + O(k \log_2 k)$, $n \geqslant 2$, see [21, Theorem 4.9] and [19, Theorem

5.2].[2] To solve our problem we can apply these methods to each $S(j)$, $j = 1, \dots, n$ separately. Taking into account that for each $j$ we need $O(nk)$ operations to find the coordinates of essentially $(n-1)$-dimensional vectors in $S(j)$, this yields an alternative method with complexity not smaller than $O(n^2k)$ and not greater than $O(n^3k(\log_2 k)^{n-3}) + O(k\log_2 k)$, $n \geqslant 3$. This method may be preferred if $\log_2 k$ is substantially larger than $n$.

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