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THE EXPRESSIVE POWER OF MALITZ QUANTIFIERS FOR LINEAR ORDERINGS

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This paper deals with the classes of linear orderings and well-orderings, respectively, which are studied using languages with additional Malitz quantifiers. For these model classes the elimination of Malitz quantifiers is shown, and the decidability of their theories is proved. The main results concerning quantifier elimination are the following. Let L^m_Δ be obtained by adding, to the elementary language L with identity, the Malitz quantifiers Q^m_α for each $\alpha \in \Delta$. Then for every formula $\varphi(\bar{x})$ of L^m_Δ there is some formula $\psi(\bar{x})$ of L^2_Δ so that $\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$ holds in all linear orderings, i.e., the quantifiers Q^m_α are eliminable with the help of the quantifiers Q^2_α. For the class of well-orderings we can do more: $\psi(\bar{x})$ can actually be chosen from L^1_Δ, i.e., the Malitz quantifiers are eliminable with the help of the unary cardinality quantifiers (the result depends on some weak hypothesis on Δ which will be introduced below). As a consequence we get that well-orderings having the same theory in L^1_Δ cannot be distinguished using Malitz quantifiers. This was already remarked by Slomson, cf. [14, 15].

Furthermore, for any finite set Δ of ordinals, the theory $\mathrm{Th}^2_\Delta(\mathrm{WO})$ of well-orderings in L^2_Δ is shown to be decidable. If in addition ω_α is regular for each $\alpha \in \Delta$, then (assuming the GCH) the theory $\mathrm{Th}^2_\Delta(\mathrm{LO})$ of all linear orderings is decidable, too. This generalizes corresponding results for unary cardinality quantifiers. Let us give some short remarks concerning the decision problems for the classes LO of linear orderings and WO of well-orderings. The elementary theories of WO and LO have been proved to be decidable by Mostowski and Tarski (see [12]) and by Ehrenfeucht (see [3]), respectively. In [9] Läuchli and Leonhard created a powerful technique for proving decidability. Their method turned out to be very useful also for treating the decision problem for non-elementary theories. Lipner was the first who proved that every well-ordering has a decidable theory in $L^1_{\{\alpha\}}$ if ω_α is regular, cf. [10]. Afterwards the decidability of the following theories has been established: (1) $\mathrm{Th}^1_{\{1\}}(\mathrm{WO})$ (cf. [14, 15]), (2) $\mathrm{Th}^1_{\{0,1\}}(\mathrm{WO})$ (cf. [7]), (3) $\mathrm{Th}^1_{\{1\}}(\mathrm{LO})$ (cf. [8] and [17]), (4) $\mathrm{Th}^1_{\{\alpha\}}(\mathrm{LO})$ for each ordinal α (assuming the GCH) (cf. [8]), (5) $\mathrm{Th}^1_\Delta(\mathrm{LO})$ for each finite set Δ of ordinals such that ω_α is regular for each $\alpha \in \Delta$ (cf. [18]).

Furthermore, we want to remark that LO possesses an undecidable theory in the language with the additional quantifier 'for almost all', cf. [16].

We have not mentioned the important results about the monadic theory of order due to Büchi, Rabin, Shelah, and others. Monadic theories of orders are studied extensively by several people. We do not review that here. The reader is referred to the papers [13], [5], and [6]. In Section 1 we give an interpretation for Q_α^m which slightly differs from the original one given by Magidor and Malitz. The basic facts of Q_α^m can be found in [11]. Ehrenfeucht introduced game-theoretical methods into the model theory of elementary logic. Badger extended these to logics containing Malitz quantifiers. Instead of games we will alternatively use equivalence relations $\overset{n}{\sim}$-mod G_Δ^m between linear orderings. Their main properties are derived in the first section. Section 2 provides us with some combinatorial facts of linear orderings. Theorems 2.3 and 2.6 are derived by modifying Shelah's proof for his Ramsey theorem for additive colourings (see [13]). Applying the combinatorial results the elimination of the m-ary Malitz quantifiers by the binary ones (with respect to the class of linear orderings) is shown in Section 3. The elimination procedure with respect to the class of well-orderings is worked out in Section 4. The last two sections are devoted to the decision problems of WO and LO, respectively. We generally work in ZFC, except in the last section where we additionally adopt the GCH. Part of the results of this paper are contained in the author's Dissertation B. We want to express our gratitude to the referee for his valuable comments.

1. Introduction

We assume familiarity with the basic concepts for linear orderings and summarize here some notions particularly important for what follows. LO is the class of all linear orderings. Its members are denoted by $A, B, C, \ldots$ instead of $\langle A, <\rangle$, $\langle B, <\rangle$, $\langle C, <\rangle, \ldots$, respectively. It can be axiomatized in an elementary language L which possesses one binary non-logical symbol $<$ only. The most important subclass is the class WO of well-orderings. Since every well-ordering is isomorphic to an ordinal, we denote them by small Greek letters $\alpha, \beta, \gamma, \ldots$ Cardinal numbers are denoted by $\aleph_\alpha$ or ω_α. The cardinality of a set X is denoted by $|X|$ or card X. Any subset of a linear ordering can be linearly ordered by restricting the order to it. We generally assume that subsets are ordered in this way.

$(a, b)_A$ is the open interval in A with the endpoints a and b. If no confusion will arise, we will omit the subscript. $A^{<a}$ is the initial segment determined by a, i.e., $A^{<a} = \{x \in A : x < a\}$. The segments $A^{>a}$, $A^{\leq a}$, and $A^{\geq a}$ are similarly defined. Intervals and segments have a particular property, they are convex.

$X \subseteq A$ is *convex (in A)* if $(a, b)_A \subseteq X$ for every $a, b \in X$.

$X \subseteq A$ is *κ-dense in A* if $|(a, b)_A \cap X| \geq \kappa$ for every $a < b$. It is easy to see that X is dense in A (in the usual sense) iff X is ω_0-dense in A.

A mapping $f:A \to B$ is *monotone* if $f(x) \leqslant f(y)$ for every $x \leqslant y$; $x, y \in A$. Obviously, the monotone maps are exactly the homomorphisms with respect to $\leqslant$. Each monotone map $f:A \to B$, therefore, defines an equivalence relation $\underset{f}{\sim}$ on $A: x \underset{f}{\sim} y$ iff $f(x) = f(y)$.

The set A/f of equivalence classes becomes a linear ordering by setting $x/f \leqslant y/f$ if $x \leqslant y$.

The sum $A + B$ and the product $A \cdot B$ are special operations to form new linear orderings. We assume the reader familiar with these constructions.

$a_1 < \cdots < a_k \in A$ abbreviates the fact that $a_1, \ldots, a_k$ are all elements of A and are ordered in the cited way. If we regard them as distinguished elements, then we indicate this by $\langle A, a_1, \ldots, a_k \rangle$. $\langle A, a_1, \ldots, a_k \rangle \models \varphi(c_1, \ldots, c_k)$ is abbreviated by $A \models \varphi(a_1, \ldots, a_k)$, where $c_1, \ldots, c_k$ are pairwise distinct new constants and $\varphi(c_1, \ldots, c_k)$ is a sentence of $L(\{c_1, \ldots, c_k\})$.

Any language L' can be extended by adding for $m \geqslant 1$ generalized quantifiers Q^m to it with the additional formation rule: if φ is a formula and $v_1, \ldots, v_m$ are pairwise distinct variables, then $Q^m v_1 \cdots v_m\, \varphi$ is again a formula.

The calculus obtained is denoted by L'^m. If for every natural number m a quantifier Q^m is added, then we obtain the language $L'^{<\omega}$. The quantifier can be interpreted in various ways. In the α-interpretation given below it is called Malitz quantifier, although it differs from the original one introduced by Magidor and Malitz [11].

However, as we will see, for ordered structures both interpretations are closely related to each other. In the 0-interpretation it is also called Ramsey quantifier.

α-interpretation of Q^m. $A \models Q^m v_1 \cdots v_m\, \varphi(v_1, \ldots, v_m)$ iff there exists some $X \subseteq A$ such that $|X| = \omega_\alpha$ and $A \models \varphi(a_1, \ldots, a_m)$ for all $a_1 < \cdots < a_m \in X$.

A subset X satisfying the right side above is called φ-orderhomogeneous. If Q^m is interpreted as above, then we write Q^m_α for it.

Let P^m_α be the Malitz quantifier with its original α-interpretation given in [11].

Proposition 1.1. *The quantifier Q^m_α and the Malitz quantifier P^m_α are definable by each other.*

Proof. The reader can easily verify the following equivalences:

$$A \models Q^m_\alpha v_1 \cdots v_m\, \varphi \quad \text{iff} \quad A \models P^m_\alpha v_1 \cdots v_m\, (v_1 < \cdots < v_m \to \varphi)$$

and

$$A \models P^m_\alpha v_1 \cdots v_m\, \varphi \quad \text{iff} \quad A \models Q^m_\alpha v_1 \cdots v_m \bigwedge_{(i_1, \ldots, i_m) \in S_m} \varphi(v_{i_1}, \ldots, v_{i_m}),$$

where S_m is the set of all permutations of $(1, \ldots, m)$. □

This gives us the right for calling Q^m_α also Malitz quantifier. With the help of the ordinal subscripts it is possible to regard several α-interpretations at the same time. Suppose Δ is an arbitrary set of ordinals; then L^m_Δ is the language arising from L by adding the quantifiers Q^m_α for all $\alpha \in \Delta$ to it. $L^{<\omega}_\Delta$ is similarly defined. If not otherwise stated, then in the following Δ is always finite. Define $\mathrm{cf}\,\Delta = \{\beta$: there is some $\alpha \in \Delta$ with $\mathrm{cf}\ \omega_\alpha = \omega_\beta\}$. Then ω_α is regular for all $\alpha \in \Delta$ iff $\mathrm{cf}\,\Delta = \Delta$. Although L^m_Δ and L^n_Δ are completely different as sets in case that $m < n$, we can identify L^m_Δ with some part of L^n_Δ, because for all models A

$$A \models Q^m_\alpha v_1 \cdots v_m\, \varphi(v_1, \ldots, v_m) \quad \text{iff} \quad A \models Q^n_\alpha v_1 \cdots v_n\, \varphi(v_1, \ldots, v_m).$$

Thus we may regard L^m_Δ as a sublanguage of L^n_Δ. For the formulae φ of L^m_Δ we can define the quantifier rank $q(\varphi)$ inductively:

(i) $q(\varphi) = 0$ if φ is quantifier free,
(ii) $q(\neg\varphi) = q(\varphi)$,
(iii) $q(\varphi \wedge \psi) = q(\varphi \vee \psi) = \max\{q(\varphi), q(\psi)\}$, and
(iv) $q(\varphi) = q(\psi) + 1$ if φ is $\exists v\, \psi$ or $Q^m v_1 \cdots v_m\, \psi$.

$L^m_\Delta(c_1, \ldots, c_k)$ denotes the language obtained from L^m_Δ by adding the constants $c_1, \ldots, c_k$ to it. The quantifier rank enables us to introduce the sublanguage $L^{m,n}_\Delta(c_1, \ldots, c_k)$, which consists of those sentences φ of $L^m_\Delta(c_1, \ldots, c_k)$ for which $q(\varphi) \leq n$. Furthermore, it gives rise to some important equivalence relations between models with distinguished constants.

Definition 1.1. $\langle A, a_1, \ldots, a_k\rangle$ and $\langle B, b_1, \ldots, b_k\rangle$ are *n-equivalent* (with respect to L^m_Δ), in terms

$$\langle A, a_1, \ldots, a_k\rangle \overset{n}{\equiv} \langle B, b_1, \ldots, b_k\rangle \mod L^m_\Delta,$$

if they are equivalent in the sublanguage $L^{m,n}_\Delta(c_1, \ldots, c_k)$.

Although $L^{m,n}_\Delta(c_1, \ldots, c_k)$ is infinite, it has a finite subset $M^{m,n}_\Delta$, which can be effectively determined, so that every sentence of $L^{m,n}_\Delta(c_1, \ldots, c_k)$ is logically equivalent to a sentence in $M^{m,n}_\Delta$. This is easily proved by induction on n. The theory of $\langle A, a_1, \ldots, a_k\rangle$ in $L^{m,n}_\Delta(c_1, \ldots, c_k)$, in terms $\mathrm{Th}^{m,n}_\Delta(\langle A, a_1, \ldots, a_k\rangle)$, is the set of all sentences of $L^{m,n}_\Delta(c_1, \ldots, c_k)$ true in $\langle A, a_1, \ldots, a_k\rangle$. The conjunction of $M^{m,n}_\Delta \cap \mathrm{Th}^{m,n}_\Delta(\langle A, a_1, \ldots, a_k\rangle)$ can be formed, because this set is finite. The conjunction is denoted by $\Phi^{\Delta,m,n}_{A,a_1\cdots a_k}(c_1, \ldots, c_k)$, or simply $\Phi^n_{A,\bar{a}}(c_1, \ldots, c_k)$ in case that Δ and m are fixed.

The following proposition is proved in the same way as the corresponding result for first-order logic.

Proposition 1.2. *Let $\langle A, a_1, \ldots, a_k\rangle$ be given, then for all $\langle B, b_1, \ldots, b_k\rangle$*

$$\langle A, a_1, \ldots, a_k\rangle \overset{n}{\equiv} \langle B, b_1, \ldots, b_k\rangle \mod L^m_\Delta(c_1, \ldots, c_k)$$

iff $B \models \Phi^{\Delta,m,n}_{A,a_1\cdots a_k}(b_1, \ldots, b_k)$.

We will next introduce the equivalence relations $\overset{n}{\sim}$-mod G^m_Δ by induction on n. If m and Δ are fixed these relations are shortly denoted by $\overset{n}{\sim}$. To reduce notation we shortly express simultaneous equivalences $A_1 \overset{n}{\sim} B_1, \ldots, A_k \overset{n}{\sim} B_k$ by $\langle A_1, \ldots, A_k \rangle \overset{n}{\sim} \langle B_1, \ldots, B_k \rangle$. Furthermore, for any sequence $a_1 < \cdots < a_k \in A$ of elements we set $\langle a_1, \ldots, a_k, A \rangle := \langle A_0, \ldots, A_k \rangle$, where $A_0 = A^{<a_1}$, $A_{i-1} = (a_{i-1}, a_i)_A$ for $1 < i \leq k$, and $A_k = A^{>a_k}$. Assume that $\overset{n}{\sim}$ is already defined, then we can derive a partial ordering for pairs $\langle X, A \rangle$ with $X \subseteq A$: $\langle X, A \rangle \lesssim^n \langle Y, B \rangle$ mod G^m_Δ iff for all $x_1 < \cdots < x_m \in X$ there are $y_1 < \cdots < y_m \in Y$ so that

$$\langle x_1, \ldots, x_m, A \rangle \overset{n}{\sim} \langle y_1, \ldots, y_m, B \rangle \quad \text{mod } G^m_\Delta.$$

Now we are ready to define $\overset{n}{\sim}$-mod G^m_Δ. For any linear orderings A and B (A or B may be empty) we set $A \overset{0}{\sim} B$ mod G^m_Δ. This immediately implies that for arbitrary $a_1 < \cdots < a_k \in A$ and $b_1 < \cdots < b_k \in B$ the relation $\langle a_1, \ldots, a_k, A \rangle \overset{0}{\sim} \langle b_1, \ldots, b_k, B \rangle$ mod G^m_Δ holds.

Suppose $A \overset{n}{\sim} B$ mod G^m_Δ is defined, then the following definition of a partial ordering between models make sense.

$A \lesssim^{n+1} B$ mod G^m_Δ iff

(i) for every $a \in A$ there is some $b \in B$ so that $\langle a, A \rangle \overset{n}{\sim} \langle b, B \rangle$ mod G^m_Δ, and

(ii) for every $X \subseteq A$ with $|X| = \omega_\alpha$, $\alpha \in \Delta$, there is some $Y \subseteq B$ of power ω_α so that $\langle Y, B \rangle \lesssim^n \langle X, A \rangle$ mod G^m_Δ.

We finally set $A \overset{n+1}{\sim} B$ mod G^m_Δ iff $A \lesssim^{n+1} B$ and $B \lesssim^{n+1} A$ mod G^m_Δ. It is easy to see that $\lesssim^{n+1}$ is reflexive and transitive. Hence, $\overset{n+1}{\sim}$ is really an equivalence relation. There is an alternative way for introducing $\overset{n}{\sim}$-mod G^m_Δ by game-theoretical means.

We refer to [1] for the connection between games and Malitz quantifiers. In the case of linear orderings it is easier to work with the equivalences $\overset{n}{\sim}$-mod G^m_Δ than with the original games for the Malitz quantifiers. $\overset{n}{\sim}$-mod G^m_Δ has the important property that it is compatible with the sum operation.

Lemma (sum property). *Let $A_1, \ldots, A_k, B_1, \ldots, B_k$ be linear orderings such that $\langle A_1, \ldots, A_k \rangle \overset{n}{\sim} \langle B_1, \ldots, B_k \rangle$ mod G^m_Δ. Then also*

$$A_1 + \cdots + A_k \overset{n}{\sim} B_1 + \cdots + B_k \quad \text{mod } G^m_\Delta.$$

We omit the proof, since it is straightforward. The sum property is of fundamental importance for the investigation of the relations $\overset{n}{\sim}$-mod G^m_Δ. It is used in the following often without mention. The next proposition shows that $\overset{n}{\sim}$-mod G^m_Δ and $\overset{n}{\equiv}$-mod L^m_Δ are closely related to each other.

Proposition 1.3. *For any $k \geq 1$ and any sequences $a_1 < \cdots < a_k \in A$ and $b_1 < \cdots < b_k \in B$, respectively, it holds*

$$\langle a_1, \ldots, a_k, A \rangle \overset{n}{\sim} \langle b_1, \ldots, b_k, B \rangle \quad \text{mod } G^m_\Delta$$

iff

$$\langle A, a_1, \ldots, a_k \rangle \overset{n}{\equiv} \langle B, b_1, \ldots, b_k \rangle \quad \text{mod } L^m_\Delta.$$

The Proposition above enables us to change $\overset{n}{\sim}$-mod G^m_Δ equivalence to $\overset{n}{\equiv}$-mod L^m_Δ equivalence and vice versa. This will be a powerful tool, since the verification of $\overset{n}{\sim}$-mod G^m_Δ equivalence is much easier. The proof of the proposition is not difficult and is left to the reader.

Often the main point consists in proving that for every $X \subseteq A$, $|X| = \omega_\alpha$, there exists some $Y \subseteq B$, $|Y| = \omega_\alpha$, with $\langle Y, B\rangle \lesssim^n \langle X, A\rangle$ mod G^m_Δ. As already remarked, for $Z \subseteq X$ the relation $\langle Y, B\rangle \lesssim^n \langle Z, A\rangle$ mod G^m_Δ implies $\langle Y, B\rangle \lesssim^n \langle X, A\rangle$ mod G^m_Δ, i.e., to prove the relation for X it is enough to prove it for some convenient subset of X. The next section is devoted to the problem of obtaining suitable subsets.

2. Combinatorics

A symmetric function $f: A^2 \to I$ into the finite set I is called a *colouring* (we disregard the values on the diagonal). The colouring f is *additive* if for every $x_i < y_i < z_i \in A$, $i = 1, 2$, $f(x_1, y_1) = f(x_2, y_2)$ and $f(y_1, z_1) = f(y_2, z_2)$ implies $f(x_1, z_1) = f(x_2, z_2)$. Additivity induces a partial operation $+$ on I. For this reason we also write $f(x, z) = f(x, y) + f(y, z)$ if $x < y < z$. $X \subseteq A$ is called *homogeneous* (with respect to f) if f is constant on X^2. Shelah proved several combinatorial theorems about the existence of homogeneous sets for additive colourings. For a proof of the following important fact we refer to [13].

Ramsey Theorem for additive colourings (Shelah). *Let κ be a regular cardinal and f be an additive colouring on κ. Then there exists a homogeneous subset $X \subseteq \kappa$ of power κ.*

There is also a generalization of the above property to dense linear orderings due to Shelah, however it is not applicable for our purposes. In the following we will derive an appropriate generalization.

Lemma 2.1. *Suppose A is κ-dense in itself, and $P_1, \ldots, P_k$ are subsets of A with $|A \setminus \bigcup_{i \leq k} P_i| < \kappa$. Then there are elements $a < b \in A$ and some P_i, so that P_i is κ-dense in (a, b).*

Proof. Assume no one of the P_i has the stated property. Then we can produce a sequence $a_1 < \cdots < a_k < b_k < \cdots < b_1$ with $|(a_i, b_i) \cap P_j| < \kappa$ for all $j \leq i$ and $i \leq k$, respectively. Suppose $a_1 < \cdots < a_m < b_m < \cdots < b_1$ are already defined. In case $m = k$ we are done. Otherwise, we can extend the sequence as follows. Since P_{m+1} is not κ-dense in (a_m, b_m), there are $a_{m+1} < b_{m+1}$ in (a_m, b_m) with $|(a_{m+1}, b_{m+1}) \cap P_{m+1}| < \kappa$. Hence $|(a_{m+1}, b_{m+1}) \cap P_j| < \kappa$ for all $j \leq m+1$. After finitely many steps the desired sequence is constructed. Since $|(a_k, b_k) \cap P_j| < \kappa$ for all $j \leq k$ and $|A \setminus \bigcup_{i \leq k} P_i| < \kappa$, it follows $|(a_k, b_k)| < \kappa$, a contradiction to the κ-density of A. Thus the lemma is proved. $\square$

For every element $a \in A$ we set

$$I(a) = \{t \in I: \text{for every } x > a \text{ there exists some } y \in (a, x) \text{ with } f(a, y) = t\}.$$

Roughly speaking, $I(a)$ is the set of those colours, which converge from above to a. Obviously, $I(a) \neq \emptyset$ for all $a \in A$ if A is a dense linear ordering without greatest element. b is said to be *minimal* for a if $a < b$ and for every $x \in (a, b)$ the colour $f(a, x)$ belongs to $I(a)$. If b is minimal for a and $c \in (a, b)$, then c is also minimal. Suppose A has no greatest element, then for every a there is some b minimal for it. For each subset $D \subseteq I$ we set $D(A) = \{a \in A : I(a) = D\}$. Clearly, $D_1(A) \cap D_2(A) = \emptyset$ if $D_1 \neq D_2$, so Pow(I), the powerset of I, induces a partition of A. These notions, which we have just introduced, are used in proving the next lemma.

Lemma 2.2. *Let A be κ-dense in itself, and $f: A^2 \to I$ be an additive colouring on A. Then there exists a homogeneous subset $X \subseteq A$ of power κ.*

Proof. The proof breaks into two parts: first we will inductively define some sequences with prescribed properties. Having established their existence, we will then derive from it the existence of a homogeneous set.

Part A. Applying Lemma 2.1 to the partition of A induced by Pow(I), we obtain $a < b \in A$ and $D \subseteq I$, so that $D(A)$ is κ-dense in (a, b). For each $k \geqslant 0$ we can define the sequences $t_0, \ldots, t_{k-1}$ of colours, $a = a_0 < a_1 < \cdots < a_k < b_k < \cdots < b_1 < b_0 = b$ of elements, and $D(A) \cap (a, b) = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_k$ of subsets, satisfying, for every $i \leqslant k$, the conditions:

(i) $X_i \subseteq (a_i, b_i)$ and X_i κ-dense in (a_i, b_i),

(ii) b_i is minimal for a_i and $a_i \in X_{i-1}$ if $i > 0$, and

(iii) $f(a_i, x) = t_i$ for all $x \in X_j$, $i < j$.

Assume the sequences defined for m, then we will extend them to $m+1$: Set $X_t = \{x \in X_m : f(a_m, x) = t\}$ for $t \in I$; then $\{X_t : t \in I\}$ is a partition of X_m. Again by Lemma 2.1 there are $a_{m+1} < c_{m+1} \in X_m$ and some t, so that X_t is κ-dense in $X_m \cap (a_{m+1}, c_{m+1})$. Choose $b_{m+1} \in (a_{m+1}, c_{m+1})$ which is minimal for a_{m+1}. Define $t_{m+1} = t$ and $X_{m+1} = X_t \cap (a_{m+1}, b_{m+1})$. From the hypothesis about X_m it follows that X_m is κ-dense in (a_{m+1}, b_{m+1}). Since X_t is κ-dense in $X_m \cap (a_{m+1}, c_{m+1})$, X_{m+1} is κ-dense in (a_{m+1}, b_{m+1}), so (i) is valid. By construction, b_{m+1} is minimal for a_{m+1} and $a_{m+1} \in X_m$, thus (ii) follows. Since $X_{m+1} \subseteq X_t$, (iii) holds for $i = m$. For $i < m$ it follows from the induction hypothesis. Hence, we can extend these sequences to any finite length.

Part B. Suppose we have constructed the sequences till length $k = |I| + 1$. Then the sequence $t_0, \ldots, t_{k-1}$ contains at least two colours which are equal, since there are $|I|$-many colours only. Let m be the least number so that $t_m = t_j$ for some $j < m$.

Now let $y < z \in X_{m+1}$. We want to prove that $f(y, z) = t_m$.

Claim A. $t_m \in D$.

By (ii), $a_m \in D(A)$ and b_m is minimal for a_m, thus $f(a_m, y) \in D$, because $y \in (a_m, b_m)$. But (iii) shows that $t_m = f(a_m, y)$.

Claim B. $t_m + t_m = t_m$.

$t_m = t_j = f(a_j, y) = f(a_j, a_m) + f(a_m, y) = t_j + t_m = t_m + t_m$, by (iii).

Claim C. $f(y, z) = t_m$.

$X_{m+1} \subseteq D(A)$, so $y \in D(A)$. Since by Claim A, $t_m \in D$, there is some $x \in (y, z)$ with $f(y, x) = t_m$. Then by Claim B and (iii):

$$\begin{aligned} f(y, z) &= f(y, x) + f(x, z) = t_m + f(x, z) \\ &= t_m + t_m + f(x, z) = f(a_m, y) + f(y, x) + f(x, z) \\ &= f(a_m, z) = t_m. \end{aligned}$$

By the last claim and (i), $X = X_{m+1}$ is homogeneous and of power κ, so the proof is finished. □

The lemma just proved, together with the Ramsey theorem for additively coloured well-orderings, yields the Ramsey theorem for arbitrary, additively coloured linear orderings. For the proof of the general theorem it is enough to show that every linear ordering of power κ possesses a subset, to which either Shelah's lemma or the preceding lemma is applicable. For that reason we introduce the equivalence relation $\underset{\kappa}{\approx}$: $x \underset{\kappa}{\approx} y$ iff $|(x, y)| < \kappa$ and $|(y, x)| < \kappa$. Obviously, each equivalence class is a convex subset, hence the set of equivalences classes A/κ can be canonically ordered. The equivalence class to which a belongs is denoted by a/κ.

Theorem 2.3. *For a regular cardinal κ let A be a linear ordering of power κ, and let $f: A^2 \to I$ be an additive colouring on A. Then there exists a homogeneous subset $X \subseteq A$ of power κ.*

Proof. We regard the linear ordering A/κ defined above.

Case 1: *There is some a/κ of power κ.* Then in a/κ there is an increasing or decreasing sequence $Y = \{y_\alpha ; \alpha < \kappa\}$ of length κ. (Hint: Let $\{u_\alpha : \alpha < \lambda_1\}$ and $\{v_\alpha : \alpha < \lambda_2\}$ be increasing and decreasing, respectively, sequences, which are both unbounded in a/κ. Then a/κ is the union of all intervals (v_α, u_β), $\beta < \lambda_1$ and $\alpha < \lambda_2$, which are all of power smaller than κ, so either λ_1 or λ_2 is equal to κ.) Restricting the colouring to Y, the hypotheses of Shelah's Lemma are satisfied, so we conclude the existence of a homogeneous set of power κ.

Case 2: *All a/κ are of power smaller than κ.* Then A/κ has cardinality κ, since κ is regular. Choose $Y \subseteq A$, so that $|Y \cap a/\kappa| = 1$ for every $a \in A$. Let $x < y \in Y$, then $|(x, y)| = \kappa$, since they are not equivalent. Also $|\bigcup \{z/\kappa : z \in Y \cap (x, y)\}| = \kappa$, so $|Y \cap (x, y)| = \kappa$, since $|z/\kappa| < \kappa$ and κ is regular. Hence Y is κ-dense in itself. Applying Lemma 2.2, we get a homogeneous subset $X \subseteq Y$ of power κ.

In any case we obtain the desired set X. □

We have shown for regular cardinals, that we always find homogeneous subsets of the same power. We may ask, whether this also holds for singular cardinals. However, it is not difficult to construct counterexamples. But we can still prove the existence of quasi-homogeneous subsets of the same power as the model, and this will be sufficient for our purposes. First, some necessary notions are introduced.

Definition 2.1. (1) Let $X, Y \subseteq A$ be arbitrary non-empty subsets. Then X *is said to be smaller than* Y, in terms $X < Y$, if each $x \in X$ is smaller than every $y \in Y$.

(2) A family $\{X_i : i \in J\}$ of non-empty subsets of A is called an *ordered family* if for all $i, j \in J$, $i \neq j$, either $X_i < X_j$ or $X_j < X_i$ is valid.

If $\{X_i : i \in J\}$ is an ordered family of subsets, then J can be canonically ordered: $i < j$ iff $X_i < X_j$. Assume X is homogeneous (with respect to a colouring f); then X is said to be *of colour* t, if for all $x < y \in X$ $f(x, y) = t$.

The index set J is called *homogeneous* (*of colour* s) if $f(x, y) = s$ for every $i < j \in J$, $x \in X_i$, and $y \in X_j$, respectively.

Definition 2.2. An ordered family $\{X_i : i \in J\}$ of infinite subsets of A is *quasi-homogeneous* if

(i) the index set J is homogeneous (of colour s) and

(ii) all X_i, $i \in J$, are homogeneous of the same colour t.

Then the family is said to be *of colour* (s, t).

For $|J| = 1$ or $s = t$ the union of a quasi-homogeneous family becomes a homogeneous subset. The u-power of $\{X_i : i \in J\}$ is the power of $\bigcup \{X_i : i \in J\}$. Suppose the ordered family $\{X_i : i \in J\}$ has u-power λ, then it is called u-singular if $|J| < \lambda$ and $|X_i| < \lambda$ for all $i \in J$. $\{Y_i : i \in J'\}$ is a subfamily of $\{X_i : i \in J\}$ if $J' \subseteq J$ and $Y_i \subseteq X_i$ for all $i \in J'$.

Lemma 2.4. *Every u-singular, ordered family* $\{X_i : i \in J\}$ *with u-power* λ *possesses a quasi-homogeneous subfamily* $\{Y_i : i \in J'\}$ *of u-power* λ.

Proof. $\{X_i : i \in J\}$ is u-singular, so λ is singular. Set $\kappa = \operatorname{cf} \lambda$, the cofinality of λ. By definition κ is regular, and there is a subset $J_1 \subseteq J$ of power κ so that $\{X_i : i \in J_1\}$ has u-power λ. Furthermore, we can choose $X_i^+ \subseteq X_i$ so that $|X_i^+|$ is regular and $\{X_i^+ : i \in J_1\}$ has still u-power λ. Restricting the colouring to X_i^+ we can apply Theorem 2.3, obtaining homogeneous subsets $Y_i^+ \subseteq X_i^+$ of the same power. Thus $\{Y_i^+ : i \in J_1\}$ has also u-power λ. Let $\max Y_i^+$ and $\min Y_i^+$ be the greatest and the least element in Y_i^+, respectively, if they exist. If one or both of $\max Y_i^+$ and $\min Y_i^+$ are defined, let Y_i be Y_i^+ without these elements. Now given

$i<j\in J_1$, let $x,y\in Y_i$ and $u,v\in Y_j$ be arbitrary. Let $y_i\in Y_i$ be so that $y_i>\max\{x,y\}$ if it exists otherwise set $y_i=\max Y_i^+$ (in this case it exists). Similarly, let $z_j\in Y_j$ so that $z_j<\min\{u,v\}$, otherwise set $z_j=\min Y_j^+$ (it also exists in this case). Then the additivity of f and the homogeneity of Y_i^+ and Y_j^+ imply:

$$\begin{aligned} f(x,u)&=f(x,y_i)+f(y_i,z_j)+f(z_j,u)\\ &=f(y,y_i)+f(y_i,z_j)+f(z_j,v)=f(y,v).\end{aligned}$$

Hence f induces an additive colouring $F:J_1^2\to I$ on J_1 by setting $F(i,j)=f(x,u)$ for some $x\in Y_i$ and $u\in Y_j$, respectively. From above it follows that the definition does not depend on the choice of x and u. Again by Theorem 2.3, since $|J_1|=\kappa$ is regular, there is a homogeneous (with respect to F) subset $J'\subseteq J_1$ of power κ. W.l.o.g. $\{Y_i:i\in J'\}$ is quasi-homogeneous. To be sure that its u-power is λ, we have to make an additional hypothesis about J_1: $|Y_i^+|=|Y_j^+|$ iff $i=j$. W.l.o.g. J_1 has this property, otherwise we could choose an appropriate subset. Then, however, $\{Y_i:i\in J_0\}$ has u-power λ for every subset $J_0\subseteq J_1$ of power κ. Hence, $\{Y_i:i\in J'\}$ is the desired, quasi-homogeneous subfamily. □

A linear ordering A is called *λ-slim* if it has power λ and $|A/\lambda|=1$, i.e., $|(a,b)|<\lambda$ for every $a<b\in A$.

Lemma 2.5. *Let λ be a singular cardinal and A be λ-slim. Then there exists a quasi-homogeneous, ordered family $\{X_i:i\in J\}$ of subsets of A with u-power λ.*

Proof. Every linear ordering A of cardinality λ is the union of a family $\{Y_i:i\in J\}$ of pairwise disjoint subsets, where in addition $|J|\leqslant \operatorname{cf}\lambda$ and all Y_i are convex and bounded. Thus $|J|<\lambda$ because λ is a singular cardinal number. Since A is λ-slim and each Y_i is bounded, we have $|Y_i|<\lambda$ for every $i\in J$. Thus $\{Y_i:i\in J\}$ is u-singular. Now, applying Lemma 2.4, we obtain the desired quasi-homogeneous, ordered family. □

Now we are ready to prove the existence of quasi-homogeneous, ordered families of sufficiently high cardinality for arbitrary, additively coloured linear orderings of singular cardinality.

Theorem 2.6. *For a singular cardinal λ let A be a linear ordering of power λ, and let $f:A^2\to I$ be an additive colouring on A. Then there exists a quasi-homogeneous, ordered family $\{X_i:i\in J\}$ of subsets of A which has the u-power λ.*

Proof. Regarding the set of equivalence classes A/λ of $\bar{\bar{\lambda}}$, we can distinguish three cases.

Case 1: $|a/\lambda|=\lambda$ *for some* $a\in A$. By definition a/λ is λ-slim. Then, applying the preceding lemma, the existence of the desired quasi-homogeneous family follows.

Case 2: *$|a/\lambda| < \lambda$ for all $a \in A$, but* $\sup\{|a/\lambda| : a \in A\} = \lambda$. Let $\kappa = \operatorname{cf} \lambda$ be the cofinality of λ. There is a subset $B \subseteq A$ of pairwise non-equivalent elements of power κ, so that $\{a/\lambda : a \in B\}$ has u-power λ. Clearly, $\{a/\lambda : a \in B\}$ is u-singular. Then the application of Lemma 2.4 yields the desired result.

Case 3: *There is some $\kappa < \lambda$ so that $|a/\lambda| \leq \kappa$ for all $a \in A$.* Choose $Y \subseteq A$, which has with each equivalence class a/λ exactly one element in common. Let $a < b \in Y$ be arbitrary; then they are non-equivalent, hence $|(a, b)| = \lambda$. One the other hand, $|(a, b)| \leq \max\{\kappa, |(a, b)_Y|\}$. Since $\kappa < \lambda$, we conclude that $|(a, b)_Y| = \lambda$. This shows that Y is λ-dense in itself. By Lemma 2.2 there is a homogeneous subset $X \subseteq Y$ of power λ. Then $\{X\}$ is a quasi-homogeneous, ordered family of u-power λ. This completes the proof of the theorem. □

3. Relative elimination of Malitz quantifiers for linear orderings

The results of the preceding section about colourings can be applied to the class of linear orderings to determine the expressive power of Malitz quantifiers. It will turn out that for linear orderings the expressive power of L^2_Δ is as strong as that of $L^{<\omega}_\Delta$. In fact we will prove that each quantifier Q^m_α is eliminable with respect to Q^2_α. Induction on n to prove $\overset{n}{\sim}$-mod G^m_Δ-equivalence will be our main tool. Firstly we will show that for $m \geq 2$ the relations $\overset{n}{\sim}$-mod G^m_Δ and $\overset{n}{\sim}$-mod G^2_Δ coincide. Here we use the existence of 'homogeneous' subsets of the same cardinality. This already yields that the language $L^{<\omega}_\Delta$ is not stronger than L^2_Δ. Finally we get the explicit elimination of Q^m_α with respect to Q^2_α.

We assume Δ to be finite.

Lemma 3.1. $\overset{n}{\sim}$-mod G^m_Δ *has finitely many equivalence classes only.*

In the following let $I^{m,n}_\Delta = \{E_1, \ldots, E_k\}$ be a set of linear orderings which is a set of *representatives for* $\overset{n}{\sim}$-mod G^m_Δ, i.e.,

(i) every linear ordering A is equivalent to some E_i in $I^{m,n}_\Delta$, i.e., there is some $1 \leq i \leq k$ so that $A \overset{n}{\sim} E_i \bmod G^m_\Delta$, and

(ii) it is minimal with respect to (i).

Sometimes m, n, or Δ are omitted if there is no misunderstanding. The elements E_i of $I^{m,n}_\Delta$ are also called colours, since $I^{m,n}_\Delta$ generates a colouring on each linear ordering.

Let A be any linear ordering; then define $f : A^2 \to I^{m,n}_\Delta$ so that

$$f(a, b) = E_i \quad \text{iff} \quad a < b \text{ and } (a, b)_A \overset{n}{\sim} E_i \bmod G^m_\Delta$$
$$\text{or} \quad b < a \text{ and } (b, a)_A \overset{n}{\sim} E_i \bmod G^m_\Delta.$$

Lemma 3.2. *f is an additive colouring.*

Definition 3.1. Let A be a linear ordering.

(i) The subset $X \subseteq A$ is called *(n, Δ)-order-homogeneous* in A iff there are linear orderings U, V, and W so that for all $a < b \in X$, $\langle a, b, A\rangle \overset{n}{\sim} \langle U, V, W\rangle$ mod G^2_Δ (remember that $\langle a, b, A\rangle$ is a shorthand for $\langle A^{<a}, (a, b)_A, A^{>b}\rangle$).

(ii) The ordered family $\{X_\alpha : \alpha \in J\}$ of subsets of A is called *(n, Δ)-quasi-homogeneous* in A iff there are linear orderings T, U, V, and W so that

(a) $\langle a, b, A\rangle \overset{n}{\sim} \langle U, T, W\rangle$ mod G^2_Δ for all α and all $a < b \in X_\alpha$, and

(b) $\langle a, b, A\rangle \overset{n}{\sim} \langle U, V, W\rangle$ mod G^2_Δ for all $\alpha < \beta \in J$ and all $a \in X_\alpha$ and $b \in X_\beta$, respectively.

The results of the preceding section now immediately yield existence theorems about (n, Δ)-order-homogeneous sets and (n, Δ)-quasi-homogeneous ordered families.

Theorem 3.3. *Let A be a linear ordering, Δ a finite set of ordinals, and $X \subseteq A$ a subset of power κ.*

(i) *If κ is regular, then there is an (n, Δ)-order-homogeneous subset $Y \subseteq X$ of power κ.*

(ii) *If κ is singular, then there is an ordered family $\{Y_\alpha : \alpha \in J\}$ of subsets of X, which has u-power κ and is (n, Δ)-quasi-homogeneous.*

Proof. Since $\overset{n}{\sim}$-mod G^2_Δ has only finitely many equivalence classes, there exists a subset $Z \subseteq X$ of the same power as X such that for all $a, b \in Z$, $\langle a, A\rangle \overset{n}{\sim} \langle b, A\rangle$ mod G^2_Δ. Then set $U := A^{<a}$ and $W := A^{>a}$ for some $a \in Z$. Let $I^{2,n}_\Delta$ be a set of representatives for $\overset{n}{\sim}$-mod G^2_Δ and $f : Z^2 \to I^{2,n}_\Delta$ the induced colouring. By the preceding lemma this colouring is additive.

Case (i). By Theorem 2.3 there is a homogeneous (with respect to f) subset Y of power κ. Then Y is (n, Δ)-order-homogeneous. Simply set $V := (a, b)_A$ for some $a < b \in Y$.

Case (ii). By Theorem 2.6 there is a quasi-homogeneous, ordered family $\{Y_\alpha : \alpha \in J\}$ of subsets of Z, which has the u-power κ. Clearly, $\{Y_\alpha : \alpha \in J\}$ is (n, Δ)-quasi-homogeneous: set $T := (a, b)_A$ for some $\alpha \in J$ and $a < b \in Y_\alpha$ and $V := (c, d)_A$ for some $\alpha < \beta \in J$ and $c \in Y_\alpha$ and $d \in Y_\beta$. □

Theorem 3.4. *For every linear orderings A and B the equivalence $A \overset{n}{\sim} B$ mod G^2_Δ implies $A \overset{n}{\sim} B$ mod G^m_Δ for every natural number m.*

Proof. The theorem is proved by induction on n. In the case $n = 0$ the theorem obviously holds. Assume it is proved for n and there are given linear orderings A and B with $A \overset{n+1}{\sim} B$ mod G^2_Δ.

Claim. $A \lesssim^{n+1} B$ mod G^m_Δ.

Let $a \in A$; then by the hypothesis there is some $b \in B$ so that $\langle a, A\rangle \overset{n}{\sim} \langle b, B\rangle$

mod G^2_Δ. Applying the induction hypothesis we get immediately $\langle a, A\rangle \overset{n}{\sim} \langle b, B\rangle$ mod G^m_Δ. Thus condition (i) for the relation $A \lesssim^{n+1} B$ mod G^m_Δ is verified.

To check the second condition let $X \subseteq A$ be a subset of power $\aleph_\alpha$ for some $\alpha \in \Delta$. We have to distinguish two cases: $\aleph_\alpha$ is singular or regular.

Case (a): $\aleph_\alpha$ *is singular*. By Theorem 3.3(ii) there is an ordered family $\{Z_\alpha : \alpha \in J\}$ of subsets of X which is (n, Δ)-quasi-homogeneous and has u-power $\aleph_\alpha$. Set $Z = \bigcup_{\alpha\in J} Z_\alpha$. Since by the hypothesis $A \lesssim^{n+1} B$ mod G^2_Δ there is some $Y \subseteq B$ of cardinality $\aleph_\alpha$ so that $\langle Y, B\rangle \lesssim^n \langle Z, A\rangle$ mod G^2_Δ. Let T, U, V, and W be suitable linear orderings, which fit in the definition of (n, Δ)-quasi-homogeneity for $\{Z_\alpha : \alpha \in J\}$. By the choice of Y we know that for every $a < b \in Y$ there are $c < d \in Z$ with $\langle a, b, B\rangle \overset{n}{\sim} \langle c, d, A\rangle$ mod G^2_Δ. The induction hypothesis implies $\langle a, b, B\rangle \overset{n}{\sim} \langle c, d, A\rangle$ mod G^m_Δ, hence $(a, b)_B \overset{n}{\sim} T$ mod G^m_Δ or $(a, b)_B \overset{n}{\sim} V$ mod G^m_Δ. Since $\{Z_\alpha : \alpha \in J\}$ is u-singular, there are $\alpha_1 < \cdots < \alpha_m$ so that Z_{α_i} are all infinite, $i = 1, \ldots, m$. Thus there exist elements $z_{ij} \in Z_{\alpha_i}$, $j = 1, \ldots, m$, with $z_{i1} < \cdots < z_{im}$. Clearly, $(z_{ij}, z_{ik})_A \overset{n}{\sim} T$ mod G^m_Δ for $j < k$ and $(z_{ij}, z_{lk})_A \overset{n}{\sim} V$ mod G^m_Δ for $i < l$. Now we check that Y also satisfies $\langle Y, B\rangle \lesssim^n \langle Z, A\rangle$ mod G^m_Δ. Suppose $y_1 < \cdots < y_m \in Y$ is an arbitrary sequence of length m. As already remarked it holds either $(y_i, y_{i+1})_B \overset{n}{\sim} T$ mod G^m_Δ or $(y_i, y_{i+1})_B \overset{n}{\sim} V$ mod G^m_Δ for each $i < m$. Obviously we can choose an increasing subsequence $z_{i_1j_1} < \cdots < z_{i_mj_m}$ so that

$$(y_k, y_{k+1})_B \overset{n}{\sim} (z_{i_kj_k}, z_{i_{k+1}j_{k+1}})_A \quad \text{mod } G^2_\Delta$$

for every $k < m$, i.e.,

$$\langle y_1, \ldots, y_m, B\rangle \overset{n}{\sim} \langle z_{i_1j_1}, \ldots, z_{i_mj_m}, A\rangle \quad \text{mod } G^2_\Delta.$$

Now by the induction hypothesis this equivalence extends to $\overset{n}{\sim}$-mod G^m_Δ.

Case (b): $\aleph_\alpha$ *is regular*. This case is solved similarly. Instead of (n, Δ)-quasi-homogeneous, ordered families of subsets we use (n, Δ)-order-homogeneous subsets. The details are left to the reader.

Hence the claim is verified. By the same argument we also obtain $B \lesssim^{n+1} A$ mod G^m_Δ, hence by definition $A \overset{n+1}{\sim} B$ mod G^m_Δ and the theorem is proved. □

From the theorem just proved we can conclude that the expressive power of L^2_Δ is as strong as that of L^m_Δ for $m > 1$. In the following we want to give this equality in strength a more explicit form. We are going to prove that Q^m_α is eliminable with the help of Q^2_α. First we give this phrase a precise meaning.

Definition 3.2. Let L_1 be a sublanguage of a language L_2 and K a class of L_2-structures. We say that L_2 *is reducible to* L_1 *with respect to the class* K if for every formula $\varphi(\bar{x})$ of L_2 there is a formula $\psi(\bar{x})$ of L_1 such that $K \vDash \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$. In case that $L_1 := L^n_\Delta$ and $L_2 := L^m_\Delta$ we say Q^m_α *is eliminable with the help of* Q^n_α *with respect to* K.

Let $\varphi(x_1, \ldots, x_k)$ be any formula of L^m_Δ and $\Phi^{\Delta,m^*,n}_{A,a_1\cdots a_k}(c_1, \ldots, c_k)$ be as defined immediately before Proposition 1.2. $\Phi^{\Delta,m^*,n}_{A,a_1\cdots a_k}(x_1, \ldots, x_k)$ is said to be consistent with φ (over K) iff there is some model B in K so that

$$B \models \varphi(b_1, \ldots, b_k) \wedge \Phi^{\Delta,m^*,n}_{A,a_1\cdots a_k}(b_1, \ldots, b_k)$$

for some elements $b_1, \ldots, b_k$ of B. For fixed Δ, m^*, n, and k there are only finitely many non-equivalent $\Phi^{\Delta,m^*,n}_{A,a_1\cdots a_k}(x_1, \ldots, x_k)$, which we denote by $\Phi_1, \ldots, \Phi_r$.

Let Δ and φ be given; then set $m^* = 2$, $n = q(\varphi)$ the quantifier rank of φ, and $H_\Delta(\varphi) = \{\Phi_i : 1 \leqslant i \leqslant r \text{ and } \Phi_i \text{ is consistent with } \varphi\}$.

Theorem 3.5. *Suppose $m > 1$. Then for every formula $\varphi(x_1, \ldots, x_k)$ of L^m_Δ it holds $\mathrm{LO} \models \varphi \leftrightarrow \bigvee H_\Delta(\varphi)$, i.e., Q^m_α is eliminable with the help of Q^2_α for the class of linear orderings.*

Proof. Let $\varphi(x_1, \ldots, x_k)$ be an arbitrary formula from L^m_Δ, B a linear ordering, and $b_1, \ldots, b_k \in B$. Suppose $B \models \varphi(b_1, \ldots, b_k)$. Then $\Phi^{\Delta,2,n}_{B,b_1\cdots b_k}(x_1, \ldots, x_k)$ and $\varphi(x_1, \ldots, x_k)$ are consistent for $n = q(\varphi)$, hence $B \models \bigvee H_\Delta(\varphi)(b_1, \ldots, b_k)$ and one direction of the theorem is proved. To prove the other direction we suppose $B \models \bigvee H_\Delta(\varphi)(b_1, \ldots, b_k)$. Then there is some Φ_i in $H_\Delta(\varphi)$ such that $B \models \Phi_i(b_1, \ldots, b_k)$. By the definition of $H_\Delta(\varphi)$ there is a linear ordering A and elements $a_1, \ldots, a_k$ in A so that

$$A \models \varphi(a_1, \ldots, a_k) \wedge \Phi_i(a_1, \ldots, a_k).$$

Since Φ_i and $\Phi^{\Delta,2,n}_{B,b_1\cdots b_k}$ are equivalent, we can apply Proposition 1.2 and can conclude that

$$\langle A, a_1, \ldots, a_k \rangle \overset{n}{\equiv} \langle B, b_1, \ldots, b_k \rangle \quad \operatorname{mod} L^2_\Delta.$$

By Proposition 1.3 and Theorem 3.4 we get

$$\langle A, a_1, \ldots, a_k \rangle \overset{n}{\equiv} \langle B, b_1, \ldots, b_k \rangle \quad \operatorname{mod} L^m_\Delta.$$

However $q(\varphi) = n$ and $A \models \varphi(a_1, \ldots, a_k)$, hence $B \models \varphi(b_1, \ldots, b_k)$. This proves the other direction of the theorem. □

We want to remark that $H_\Delta(\varphi)$ can be effectively determined for given φ.

Corollary 3.6. *Let Δ be an arbitrary set of ordinals. Then $L^{<\omega}_\Delta$ is reducible to L^2_Δ with respect to the class of linear orderings.*

The preceding corollary shows that for the class of linear orderings the binary Malitz quantifier is sufficient to express everything expressible by arbitrary Malitz quantifiers. This generalizes some weaker results obtained in [19] and confirms some conjectures made there.

For linear orderings the language L^1_Δ is obviously stronger than the elementary language. However we may ask whether L^2_Δ is really more expressive than L^1_Δ with respect to LO. The following theorem answers this question.

Theorem 3.7. *L^2_Δ is not reducible to L^1_Δ with respect to the class of linear orderings.*

Proof. Clearly, we assume Δ to be non-empty. We will cite two linear orderings A and B, which are equivalent with respect to L^1_Δ, whereas they can be distinguished in L^2_Δ. Let α be the least ordinal in Δ. Then set

$$A := ((\omega^* + \omega) \cdot \omega_\alpha) \cdot (\omega^* + \omega)$$

and

$$B := ((\omega^* + \omega) \cdot \omega_\alpha) \cdot ((\omega^* + \omega) \cdot \omega_\alpha).$$

A and B can be distinguished in L^2_Δ by the sentences

$$Q^2_\alpha xy\ (Q_\alpha z\ (x < z < y))$$

in case of $\alpha > 0$ and

$$\exists w_0\, \exists w_1\, Q^2_0 uv\, Q^2_0 xy\ (w_0 < u < x \wedge y < v < w_1 \wedge Q_0 z\ (x < z < y))$$

if $\alpha = 0$. To show their equivalence with respect to L^1_Δ we establish a more general fact, which is stated in the next lemma.

A linear ordering A is said to be n-α-rich iff for every $a \in A$ there is a subset $X \subseteq A$ of cardinality $\aleph_\alpha$ so that for all $x \in X$, $\langle a, A\rangle \overset{n}{\sim} \langle x, A\rangle \bmod G^1_{\{\alpha\}}$.

Lemma 3.8. *Let D be a n-α-rich linear ordering. For any linear orderings E and F the equivalence $E \overset{n}{\sim} F \bmod G_\emptyset$ implies*

$$D \cdot E \overset{n}{\sim} D \cdot F \quad \bmod G^1_{\{\alpha\}}.$$

Proof. The lemma is proved by induction on n. For $n = 0$ the lemma is obviously true. Suppose now it is proved for k and we are going to prove it for $k+1$. Let $a \in D \cdot E$; then there are elements $x \in D$ and $c \in E$ such that $a = \langle x, c\rangle$. By the hypothesis there is some $d \in F$ such that $\langle c, E\rangle \overset{k}{\sim} \langle d, F\rangle \bmod G_\emptyset$. Set $b = \langle x, d\rangle$. Then $(D \cdot E)^{<a}$ is isomorphic to the sum $D \cdot E^{<c} + D^{<x}$, similarly $(D \cdot F)^{<b}$ is isomorphic to $D \cdot F^{<d} + D^{<x}$. By the induction hypothesis and the sum property we get $(D \cdot E)^{<a} \overset{k}{\sim} (D \cdot F)^{<b} \bmod G^1_{\{\alpha\}}$. By the same argument we can conclude $(D \cdot E)^{>a} \overset{k}{\sim} (D \cdot F)^{>b} \bmod G^1_{\{\alpha\}}$, hence

$$\langle a, D \cdot E\rangle \overset{k}{\sim} \langle b, D \cdot F\rangle \quad \bmod G^1_{\{\alpha\}},$$

thus the first condition in order to show $D \cdot E \overset{k+1}{\sim} D \cdot F \bmod G^1_{\{\alpha\}}$ is verified. Now let $X \subseteq D \cdot E$ be a subset of cardinality $\aleph_\alpha$ and $a \in X$. Then as above there are

$x \in D$ and $c \in E$ such that $a = \langle x, c\rangle$. Choose $d \in F$ also as above. Since D is $k+1$-α-rich, there is a subset $Z \subseteq D$ of cardinality $\aleph_\alpha$ such that for all $z \in Z$

$$\langle z, D\rangle \overset{k}{\sim} \langle x, D\rangle \mod G^1_{\{\alpha\}}.$$

Let $z \in Z$ be arbitrary, set $b = \langle z, d\rangle$; then again as above we obtain $\langle a, D \cdot E\rangle \overset{k}{\sim} \langle b, D \cdot F\rangle \mod G^1_{\{\alpha\}}$, hence $\langle Y, D \cdot F\rangle \lesssim^k \langle X, D \cdot E\rangle \mod G^1_{\{\alpha\}}$ for $Y = \{\langle z, d\rangle : z \in Z\}$. This proves $D \cdot E \lesssim^{k+1} D \cdot F \mod G^1_{\{\alpha\}}$. By symmetry the reverse relation also holds, thus $D \cdot E \overset{k+1}{\sim} D \cdot F \mod G^1_{\{\alpha\}}$. □

Examples. (1) Let $\alpha > 0$; then $D := ((\omega^* + \omega) \cdot \omega_\alpha)$ is n-α-rich for every n. This easily follows by induction on n using the following fact (which can be proved also by induction): let A be a finite linear ordering with at least $2^k - 1$ elements and β an arbitrary ordinal, then it holds

$$A \overset{k}{\sim} \omega + (\omega^* + \omega) \cdot \beta + \omega^* \mod G_\emptyset.$$

If β is countable, then the above equivalence also holds mod $G^1_{\{\alpha\}}$, because $\alpha > 0$ and hence no subsets of power $\aleph_\alpha$ exist. Now set $E := (\omega^* + \omega)$ and $F := (\omega^* + \omega) \cdot \omega_\alpha$. As discrete open orderings E and F are elementarily equivalent. Thus by Lemma 3.8,

$$D \cdot E \overset{n}{\sim} D \cdot F \mod G^1_{\{\alpha\}}$$

for every natural number n. By Proposition 1.3, the linear orderings $D \cdot E$ and $D \cdot F$ satisfy the same sentences of $L^1_{\{\alpha\}}$. Since both orderings have cardinality $\aleph_\alpha$ and α is the least ordinal in Δ, both orderings are also equivalent with respect to L^1_Δ. This completes the proof of Theorem 3.7 for $\alpha > 0$.

(2) Let $D := (\omega^* + \omega)$; then D is n-0-rich. Set $E := \omega \cdot (\omega^* + \omega)$ and $F : \omega \cdot (\omega^* + \omega) \cdot \omega$; then again E and F are elementarily equivalent. By Lemma 3.8, $D \cdot E \overset{n}{\sim} D \cdot F \mod G^1_{\{0\}}$. This completes the proof of Theorem 3.7 for $\alpha = 0$. The details are left to the reader.

4. Relative elimination of Malitz quantifiers for well-orderings

For linear orderings the binary quantifiers Q^2_α are already sufficient to eliminate the quantifiers Q^m_α. This result can be strengthened for the class WO of well-orderings. We will prove that L^2_Δ is even reducible to L^1_Δ with respect to WO. Thus the Malitz quantifiers are eliminable with the help of the unary cardinality quantifiers. However, to obtain this result we have to make an additional assumption about Δ, namely cf $\Delta \subseteq \Delta$ (see the first section for the definition of cf Δ). We adopt this additional supposition throughout the section.

Every well-ordering is isomorphic to some ordinal. In the following we will identify well-orderings with the ordinals to which they are isomorphic. The proof of our main result is based on the representation of ordinals as sums of certain products given in the first lemma. We are mainly concerned with a detailed study of conditions to obtain equivalent products or factors, respectively. The most important fact is stated in Lemma 4.3. It allows to give effective bounds for the ordinals we have to consider.

Lemma 4.1. *Let α, β, and γ be arbitrary ordinals.*

(i) *If $\beta < \gamma$, then also $\alpha + \beta < \alpha + \gamma$.*

(ii) *$\alpha < \beta$ iff there is an ordinal $\delta > 0$ such that $\alpha + \delta = \beta$.*

(iii) *For every natural number n and every ordinal $\beta > 0$ there are uniquely determined ordinals $\beta_0, \beta_1, \ldots, \beta_n$ so that $\alpha = \beta^n \cdot \beta_n + \cdots + \beta^1 \cdot \beta_1 + \beta_0$ and $\beta_i < \beta$ for $i < n$.*

Proof. These properties are easily proved by transfinite induction. The details are left to the reader.

Lemma 4.2. *Let A and B be any two finite linear orderings with cardinality at least $2^n - 1$. Then $A \overset{n+1}{\sim} B \bmod G_\emptyset$.*

For a proof we refer to the preceding section (Example (1)).

Now we are going to make the most important step to prove our main result.

Lemma 4.3. *Let α and β be non-zero ordinals, $\mu = \max \Delta$. Then for every natural number n:*

(i) $\omega_\mu^n \overset{n}{\sim} \omega_\mu^n \cdot \beta \bmod G_\Delta^2$, *and*

(ii) $\omega_\mu^n \cdot \alpha \overset{n}{\sim} \omega_\mu^n \cdot \beta \bmod G_\Delta^2$.

Proof. By induction on n. For $n = 0$ the proposition holds for trivial reasons. Assume the lemma is true for n. We first prove (i) for $n+1$. We regard ω_μ^{n+1} as an initial segment of $\omega_\mu^{n+1} \cdot \beta$. Let $a \in \omega_\mu^{n+1}$. Then $(\omega_\mu^{n+1})^{>a}$ and $(\omega_\mu^{n+1} \cdot \beta)^{>a}$ are isomorphic to ω_μ^{n+1} and $\omega_\mu^{n+1} \cdot \beta$, respectively. Applying the induction hypothesis we obtain

$$\langle a, \omega_\mu^{n+1} \rangle \overset{n}{\sim} \langle a, \omega_\mu^{n+1} \cdot \beta \rangle \quad \bmod G_\Delta^2.$$

If $X \subseteq \omega_\mu^{n+1}$, then for the same reason it holds

$$\langle X, \omega_\mu^{n+1} \cdot \beta \rangle \lesssim^n \langle X, \omega_\mu^{n+1} \rangle \quad \bmod G_\Delta^2,$$

hence $\omega_\mu^{n+1} \lesssim^n \omega_\mu^{n+1} \cdot \beta \bmod G_\Delta^2$. Now we are going to prove the reverse relation.

Let $a \in \omega_\mu^{n+1} \cdot \beta$. We have to distinguish two cases: either $a \in \omega_\mu^{n+1}$ or $a \notin \omega_\mu^{n+1}$. The first case is treated as above. Suppose $a \notin \omega_\mu^{n+1}$. According to Lemma 4.1, the ordinal a can be represented as follows

$$a = \omega_\mu^n \cdot \alpha_n + \cdots + \omega_\mu \cdot \alpha_1 + \alpha_0$$

with $\alpha_i < \omega_\mu$ for all $i < n$. Furthermore $\alpha_n > 0$, since a does not belong to ω_μ^{n+1}. Then set $b = \omega_\mu^n + \omega_\mu^{n-1} \cdot \alpha_{n-1} + \cdots + \alpha_0$ which belongs to the initial segment ω_μ^{n+1}. By the induction hypothesis and the sum property for the ordinals a and b (considered as linear orderings) the equivalence $a \overset{n}{\sim} b \bmod G_\Delta^2$ follows. Moreover, $(\omega_\mu^{n+1})^{>b}$ and $(\omega_\mu^{n+1} \cdot \beta)^{>a}$ are isomorphic to $\omega_\mu^n \cdot \gamma$ and $\omega_\mu^n \cdot \delta$, respectively, for some non-zero ordinals γ and δ. Using the induction hypothesis for (ii) we get

$$(\omega_\mu^{n+1})^{>b} \overset{n}{\sim} (\omega_\mu^{n+1} \cdot \beta)^{>a} \mod G_\Delta^2$$

and finally

$$\langle b, \omega_\mu^{n+1} \rangle \overset{n}{\sim} \langle a, \omega_\mu^{n+1} \cdot \beta \rangle \mod G_\Delta^2.$$

This verifies the first condition of the desired relation. Now let $X \subseteq \omega_\mu^{n+1} \cdot \beta$ be a subset of cardinality $\aleph_\nu$ for some $\nu \in \Delta$. If $X \cap \omega_\mu^{n+1}$ has cardinality $\aleph_\nu$, we proceed as in the proof of $\omega_\mu^{n+1} \lesssim^{n+1} \omega_\mu^{n+1} \cdot \beta \bmod G_\Delta^2$. Otherwise we may suppose that X and ω_μ^{n+1} are disjoint. Since $\overset{n}{\sim}$-mod G_Δ^2 has only finitely many equivalence classes, there exists a subset $Z \subseteq X$ of the same power as X such that for all $a < b \in Z$, $\langle a, \omega_\mu^{n+1} \cdot \beta \rangle \overset{n}{\sim} \langle b, \omega_\mu^{n+1} \cdot \beta \rangle \bmod G_\Delta^2$. Then let $x_1 < x_2 \in Z$. According to Lemma 4.1, x_1 and x_2 are represented in the following form $x_1 = \omega_\mu^n \cdot \alpha_n + \omega_\mu^{n-1} \cdot \alpha_{n-1} + \cdots + \alpha_0$ and $x_2 = \omega_\mu^n \cdot \beta_n + \omega_\mu^{n-1} \cdot \beta_{n-1} + \cdots + \beta_0$, respectively. Since the elements of Z are greater than ω_μ^{n+1} the ordinals α_n and β_n are non-zero. Now we distinguish two cases.

Case 1: *$\alpha_n = \beta_n$ for all $x_1 < x_2 \in Z$.* Clearly, the intervals $(\omega_\mu^n \cdot \alpha_n, \omega_\mu^n \cdot (\alpha_n + 1))$ and $(\omega_\mu^n, \omega_\mu^n \cdot 2)$ are isomorphic. Let f be the canonical isomorphism between these intervals and Y the image of Z under this isomorphism. Suppose $y_1 < y_2 \in Y$ are arbitrary elements. Let $u_1 = f^{-1}(y_1)$ and $u_2 = f^{-1}(y_2)$. There is an ordinal $\gamma < \omega_\mu^n$ such that $y_1 = \omega_\mu^n + \gamma$ and $u_1 = \omega_\mu^n \cdot \alpha_n + \gamma$. By the induction hypothesis and the sum property $y_1 \overset{n}{\sim} u_1 \bmod G_\Delta^2$. As above for a and b we also have

$$(\omega_\mu^{n+1})^{>y_2} \overset{n}{\sim} (\omega_\mu^{n+1} \cdot \beta)^{>u_2} \mod G_\Delta^2.$$

By the choice of u_1 and u_2 the intervals (u_1, u_2) and (y_1, y_2) are isomorphic. Hence we can conclude that

$$\langle y_1, y_2, \omega_\mu^{n+1} \rangle \overset{n}{\sim} \langle u_1, u_2, \omega_\mu^{n+1} \cdot \beta \rangle \mod G_\Delta^2$$

and the second condition is verified.

Case 2: *$\alpha_n \neq \beta_n$ for some $x_1 < x_2 \in Z$.* Define

$$y_\lambda = \omega_\mu^n \cdot \lambda + \omega_\mu^{n-1} \cdot \beta_{n-1} + \cdots + \beta_0$$

for every ordinal λ and $Y = \{y_\lambda : \lambda < \omega_\nu\}$. Then clearly $Y \subseteq \omega_\mu^{n+1}$ and Y has cardinality $\aleph_\nu$. For any $\lambda_1 < \lambda_2$ the interval $(y_{\lambda_1}, y_{\lambda_2})$ is isomorphic to $\omega_\mu^n \cdot \gamma + \omega_\mu^{n-1} \cdot \beta_{n-1} + \cdots + \beta_0$ for some ordinal γ. For the same reason the interval (x_1, x_2) is isomorphic to $\omega_\mu^n \cdot \delta + \omega_\mu^{n-1} \cdot \beta_{n-1} + \cdots + \beta_0$ for some ordinal $\delta > 0$. Then by the induction hypothesis and the sum property $(x_1, x_2) \overset{n}{\sim} (y_{\lambda_1}, y_{\lambda_2})$

mod G^2_Δ. For the same reason we have $x_2 \overset{n}{\sim} y_{\lambda_1}$ mod G^2_Δ. By the choice of Z then $x_1 \overset{n}{\sim} y_{\lambda_1}$ mod G^2_Δ follows. Again as above we also get

$$(\omega_\mu^{n+1} \cdot \beta)^{>x_2} \overset{n}{\sim} (\omega_\mu^{n+1})^{>y_{\lambda_2}} \mod G^2_\Delta.$$

All relations together yield

$$\langle y_{\lambda_1}, y_{\lambda_2}, \omega_\mu^{n+1} \rangle \overset{n}{\sim} \langle x_1, x_2, \omega_\mu^{n+1} \cdot \beta \rangle \mod G^2_\Delta$$

and the induction step is accomplished.

The property (ii) follows immediately from (i). □

The lemma just proved shows that arbitrary non-empty products with left factor ω_μ^n are not distinguishable from each other by sentences of quantifier rank n. We remark that for this lemma the assumption $\operatorname{cf} \Delta \subseteq \Delta$ is unnecessary. Moreover, no special assumptions about the right factors are made. But in contrast in the next lemma no suppositions about the left factor are demanded.

Lemma 4.4. *Let α and β be arbitrary ordinals with $\alpha \overset{n}{\sim} \beta$ mod G^2_Δ. Then for every ordinal $\delta > 0$ $\delta \cdot \alpha \overset{n}{\sim} \sigma \cdot \beta$ mod G^2_Δ.*

Because of later applications let us note that the proof can also be done with "mod G^2_Δ" replaced by "mod G^1_Δ".

Proof. By induction on n. For $n = 0$ there is nothing to prove. Assume the lemma is proved for n and we are given the ordinals α and β with $\alpha \overset{n+1}{\sim} \beta$ mod G^2_Δ. By symmetry it is enough to show $\delta \circ \alpha \lesssim^{n+1} \delta \cdot \beta$ mod G^2_Δ. Let $a \in \delta \cdot \alpha$, i.e., $a = \langle a_1, a_2 \rangle$ with $a_1 \in \delta$ and $a_2 \in \alpha$. Set $b = \langle b_1, b_2 \rangle$, where b_1 and b_2 are chosen so that $b_1 = a_1$ and $\langle a_2, \alpha \rangle \overset{n}{\sim} \langle b_2, \beta \rangle$ mod G^2_Δ. Since α and β are equivalent with respect to $\overset{n+1}{\sim}$-mod G^2_Δ, b_2 exists. $(\delta \cdot \alpha)^{<a}$ and $(\delta \cdot \beta)^{<b}$ are isomorphic to $\delta \cdot \alpha^{<a_2} + \delta^{<a_1}$ and $\delta \cdot \beta^{<b_2} + \delta^{<a_1}$, respectively. Then by the induction hypothesis and the sum property $(\delta \cdot \alpha)^{<a} \overset{n}{\sim} (\delta \cdot \beta)^{<b}$ mod G^2_Δ. For the same reason $(\delta \cdot \alpha)^{>a} \overset{n}{\sim} (\delta \cdot \beta)^{>b}$ mod G^2_Δ, hence

$$\langle a, \delta \cdot \alpha \rangle \overset{n}{\sim} \{b, \delta \cdot \beta \rangle \mod G^2_\Delta.$$

Now suppose $X \subseteq \delta \cdot \alpha$ is a subset of cardinality $\aleph_\mu$ for some $\mu \in \Delta$. We are searching for a subset $Y \subseteq \delta \cdot \beta$ of cardinality $\aleph_\mu$ so that $\langle Y, \delta \cdot \beta \rangle \lesssim^n \{X, \delta \cdot \alpha \rangle$ mod G^2_Δ.

Case 1. There is a subset $U \subseteq \delta$ of power $\aleph_\mu$ and an element $d \in \alpha$ so that $\{\langle c, d \rangle : c \in U\}$ is contained in X. Choose $e \in \beta$ so that $\langle e, \beta \rangle \overset{n}{\sim} \langle d, \alpha \rangle$ mod G^2_Δ. By the induction hypothesis and the sum property $Y = \{\langle c, e \rangle : c \in U\}$ has the desired property.

Case 2. There is a subset $D \subseteq \alpha$ of power $\aleph_\mu$ so that for each $d \in D$ there is some $x \in \delta$ with $\langle x, d \rangle \in X$. We may suppose that there is some $y \in \delta$ so that $\langle x, \delta \rangle \overset{n}{\sim} \langle y, \delta \rangle$ mod G^2_Δ for every $\langle x, d \rangle \in X$, otherwise we choose a suitable subset. Since $\alpha \overset{n+1}{\sim} \beta \bmod G^2_\Delta$, we find $E \subseteq \beta$ with $\langle E, \beta\rangle \lesssim^n \langle D, \alpha\rangle \bmod G^2_\Delta$. By the induction hypothesis and the sum property $Y = \{\langle y, e\rangle : e \in E\}$ has the desired property. Let $y_1 < y_2 \in Y$ and $e_1 < e_2 \in E$ with $y_1 = \langle y, e_1\rangle$ and $y_2 = \langle y, e_2\rangle$. Then there are $d_1 < d_2 \in D$ and $x_1, x_2 \in \delta$ so that $\langle d_1, d_2, \alpha\rangle \overset{n}{\sim} \langle e_1, e_2, \beta\rangle \bmod G^2_\Delta$ and $u_1 = \langle x_1, d_1\rangle$ and $u_2 = \langle x_2, d_2\rangle$ belong to X. Finally we get $\langle y_1, y_2, \delta\cdot\beta\rangle \overset{n}{\sim} \langle u_1, u_2, \delta\cdot\alpha\rangle \bmod G^2_\Delta$. Hint: the intervals $(y_1, y_2)_{\delta\cdot\beta}$ and $(u_1, u_2)_{\delta\cdot\alpha}$ are isomorphic to $\delta^{>y} + \delta\cdot(e_1, e_2)_\beta + \delta^{<y}$ and $\delta^{>x_1} + \delta\cdot(d_1, d_2)_\alpha + \delta^{<x_2}$, respectively. By the choice of y, e_1, and e_2, the induction hypothesis, and the sum property $(y_1, y_2)_{\delta\cdot\beta} \overset{n}{\sim} (u_1, u_2)_{\delta\cdot\alpha} \bmod G^2_\Delta$ follows. Similarly the other equivalences are verified.

Clearly, if $\aleph_\mu$ is regular, no further cases occur. Assume now $\aleph_\mu$ is singular. Let $\operatorname{cf}\aleph_\mu = \aleph_\nu$.

Case 3. There is a subset $D \subseteq \alpha$ so that $X = \bigcup\{X_d : d \in D\}$, $\operatorname{card} D < \aleph_\mu$, $\operatorname{card} X_d < \aleph_\mu$ for every $d \in D$, and the elements of X_d are of the form $\langle x, d\rangle$. We may assume that $\operatorname{card} D = \aleph_\nu$, for all $d \in D$, $\operatorname{card} X_d$ is regular, and by Theorem 3.3 all $p_1(X_d) = \{x \in \delta : \langle x, d\rangle \in X_d\}$ are (n, Δ)-order-homogeneous of the same type, i.e., for any $x_1 < x_2 \in p_1(X_c)$ and $y_1 < y_2 \in p_1(X_d)$, respectively, $\langle x_1, x_2, \delta\rangle \overset{n}{\sim} \langle y_1, y_2, \delta\rangle \bmod G^2_\Delta$ holds. Otherwise we choose a subset of X which has these properties. Since $\operatorname{cf}\Delta \subseteq \Delta$ we find a subset $E \subseteq \beta$ with $|E| = \aleph_\nu$ and $\langle E, \beta\rangle \lesssim^n \langle D, \alpha\rangle \bmod G^2_\Delta$. Let $f : E \to D$ be a bijection between these two sets. Define $Y_e = \{\langle x, e\rangle : x \in p_1(X_{f(e)})\}$ for $e \in E$ and $Y = \bigcup_{e\in E} Y_e$. Then Y has cardinality $\aleph_\mu$ and satisfies $\langle Y, \delta\cdot\beta\rangle \lesssim^n \langle X, \delta\cdot\alpha\rangle \bmod G^2_\Delta$. Hint: Let $y_1 < y_2 \in Y$. There are $z_1, z_2 \in \delta$ and $e_1 \leqslant e_2 \in E$ with $y_1 = \langle z_1, e_1\rangle$ and $y_2 = \langle z_2, e_2\rangle$. Two possibilities arise. If $e_1 = e_2$, then set $u_1 = \langle z_1, d\rangle$ and $u_2 = \langle z_2, d\rangle$ with $d = f(e_1)$ and argue as in the first case. (For simplicity we assume also that D and E are (n, Δ)-order-homogeneous.) If $e_1 < e_2$, then let $d_1 < d_2 \in D$ be arbitrary and $u_1 = \langle z_1, d_1\rangle$ and $u_2 = \langle z_2, d_2\rangle$. To show $\langle y_1, y_2, \delta\cdot\beta\rangle \overset{n}{\sim} \langle u_1, u_2, \delta\cdot\alpha\rangle \bmod G^2_\Delta$ proceed as in the second case.

This completes the proof of the lemma. □

Both the preceding lemmata enable us to conclude the equivalence of products if the factors satisfy certain assumptions. Now, vice versa, we try to find conditions about products from which the equivalence of corresponding factors follows. For that reason we introduce the notion "α divides γ", in terms α/γ, which means that there is some ordinal β with $\alpha\cdot\beta = \gamma$. The unary predicate ω^n_μ/x has the following properties.

Lemma 4.5. *Let k be a natural number, x and μ arbitrary ordinals. Then*:

(i) ω_μ/x *iff* $\forall y < x\, Q_\mu z\, (y < z < x)$,

(ii) ω^{k+1}_μ/x *iff* $\omega^k_\mu/x \wedge \forall y < x\, Q_\mu z\, (y < z < x \wedge \omega^k_\mu/z)$, *and*

(iii) $q(\omega^k_\mu/x) = 2k$.

Proof. If $x = \omega_\mu \cdot a$, then the formula on the right-hand side of (i) is true.

Assume now that x satisfies this formula. By Lemma 4.1, $x = \omega_\mu \cdot a + b$, where $b < \omega_\mu$. Then clearly $b = 0$, hence ω_μ / x. This proves (i). The right side of (ii) follows immediately from the left one. In order to prove the reverse direction let x satisfy the formula on the right side. Again by Lemma 4.1, x has a representation $x = \omega_\mu^{k+1} \cdot a_{k+1} + \omega_\mu^k \cdot a_k + \cdots + a_0$ where $a_i = 0$ for $i < k$ (because ω_μ^k divides x) and $a_k < \omega_\mu$. In addition $a_k = 0$ since x satisfies the given formula. By recursion we obtain from (i) and (ii) a formula of $L^1_{\{\mu\}}$ which defines the relation ω_μ^k / x. For short we also use ω_μ^k / x to denote this formula. (iii) says that it has quantifier rank $2k$. As $q(\omega_\mu^{k+1}/x) = q(\omega_\mu^k / x) + 2$ this is easily proved by induction. □

Let $\mu \in \Delta$. Suppose we are given the following ordinals: $a = \omega_\mu^k \cdot \alpha_1$, $\alpha = a + \alpha_2$ with $0 < \alpha_2 < \omega_\mu^k$, and $\beta = \omega_\mu^k \cdot \beta_1 + \beta_2$ with $\beta_2 < \omega_\mu^k$.

In the next lemma we consider α and β as linear orderings, whereas a is regarded as an element of α.

Lemma 4.6. *Let α, β, and a as above, $b \in \beta$. Then $\langle a, \alpha \rangle \overset{2k+1}{\sim} \langle b, \beta \rangle \bmod G^1_\Delta$ implies $b = \omega_\mu^k \cdot \beta_1$.*

Proof. Since $q(\omega_\mu^k / x) = 2k$ and ω_μ^k / a, we conclude by Proposition 1.3 that ω_μ^k / b. Furthermore, a is the greatest element in α with this property, i.e., in $\alpha^{>a}$ the sentence $\neg\exists x\, (\omega_\mu^k / x)$ holds. However the quantifier rank of $\neg\exists x\, (\omega_\mu^k / x)$ is $2k + 1$, thus again by Proposition 1.3, b is the greatest element of β satisfying the formula (ω_μ^k / x). Then there are b_1 and b_2 so that $b = \omega_\mu^k \cdot b_1$ and $\beta = \omega_\mu^k \cdot b_1 + b_2$, where $b_2 < \omega_\mu^k$. Since this representation is unique, $b_1 = \beta_1$ follows. □

Lemma 4.7. *Let α and β be arbitrary ordinals, $k > 1$ a natural number, and $\mu \in \Delta$. If $\omega_\mu \cdot \alpha \overset{2k}{\sim} \omega_\mu \cdot \beta \bmod G^1_\Delta$, then also $\alpha \overset{k}{\sim} \beta \bmod G^1_\Delta$.*

Proof. First, we remark that for arbitrary ordinals γ and δ, $\gamma \overset{1}{\sim} \delta \bmod G^1_\Delta$ iff (1) ($\gamma = 0$ iff $\delta = 0$) and (2) for all $\nu \in \Delta$ ($\gamma \geq \omega_\nu$ iff $\delta \geq \omega_\nu$). Now the lemma is proved by induction on k. Let $k = 2$ and $\omega_\mu \cdot \alpha \overset{4}{\sim} \omega_\mu \cdot \beta \bmod G^1_\Delta$. By symmetry it is enough to show $\alpha \lesssim^2 \beta \bmod G^1_\Delta$.

(i) Let $a \in \alpha$. Set $x = \omega_\mu \cdot a$. Choose $y \in \omega_\mu \cdot \beta$ so that $\langle x, \omega_\mu \cdot \alpha \rangle \overset{3}{\sim} \langle y, \omega_\mu \cdot \beta \rangle \bmod G^1_\Delta$. By the preceding lemma $y = \omega_\mu \cdot b$ for some $b \in \beta$. Now $\alpha^{>a} \geq \omega_\nu$ iff there are at least ω_ν-many elements in $(\omega_\mu \cdot \alpha)^{>x}$ which are divisible by ω_μ iff the sentence $Q_\nu u\, \forall v < u\, Q_\mu z\, (v < z < u)$ holds in $(\omega_\mu \cdot \alpha)^{>x}$. Since this sentence has quantifier rank 3, it then holds also in $(\omega_\mu \cdot \beta)^{>y}$, hence $\beta^{>b} \geq \omega_\nu$. Similarly we can show that $\alpha^{>a} > 0$ implies $\beta^{>b} > 0$. For the same reason $\beta^{>b} \geq \omega_\nu$ and $\beta^{>b} > 0$ imply $\alpha^{>a} \geq \omega_\nu$ and $\alpha^{>a} > 0$, respectively. The same relations are derivable for $\alpha^{<a}$ and $\beta^{<b}$, hence by the introductory remark we have $\langle a, \alpha\rangle \overset{1}{\sim} \langle b, \beta\rangle \bmod G^1_\Delta$ as desired.

(ii) Let $X \subseteq \alpha$ be a subset of power $\aleph_\nu$ for some $\nu \in \Delta$. Set $X' = \{\omega_\mu \cdot a : a \in X\}$. Let $Y' \subseteq \omega_\mu \cdot \beta$ be a subset of power $\aleph_\nu$ so that $\langle Y', \omega_\mu \cdot \beta\rangle \lesssim^3 \langle X', \omega_\mu \cdot \alpha\rangle \bmod G^1_\Delta$. Then again by the preceding lemma the elements of Y' are divisible by ω_μ, thus there is a subset $Y \subseteq \beta$ of power $\aleph_\nu$ such that $Y' = \{\omega_\mu \cdot b : b \in Y\}$. Then proceed as in case (i) to show $\langle Y, \beta\rangle \lesssim^1 \langle X, \alpha\rangle \bmod G^1_\Delta$. This completes the proof for $k = 2$.

Now we are going to verify the induction step. Assume the lemma is proved for $k \geqslant 2$ and we are given α and β such that $\omega_\mu \cdot \alpha \overset{2(k+1)}{\sim} \omega_\mu \cdot \beta \bmod G^1_\Delta$. Again by symmetry it is enough to show $\alpha \lesssim^{k+1} \beta \bmod G^1_\Delta$.

(i) Let $a \in \alpha$. Set $x = \omega_\mu \cdot a$ and choose $y \in \omega_\mu \cdot \beta$ so that $\langle x, \omega_\mu \cdot \alpha\} \overset{2k+1}{\sim} \langle y, \omega_\mu \cdot \beta\rangle \bmod G^1_\Delta$. As before $y = \omega_\mu \cdot b$ for some $b \in \beta$. Then by the induction hypothesis we obtain $\alpha^{<a} \overset{k}{\sim} \beta^{<b} \bmod G^1_\Delta$. (Hint: In case that a and b are successor ordinals choose the predecessors before applying the induction hypothesis.) Now $(\omega_\mu \cdot a)^{>x}$ and $(\omega_\mu \cdot b)^{>y}$ are obviously isomorphic to $\omega_\mu + \omega_\mu \cdot \alpha^{>a}$ and $\omega_\mu + \omega_\mu \cdot \beta^{>b}$, respectively, hence $\omega_\mu \cdot \alpha^{>a} \overset{2k}{\sim} \omega_\mu \cdot \beta^{>b} \bmod G^1_\Delta$ and finally by the induction hypothesis $\alpha^{>a} \overset{k}{\sim} \beta^{>b} \bmod G^1_\Delta$. Thus $\langle a, \alpha\rangle \overset{k}{\sim} \langle b, \beta\rangle \bmod G^1_\Delta$.

(ii) Now let $X \subseteq \alpha$ be a subset of power $\aleph_\nu$ for some $\nu \in \Delta$. Set $X' = \{\omega_\mu \cdot a : a \in X\}$ and choose Y' so that $\langle Y', \omega_\mu \cdot \beta\rangle \lesssim^{2k+1} \langle X', \omega_\mu \cdot \alpha\rangle \bmod G^1_\Delta$ and $|Y'| = \aleph_\nu$. As before all elements of Y' are divisible by ω_μ. Let $Y \subseteq \beta$ be such that $Y' = \{\omega_\mu \cdot b : b \in Y\}$. To prove $\langle Y, \beta\rangle \lesssim^k \langle X, \alpha\rangle \bmod G^1_\Delta$ we proceed as in the case (i). □

Lemma 4.8. *Let α and β be any ordinals, $\mu \in \Delta$, k and m natural numbers, $m > 1$. If*

$$\omega^k_\mu \cdot \alpha \overset{2^k \cdot m}{\sim} \omega^k_\mu \cdot \beta \quad \bmod G^1_\Delta,$$

then also $\alpha \overset{m}{\sim} \beta \bmod G^1_\Delta$.

Proof. By induction on k using the preceding lemma. □

Now we have a series of criterions for the equivalence of products and factors, respectively. We use them to eliminate Q^2_α with the help of Q_α with respect to the class of well-orderings. For that reason we introduce the recursive function $F(k, m)$ defined as follows:

$$F(0, m) = m + 2 \quad \text{and} \quad F(k+1, m) = (2m+1) \cdot F(k, m).$$

Obviously $F(k, m)$ is recursive and always greater than 1.

Now we are ready to prove the main theorem of the section.

Theorem 4.9. *Let α and β be any ordinals and k be the number of elements of Δ. If $\alpha \overset{F(k,m)}{\sim} \beta \bmod G^1_\Delta$, then also $\alpha \overset{m}{\simeq} \beta \bmod G^2_\Delta$.*

Proof. By induction on k. If Δ is empty, then there is nothing to prove. Now let Δ be a set of $k+1$ ordinals satisfying $\mathrm{cf}\,\Delta \subseteq \Delta$. Suppose $\mu = \max \Delta$. Then set $\Gamma = \Delta \setminus \{\mu\}$. Clearly, $cf\,\Gamma \subseteq \Gamma$. By the induction hypothesis the theorem holds for Γ. The proof is accomplished in three steps:

Step 1. By Lemma 4.1, α and β can be represented as follows:

$$\alpha = \omega_\mu^m \cdot \alpha_m + \omega_\mu^{m-1} \cdot \alpha_{m-1} + \cdots + \alpha_0$$

and

$$\beta = \omega_\mu^m \cdot \beta_m + \omega_\mu^{m-1} \cdot \beta_{m-1} + \cdots + \beta_0,$$

where $\alpha_i, \beta_i < \omega_\mu$ for $i < m$. For $0 \leqslant n < m$ set

$$x_n = \omega_\mu^m \cdot \alpha_m + \cdots + \omega_\mu^{m-n} \cdot \alpha_{m-n}.$$

To begin with, we assume $x_0 < x_1 < \cdots < x_{m-1}$. Then there are $y_0 < y_1 < \cdots < y_{m-1} \in \beta$ satisfying

$$\langle x_0, \ldots, x_{m-1}, \alpha \rangle \overset{h}{\sim} \langle y_0, \ldots, y_{m-1}, \beta \rangle \quad \bmod G^1_\Delta$$

where $h = F(k+1, m) - m$. Obviously, we have

$$F(k+1, m) = (2^m + 1) \cdot F(k, m) \geqslant (2^m + 1)(m + 2) \geqslant 3m + 1,$$

thus $h \geqslant 2m + 1$. By a successive application of Lemma 4.6 we get

$$y_n = \omega_\mu^m \cdot \beta_m + \cdots + \omega_\mu^{m-n} \cdot \beta_{m-n} \quad \text{for } 0 \leqslant n < m.$$

Now suppose $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{m-1}$ is not strictly increasing. Then, choosing a maximal, strictly increasing subsequence, we can proceed as above.

Step 2. In addition we have:

$$h = F(k+1, m) - m = (2^m + 1) \cdot F(k, m) - m \geqslant 2^m \cdot F(k, m).$$

Together with the results in the first step this yields

$$\omega_\mu^{m-n} \cdot \alpha_{m-n} \overset{2^m \cdot F(k,m)}{\sim} \omega_\mu^{m-n} \cdot \beta_{m-n} \quad \bmod G^1_\Delta$$

for $0 \leqslant n \leqslant m$. Since $F(k, m) > 1$, Lemma 4.8 is applicable, and we conclude $\alpha_n \overset{F(k,m)}{\sim} \beta_n \bmod G^1_\Delta$ for $0 \leqslant n \leqslant m$. By the induction hypothesis it follows that $\alpha_n \overset{m}{\simeq} \beta_n \bmod G^2_\Gamma$. For $n < m$ this implies $\alpha_n \overset{m}{\simeq} \beta_n \bmod G^2_\Delta$, because $\alpha_n, \beta_n < \omega_\mu$ and $\Delta = \Gamma \cup \{\mu\}$. Furthermore, using Lemma 4.4, we can derive

$$\omega_\mu^n \cdot \alpha_n \overset{m}{\simeq} \omega_\mu^n \cdot \beta_n \quad \bmod G^2_\Delta.$$

However, $\omega_\mu^m \cdot \alpha_m \overset{m}{\simeq} \omega_\mu^m \cdot \beta_m \bmod G^2_\Delta$ is also valid by Lemma 4.3. (It is easily shown that α_m and β_m are either both zero or non-zero, respectively.)

Step 3. The sequences $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{m-1} \in \alpha$ and $y_0 \leqslant y_1 \leqslant \cdots \leqslant y_{m-1} \in \beta$ partition α and β, respectively, so that corresponding parts are equivalent with respect to $\overset{m}{\simeq}$-mod G^2_Δ. Using the sum property we conclude that $\alpha \overset{m}{\simeq} \beta$ mod G^2_Δ. □

From the theorem just proved we derive the elimination of Malitz quantifiers for well-orderings in the same way as the corresponding result for linear orderings. We introduce a similar notation as for the proof of the elimination result for linear orderings. Let φ be a formula of L^m_Δ and $\Phi^{\Delta,m^*,n}_{A,a_1\cdots a_k}(c_1, \ldots, c_k)$ as defined before Proposition 1.2. Now we fix $m^* = 1$, r the number of elements of Δ, $s = q(\varphi)$ the quantifier rank, and $n = F(r, s)$. For these fixed values and fixed k there are only finitely many non-equivalent $\Phi^{\Delta,m^*,n}_{A,a_1\cdots a_k}(x_1, \ldots, x_k)$, which we denote by $\Phi_1, \ldots, \Phi_p$. Then define $H^1_\Delta(\varphi) = \{\Phi_i : 1 \leqslant i \leqslant p$ and Φ_i is consistent with φ over WO$\}$. This notation is very similar to that following Definition 3.2.

Theorem 4.10. *For every formula $\varphi(x_1, \ldots, x_k)$ of L^m_Δ,* WO $\vDash \varphi \leftrightarrow \bigvee H^1_\Delta(\varphi)$ *holds, i.e., Q^m_α is eliminable with the help of the unary cardinality quantifier Q_α for the class of well-orderings.*

Proof. With some obvious changes it is completely the same (nearly word by word) as that of Theorem 3.5. As an additional argument we have to use Theorem 4.9. □

Corollary 4.11. *Let Δ be an arbitrary set of ordinals satisfying* cf $\Delta \subseteq \Delta$. *Then $L^{<\omega}_\Delta$ is reducible to L^1_Δ with respect to the class of well-orderings.*

Now we have finished the proof of the desired elimination result. It has been derived under the assumption cf $\Delta \subseteq \Delta$. We do not know whether this assumption is really necessary. However, it seems to be no proper restriction, for it is a desirable property of Δ.

5. Decidability for the class of well-orderings

An important property of a theory is its decidability. Since the elementary theory of well-orderings is decidable, it is natural to ask whether the theory of WO in the extended language $L^{<\omega}_\Delta$ remains decidable or not. Our main result is an affirmative answer to this question. In particular we get the decidability of Th^1_Δ(WO), which can be already derived from the results of Chapter 5 of [2] (under the hypothesis cf $\Delta = \Delta$). When proving decidability of theories in L^m_Δ some additional difficulties arise, because we cannot use the axiomatizability of the logic of L^m_Δ. The decision procedure, which we shall devise in the following, is based on a generalization of the quantifier elimination method.

As before let Δ be an arbitrary finite set of ordinals which satisfies the closure condition $\mathrm{cf}\,\Delta \subseteq \Delta$. For notational simplicity we carry out the proof for L^2_Δ only. No new ideas are required to generalize it to arbitrary L^m_Δ, $m > 2$.

First we reduce the decision problem for the class WO to the more manageable subset R^Δ, which is defined by induction on the number of elements of Δ. However, the definition of R^Δ requires some more notation.

Definition 5.1. An *order polynomial* $t(x)$ is an expression

$$t(x) = x^k \cdot p_k + x^{k-1} \cdot p_{k-1} + \cdots + p_0$$

where $p_0, \ldots, p_k$ are given ordinals, called the *coefficients* of $t(x)$.

If $p_k \neq 0$, then k is called the *degree* of $t(x)$.

Let H be a set of ordinals. The set of all order polynomials with coefficients from H is denoted by $H[x]$. Usually we assume H to be closed by addition and multiplication. Now let $t(x)$ be an order polynomial from $H[x]$ and β an arbitrary ordinal. The value of $t(x)$ at β is the ordinal $t(\beta)$ obtained from $t(x)$ by substituting β for x. The set of all values of polynomials from $H[x]$ at β is denoted by $H[\beta]$.

Now we give the definition of R^Δ by induction on the number of elements of Δ. Let N be the set of natural numbers (including 0). Clearly, N is closed by addition and multiplication. We put $R^\emptyset = N[\omega]$. Now suppose Δ is non-empty, $\alpha = \max \Delta$, and $\Gamma = \Delta \setminus \{\alpha\}$. By the induction hypothesis R^Γ is already defined. Then set $R^\Delta = R^\Gamma[\omega_\alpha]$. The elements of R^Δ are also called Δ-polynomials. Obviously, if $\Delta_1 \subseteq \Delta_2$, then $R^{\Delta_1} \subseteq R^{\Delta_2}$. The Δ-polynomials carry additional informations arising from their representations as order polynomials. To emphasize the representation the elements of R^Δ are also denoted by $r, s, t, \ldots$ (possibly with indices). On the other hand, the notation r^0 indicates that the additional structure of r is disregarded.

Before we will study R^Δ in detail let us remind the reader of some laws of ordinal arithmetic: the associative laws for both addition and multiplication and the (left) distributive law. In the following they are used without mention.

Moreover, for $\beta < \omega_\alpha$ we have $\beta + \omega_\alpha = \beta \cdot \omega_\alpha = \omega_\alpha$.

The basic properties of R^Δ are expressed in the following proposition.

Proposition 5.1. (i) *The representation of each element of R^Δ as a Δ-polynomial is unique.*

(ii) *R^Δ is closed by addition and multiplication. Moreover, for given Δ-polynomials r and s, the sum $r + s$ and the product $r \cdot s$ can be calculated effectively.*

Proof. To prove (i) it is sufficient to show that different order polynomials from $R^\Gamma[x]$ have different values at ω_α (assuming $\alpha = \max \Delta$ and $\Gamma = \Delta \setminus \{\alpha\}$). The details are left to the reader. Also the second part of the proposition is proved by induction on the number of elements of Δ. We verify the induction step for $\alpha = \max \Delta$ and $\Gamma = \Delta \setminus \{\alpha\}$. Suppose r and s are given non-zero Δ-polynomials of degree k and m, respectively. Then $r = \omega_\alpha^k \cdot r_k + \omega_\alpha^{k-1} \cdot r_{k-1} + \cdots + r_0$ and $s = \omega_\alpha^m \cdot s_m + \omega_\alpha^{m-1} \cdot s_{m-1} + \cdots + s_0$ for some $r_0, \ldots, r_k, s_0, \ldots, s_m$ from R^Γ. First we calculate the sum $r + s$.

Case 1: $k = m = 0$. Then r and s belong to R^Γ, hence $r + s \in R^\Gamma$.

Case 2: $k < m$. In this case we have $r + s = s$.

Case 3: $0 < k = m$. Then

$$r + s = \omega_\alpha^m \cdot (r_m + s_m) + \omega_\alpha^{m-1} \cdot s_{m-1} + \cdots + s_0.$$

Case 4: $m < k$. An easy calculation shows that

$$r + s = \omega_\alpha^k \cdot r_k + \cdots + \omega_\alpha^{m+1} \cdot r_{m+1} + \omega_\alpha^m \cdot (r_m + s_m) + \omega_\alpha^{m-1} \cdot s_{m-1} + \cdots + s_0.$$

In any case the sum is again an element of R^Δ.

Using the distributive law the calculation of $r \cdot s$ can be reduced to the calculations of the products $r \cdot (\omega_\alpha^l \cdot s_l)$, $0 \leqslant l \leqslant m$.

Case 1: $l = 0$.

Case 1.1: *s_0 is a limit ordinal.*

$$\begin{aligned} r \cdot s_0 &= (\omega_\alpha^k \cdot r_k + \omega_\alpha^{k-1} \cdot r_{k-1} + \cdots + r_0) \cdot s_0 \\ &= (\omega_\alpha^k \cdot r_k) \cdot s_0 = \omega_\alpha^k \cdot (r_k \cdot s_0). \end{aligned}$$

Hence $r \cdot s_0$ is a Δ-polynomial since $r_k \cdot s_0 \in R^\Gamma$ by the induction hypothesis.

Case 1.2: *s_0 is a successor ordinal.* A similar calculation as in the preceding case yields

$$r \cdot s_0 = \omega_\alpha^k \cdot (r_k \cdot s_0) + \omega_\alpha^{k-1} \cdot r_{k-1} + \cdots + r_0.$$

Case 2: $l > 0$. Again, the desired result follows from an easy calculation:

$$\begin{aligned} r \cdot (\omega_\alpha^l \cdot s_l) &= ((\omega_\alpha^k \cdot r_k + \cdots + r_0) \cdot \omega_\alpha^l) \cdot s_l \\ &= ((\omega_\alpha^k \cdot r_k) \cdot \omega_\alpha^l) \cdot s_l = \omega_\alpha^k \cdot (r_k \cdot \omega_\alpha^l) \cdot s_l = \omega_\alpha^{k+l} \cdot s_l. \end{aligned}$$

The procedure described above shows that R^Δ is closed by addition and multiplication. In addition it enables us to calculate sums and products effectively. □

By the preceding proposition R^Δ may be also characterized as the least set of ordinals, which contains $\{\omega_\alpha : \alpha \in \Delta\} \cup \{0, 1, \omega\}$ and is closed by addition and multiplication. Now we are going to prove that the decision problem for the class WO can be reduced to R^Δ. For that purpose we introduce the operators $\sigma_n^\Delta : \mathrm{On} \to R^\Delta$ mapping the class of ordinals into the set of Δ-polynomials. They are defined by induction on the number of elements of Δ. Suppose Δ is the empty set. According to Lemma 4.1 each ordinal β admits a decomposition

$$\beta = \omega^n \cdot \beta_n + \omega^{n-1} \cdot \beta_{n-1} + \cdots + \beta_0$$

where $\beta_0, \ldots, \beta_{n-1}$ are finite ordinals. In the following we will freely use this decomposition lemma without mention. Set

$$\sigma_n^{\emptyset}(\beta) = \omega^n \cdot \varepsilon + \omega^{n-1} \cdot p_{n-1} + \cdots + p_0$$

where $\varepsilon = 0$ iff $\beta_n = 0$, $\varepsilon = 1$ iff $\beta_n > 0$, and for $i < n$

$$p_i = \begin{cases} \beta_i & \text{iff } \beta_i < 2^n - 1, \\ 2^n - 1 & \text{iff } \beta_i \geq 2^n - 1. \end{cases}$$

Clearly, $\sigma_n^{\emptyset}(\beta)$ is the value of an order polynomial from $N[x]$ at ω. The range of $\sigma_n^{\emptyset}$ is denoted by $R_n^{\emptyset}$. We have an explicit description of $R_n^{\emptyset}$:

$$R_n^{\emptyset} = \{\beta : \beta = \omega^n \cdot p_n + \cdots + p_0 \text{ where } p_n = 0 \text{ or } p_n = 1,$$
$$\text{and } p_0, \ldots, p_{n-1} \text{ are finite ordinals smaller then } 2^n\}.$$

It is not difficult to derive $\sigma_n^{\emptyset}(\beta) \overset{n}{\sim} \beta \bmod G_{\emptyset}$. Hence the set $R_n^{\emptyset}$ is a finite set of $\emptyset$-polynomials such that each elementary sentence with at most n quantifiers, which is satisfiable in some well-ordering, has already a model in the set $R_n^{\emptyset}$.

Now suppose Δ is non-empty, $\alpha = \max \Delta$, and $\Gamma = \Delta \setminus \{\alpha\}$. Then again decompose β according to the lemma:

$$\beta = \omega_\alpha^n \cdot \beta_n + \omega_\alpha^{n-1} \cdot \beta_{n-1} + \cdots + \beta_0,$$

where $\beta_0, \ldots, \beta_{n-1}$ are ordinals smaller than ω_α. Set

$$\sigma_n^{\Delta}(\beta) = \omega_\alpha^n \cdot \varepsilon + \omega_\alpha^{n-1} \cdot \sigma_n^{\Gamma}(\beta_{n-1}) + \cdots + \sigma_n^{\Gamma}(\beta_0),$$

where $\varepsilon = 0$ iff $\beta_n = 0$, and $\varepsilon = 1$ iff $\beta_n > 0$.

The operators σ_n^{Δ} have the following important property, which is proved by induction on the number of elements of Δ using Lemma 4.3, Lemma 4.4, and the sum property.

Lemma 5.2. *For arbitrary finite* Δ, $\operatorname{cf} \Delta \subseteq \Delta$, *and every ordinal* β, *and every natural number* n:

$$\sigma_n^{\Delta}(\beta) \overset{n}{\sim} \beta \quad \bmod G_\Delta^2.$$

The range of σ_n^{Δ} is denoted by R_n^{Δ}. There is an explicit description of R_n^{Δ} using R_n^{Γ}:

$$R_n^{\Delta} = \{\beta : \beta = \omega_\alpha^n \cdot p_n + \omega_\alpha^{n-1} \cdot p_{n-1} + \cdots + p_0 \text{ where}$$
$$p_k \in R_n^{\Gamma} \text{ for } k < n, \text{ and } p_n = 0 \text{ or } p_n = 1\}.$$

We want to remark that for every n the subsets R_n^{Δ} are finite. Furthermore, we have $R^{\Delta} = \bigcup_{n<\omega} R_n^{\Delta}$. Thus the sets R_n^{Δ} may be considered as finite approximations of R^{Δ}. By Lemma 5.2 we may conclude that the decision problem for WO can be reduced to R^{Δ}. Thus R^{Δ} is sufficiently large to serve as a substitute for WO (with respect to satisfaction of sentences from L_Δ^2). However R^{Δ} has the advantage over WO that its elements are clearly arranged. To get a decision procedure for R^{Δ} we require the notion of a composite Δ-polynomial.

Definition 5.2. (1) Every Δ-polynomial t is a composite Δ-polynomial with respect to the empty sequence.

(2) Let $\bar{c} = (c_1, \ldots, c_m)$ be a sequence of pairwise different constants, s a composite Δ-polynomial in $\bar{c}$, and t a Δ-polynomial. Then for every constant c, $c \neq c_i$ for all $i \leq m$, the formal sum $s + c + t$ is a composite Δ-polynomial in $(c_1, \ldots, c_m, c)$.

The set of composite Δ-polynomials in $(c_1, \ldots, c_m)$ is denoted by $C^{\Delta}(c_1, \ldots, c_m)$. If in the definition above all occurring Δ-polynomials belong to R_n^{Δ}, then the resulting set of composite Δ-polynomials is denoted by $C_n^{\Delta}(c_1, \ldots, c_m)$. It is convenient to think of composite Δ-polynomials $t(c_1, \ldots, c_m)$ as sums $t_1 + c_1 + t_2 + \cdots + c_m + t_{m+1}$, where $t_1, \ldots, t_{m+1}$ are Δ-polynomials which are called the parts of t. More precisely, t_i is the i-th part, $1 \leq i \leq m+1$, and by defining $q_i(t) = t_i$ we introduce the part-functions q_i.

Now we want to relate the composite Δ-polynomials to linear orderings with distinguished constants. Let us recall that r^0 denotes the pure linear ordering determined by the Δ-polynomial r (i.e., the additional structure arising from the representation of r is completely disregarded). If c is a constant, then c^0 is the one-element linear ordering with domain $\{c\}$. Assume we are given a composite Δ-polynomial $t = t_1 + c_1 + t_2 + \cdots + c_m + t_{m+1}$. Then t^0 is the linear ordering $t_1^0 + c_1^0 + t_2^0 + \cdots + c_m^0 + t_{m+1}^0$ with $c_1, \ldots, c_m$ as distinguished constants. If there is no misunderstanding, we also write t instead of t^0.

The most important property of the set of composite Δ-polynomials is expressed in the following lemma.

Lemma 5.3. *Let A be a well-ordering with the distinguished elements $a_1 < a_2 < \cdots < a_m \in A$, Δ a finite set of ordinals,* $\mathrm{cf}\,\Delta \subseteq \Delta$, *and n a natural number. Then there is a composite Δ-polynomial $t \in C_n^{\Delta}(c_1, \ldots, c_m)$ such that*

$$\langle a_1, \ldots, a_m, A \rangle \overset{n}{\simeq} \langle c_1, \ldots, c_m, t \rangle \mod G_{\Delta}^2.$$

Proof. the elements $a_1, \ldots, a_m$ partition the set A. Apply Lemma 5.2 to each part, and then combine the desired t. □

Let $t(c_1, \ldots, c_m)$ be a composite Δ-polynomial and $\varphi(d_1, \ldots, d_k)$ a sentence of $L_{\Delta}^2(d_1, \ldots, d_k)$. The sentence φ is said to be defined in t iff the set $\{d_1, \ldots, d_k\}$ is contained in the set $\{c_1, \ldots, c_m\}$. If φ is defined in t, then $t \models \varphi$ indicates that φ is valid in t^0, where the constants are interpreted in the natural way. From the next lemma the importance of the composite Δ-polynomials will become apparent.

Lemma 5.4. *Let $\varphi(c_1, \ldots, c_m)$ be a sentence of $L^2_\Delta(c_1, \ldots, c_m)$ which is satisfiable in some well-ordering A with the distinguished elements $a_1 < \cdots < a_m \in A$. Then there are a natural number n, which can be calculated efficiently, and a composite Δ-polynomial $t \in C^\Delta_n(c_1, \ldots, c_m)$ such that $t \models \varphi(c_1, \ldots, c_m)$.*

Proof. Assume A is a well-ordering with the distinguished constants $a_1 < \cdots < a_m \in A$, in which $\varphi(c_1, \ldots, c_m)$ is valid. It is sufficient to put for n the number of quantifiers occurring in φ. By the previous lemma there is a composite Δ-polynomial $t \in C^\Delta_n(c_1, \ldots, c_m)$ such that

$$\langle a_1, \ldots, a_m, A \rangle \overset{n}{\simeq} \langle c_1, \ldots, c_m, t \rangle \mod G^2_\Delta.$$

Using Proposition 1.3 we can conclude $t \models \varphi(c_1, \ldots, c_m)$, because $A \models \varphi(a_1, \ldots, a_m)$ by the hypothesis on A. □

In the following the power set of a set Z is denoted by $P(Z)$. For every natural number $n \geq 1$ and every ordinal $\alpha \in \Delta$, we define the mappings $f^n: R^\Delta \to P(C^\Delta_{n-1}(c))$, $g^n_\alpha: R^\Delta \to P(C^\Delta_{n-1}(c))$, $h^n_\alpha: R^\Delta \to P(C^\Delta_{n-1}(c, d))$ in case ω_α is regular, and $h^n_\alpha: R^\Delta \to P(C^\Delta_{n-1}(c, d) \times C^\Delta_{n-1}(c, d))$ for ω_α singular. Before we give the definitions we state the intended properties which the functions will have

(I) for each Δ-polynomial t and every $a \in t^0$ there is some $s(c) \in f^n(t)$ such that

$$\langle a, t^0 \rangle \overset{n-1}{\simeq} \langle c^0, s^0 \rangle \mod G^2_\Delta, \tag{1}$$

and, conversely, for every $s(c) \in f^n(t)$ there is some $a \in t^0$ which satisfies (1).

(II) For each Δ-polynomial t and every subset $X \subseteq t^0$ of cardinality ω_α, $\alpha \in \Delta$, there are some $s(c) \in g^n_\alpha(t)$ and a subset $X' \subseteq X$ of cardinality ω_α such that

$$\text{for all } a \in X' \quad \langle a, t^0 \rangle \overset{n-1}{\simeq} \langle c^0, s^0 \rangle \mod G^2_\Delta, \tag{2}$$

and, conversely, for every $s(c) \in g^n_\alpha(t)$ there is a subset $X' \subseteq t^0$ of cardinality ω_α which satisfies (2).

(IIIa) for each Δ-polynomial t and every subset $X \subseteq t^0$ of cardinality ω_α, $\alpha \in \Delta$ and ω_α regular, there are some $s(c, d) \in h^n_\alpha(t)$ and a subset $X' \subseteq X$ of cardinality ω_α such that

$$\text{for all } a < b \in X' \quad \langle a, b, t^0 \rangle \overset{n-1}{\simeq} \langle c^0, d^0, s^0 \rangle \mod G^2_\Delta, \tag{3a}$$

and, conversely, for every $s(c, d) \in h^n_\alpha(t)$ there is a subset $X' \subseteq t^0$ of cardinality ω_α which satisfies (3a).

(IIIb) For each Δ-polynomial t and every subset $X \subseteq t^0$ of cardinality ω_α, $\alpha \in \Delta$ and ω_α singular, there are some $(r(c, d), (s(c, d)) \in h^n_\alpha(t)$ and a subset

$X' \subseteq X$ of cardinality ω_α such that

$$\text{for all } a<b \in X' \quad \text{either}$$

$$\langle a, b, t^0\rangle \overset{n-1}{\simeq} \langle c^0, d^0, r^0\rangle \mod G^2_\Delta \quad \text{or} \tag{3b}$$

$$\langle a, b, t^0\rangle \overset{n-1}{\simeq} \langle c^0, d^0, s^0\rangle \mod G^2_\Delta,$$

and, conversely, for every $(r(c, d), s(c, d)) \in h^n_\alpha(t)$ there is a subset $X' \subseteq t^0$ of cardinality ω_α which satisfies (3b).

First we define the values of f^n, g^n_α, and h^n_α for the elements 0, 1, ω, and ω_β for $\beta \in \Delta$. For every ordinal β let $\Delta \cap \beta = \{\gamma \in \Delta : \gamma < \beta\}$. Furthermore, the Δ-polynomials $t = \omega^n_\alpha \cdot p_n + \cdots + p_0$ with $p_n = 1$ are called monic. The set of monic Δ-polynomials contained in R^Δ_n is denoted by Q^Δ_n.

Definition of f^n

$$f^n(0) = \emptyset, \quad f^n(1) = \{c\}, \quad f^n(\omega) = \{k + c + \omega : k < 2^{n-1}\},$$
$$f^n(\omega_\beta) = \{s + c + \omega_\beta : s \in R^{\Delta\cap\beta}_{n-1}\} \quad \text{for } \beta \in \Delta.$$

Remark. $c + 0$ and $0 + c$ are usually shortened to c.

Definition of g^n_α ($\alpha \in \Delta$)

$$g^n_\alpha(0) = \emptyset \quad \text{and} \quad g^n_\alpha(1) = \emptyset.$$

For the definition of $g^n_\alpha(\omega)$ we distinguish two cases: put $g^n_\alpha(\omega) = \emptyset$ for $\alpha > 0$ and $g^n_0(\omega) = \{2^{n-1} + c + \omega\}$ for $\alpha = 0$. Now let $\beta > 0$ and $\beta \in \Delta$.

1. $\alpha > \beta$. Set $g^n_\alpha(\omega_\beta) = \emptyset$.
2. $\alpha = \beta$. Define $g^n_\alpha(\omega_\alpha) = \{s + c + \omega_\alpha : s \in Q^{\Delta\cap\alpha}_{n-1}\}$.
3. $\alpha < \beta$. Suppose $\mu = \max \Delta \cap \beta$. Then $g^n_\alpha(\omega_\beta)$ is obtained by recursion:

$$g^n_\alpha(\omega_\beta) = \{s + c + \omega_\beta : s \in Q^{\Delta\cap\beta}_{n-1}\} \cup \{s + \omega_\beta : s \in g^n_\alpha(\omega^{n-1}_\mu)\}.$$

Definition of h^n_α (ω_α regular and $\alpha \in \Delta$)

$$h^n_\alpha(0) = \emptyset \quad \text{and} \quad h^n_\alpha(1) = \emptyset.$$

Set $h^n_\alpha(\omega) = \emptyset$ for $\alpha > 0$ and

$$h^n_\alpha(\omega) = \{2^{n-1} + c + 2^{n-1} + d + \omega\} \quad \text{for } \alpha = 0.$$

Now let $\beta > 0$ and $\beta \in \Delta$.

1. $\alpha > \beta$. Put $h^n_\alpha(\omega_\beta) = \emptyset$.
2. $\alpha = \beta$. Set $h^n_\alpha(\omega_\alpha) = \{s + c + s + d + \omega_\alpha : s \in Q^{\Delta\cap\alpha}_{n-1}\}$.

3. $\alpha < \beta$. Suppose $\mu = \max \Delta \cap \beta$. Then $h^n_\alpha(\omega_\beta)$ is obtained by recursion:

$$h^n_\alpha(\omega_\beta) = \{s + \omega_\beta : s \in h^n_\alpha(\omega^{n-1}_\mu)\}$$
$$\cup \{\omega^{n-1}_\mu + s + \omega_\beta : s \in h^n_\alpha(\omega^{n-1}_\mu)\}$$
$$\cup \{s + c + s + d + \omega_\beta : s \in Q^{\Delta\cap\beta}_{n-1}\}.$$

Definition of h^n_α (ω_α singular)

$$h^n_\alpha(0) = \emptyset, \qquad h^n_\alpha(1) = \emptyset, \quad \text{and} \quad h^n_\alpha(\omega) = \emptyset.$$

Let $\beta > 0$ and $\beta \in \Delta$.

1. $\alpha > \beta$. Put $h^n_\alpha(\omega_\beta) = \emptyset$.
2. $\alpha = \beta$. Define

$$h^n_\alpha(\omega_\alpha) = \{(s + c + s + d + \omega_\alpha, s + c + s + d + \omega_\alpha) : s \in Q^{\Delta\cap\alpha}_{n-1}\}.$$

3. $\alpha < \beta$. Suppose $\mu = \max \Delta \cap \beta$. Then $h^n_\alpha(\omega_\beta)$ is defined by recursion:

$$h^n_\alpha(\omega_\beta) = Z_1 \cup Z_2 \cup Z_3 \cup Z_4,$$

where the sets Z_1, Z_2, Z_3, and Z_4 have the following meaning:

$$Z_1 = \{(r + \omega_\beta, s + \omega_\beta) : (r, s) \in h^n_\alpha(\omega^{n-1}_\mu)\},$$
$$Z_2 = \{(\omega^{n-1}_\mu + r + \omega_\beta, \omega^{n-1}_\mu + s + \omega_\beta) : (r, s) \in h^n_\alpha(\omega^{n-1}_\mu)\},$$
$$Z_3 = \{(s + c + s + d + \omega_\beta, s + c + s + d + \omega_\beta) : s \in Q^{\Delta\cap\beta}_{n-1}\},$$
$$Z_4 = \{(\omega^{n-1}_\mu + r + \omega_\beta, \omega^{n-1}_\mu + r_1 + c + \omega^{n-1}_\mu + r_1 + d + \omega_\beta) :$$
$$(r, s) \in h^n_\alpha(\omega^{n-1}_\mu) \text{ for some } s \text{ and } r_1 = q_1(r)\}.$$

In the definition above the values $g^n_\alpha(\omega_\beta)$ and $h^n_\alpha(\omega_\beta)$, for $\alpha < \beta$, are given only by recursion. To calculate them we require further recursions which will be given below. To state these recursions readily, we use addition and multiplication of elements from R^Δ_{n-1}. In general, the resulting ordinals will not belong to R^Δ_{n-1}. In such cases, we shall apply the operator σ^Δ_{n-1} to each component of the composite Δ-polynomials under discussion. Since the elements of $R^\Delta_{n-1}(c)$ are left fixed by σ^Δ_{n-1}, we may apply σ^Δ_{n-1} in any case. The application of σ^Δ_{n-1} to a set Z is indicated by $\sigma^\Delta_{n-1}Z$. For sums and products, the values of f^n, g^n_α, and h^n_α are calculated according to the following rules:

Case 1: $t = t_1 + t_2$.

$$f^n(t) = \sigma^\Delta_{n-1}(\{t_1 + s : s \in f^n(t_2)\} \cup \{s + t_2 : s \in f^n(t_1)\}),$$
$$g^n_\alpha(t) = \sigma^\Delta_{n-1}(\{t_1 + s : s \in g^n_\alpha(t_2)\} \cup \{s + t_2 : s \in g^n_\alpha(t_1)\}),$$
$$h^n_\alpha(t) = \sigma^\Delta_{n-1}(\{t_1 + s : s \in h^n_\alpha(t_2)\} \cup \{s + t_2 : s \in h^n_\alpha(t_1)\})$$

for ω_α regular, and

$$h^n_\alpha(t) = \sigma^\Delta_{n-1}\{(t_1 + r, t_1 + s) : (r, s) \in h^n_\alpha(t_2)\}$$
$$\cup\, \sigma^\Delta_{n-1}\{(r + t_2, s + t_2) : (r, s) \in h^n_\alpha(t_1)\}$$

for ω_α singular.

Case 2: $t = \omega_\beta \cdot t_1$, $\beta \in \Delta$ or $\beta = 0$.

$$f^n(t) = \sigma^\Delta_{n-1}\{\omega_\beta \cdot s_1 + r + \omega_\beta \cdot s_2 : r \in f^n(\omega_\beta) \text{ and } s_1 + c + s_2 = s \in f^n(t_1)\},$$
$$g^n_\alpha(t) = \sigma^\Delta_{n-1}\{\omega_\beta \cdot s_1 + r + \omega_\beta \cdot s_2 : [r \in g^n_\alpha(\omega_\beta) \text{ and } s_1 + c + s_2 = s \in f^n(t_1)]$$
$$\text{or } [r \in f^n(\omega_\beta) \text{ and } s_1 + c + s_2 = s \in g^n_\alpha(t_1)]\},$$
$$h^n_\alpha(t) = \sigma^\Delta_{n-1}\{\omega_\beta \cdot s_1 + r + \omega_\beta \cdot s_2 : r \in h^n_\alpha(\omega_b) \text{ and } s_1 + c + s_2 = s \in f^n(t_1)\}$$
$$\cup\, \sigma^\Delta_{n-1}\{\omega_\beta \cdot s_1 + r(c) + \omega_\beta \cdot s_2 + r(d) + \omega_\beta \cdot s_3 :$$
$$r(c) \in f^n(\omega_\beta) \text{ and } s_1 + c + s_2 + d + s_3 = s \in h^n_\alpha(t_1)\}$$

for ω_α regular, and

$$h^n_\alpha(t) = \sigma^\Delta_{n-1}U_1 \cup \sigma^\Delta_{n-1}U_2 \cup \sigma^\Delta_{n-1}U_3$$

for ω_α singular, where

$$U_1 = \{(\omega_\beta \cdot s_1 + u + \omega_\beta \cdot s_2, \omega_\beta \cdot s_1 + v + \omega_\beta \cdot s_2):$$
$$(u, v) \in h^n_\alpha(\omega_\beta) \text{ and } s_1 + c + s_2 = s \in f^n(t_1)\},$$
$$U_2 = \{(\omega_\beta \cdot u_1 + s(c) + \omega_\beta \cdot u_2 + s(d) + \omega_\beta \cdot u_3,$$
$$\omega_\beta \cdot v_1 + s(c) + \omega_\beta \cdot v_2 + s(d) + \omega_\beta \cdot v_3):$$
$$s(c) \in f^n(\omega_\beta) \text{ and } (u, v) \in h^n_\alpha(t_1) \text{ where}$$
$$u = u_1 + c + u_2 + d + u_3 \text{ and } v = v_1 + c + v_2 + d + v_3\},$$

and

$$U_3 = \{(\omega_\beta \cdot s_1 + u + \omega_\beta \cdot s_3, \omega_\beta \cdot s_1 + u_1 + c + u_3 + \omega_\beta \cdot s_2 + u_1$$
$$+ d + u_3 + \omega_\beta \cdot s_3 : (u, v) \in h^n_\alpha(\omega_\beta) \text{ and } s \in h^n_\alpha(t_1) \text{ where}$$
$$u = u_1 + c + u_2 + d + u_3, s = s_1 + c + s_2 + d + s_3, \text{ and } \omega_\delta = \operatorname{cf} \omega_\alpha\}.$$

As an example we demonstrate the calculation of $f^3(\omega^2 + 2)$. We have to calculate the following values: $f^3(1)$, $f^3(2)$, $f^3(\omega)$, $f^3(\omega^2)$, and $f^3(\omega^2 + 2)$. Using the definition of $f^3(t)$ for $t = 1$ and $t = \omega$ we easily get $f^3(1) = \{c\}$ and $f^3(\omega) = \{k + c + \omega : k < 4\}$. Applying the recursions for addition and multiplication we obtain:

$$f^3(2) = f^3(1 + 1) = \{c + 1, 1 + c\},$$
$$f^3(\omega^2) = f^3(\omega \cdot \omega)$$
$$= \sigma^\Delta_2\{\omega \cdot s_1 + r + \omega^2 : r \in f^3(\omega)$$

$$\text{and } s_1 = q_1(s) \text{ for some } s \in f^3(\omega)\}$$
$$= \{\omega \cdot k + l + c + \omega^2 : 0 \leqslant k,l \leqslant 3\},$$
$$f^3(\omega^2+2) = \{\omega \cdot k + l + c + \omega^2 + 2 : 0 \leqslant k,l \leqslant 3\}$$
$$\cup \{\omega^2 + c + 1,\ \omega^2 + 1 + c\}.$$

The existence of functions f^n, g^n_α, and h^n_α with the properties (I)–(III), respectively, already follows from Lemma 5.2. The advantage of the definition above consists in the constructive way it is given.

Lemma 5.5. *For each $n \geqslant 1$ and $\alpha \in \Delta$, the functions f^n, g^n_α, and h^n_α are recursive.*

Proof. The operations + and · are recursive by Proposition 5.1. Furthermore, σ^Δ_{n-1} restricted to R^Δ is also recursive. Hence f^n, g^n_α, and h^n_α are recursive. □

It remains to check that f^n, g^n_α, and h^n_α have indeed the properties (I)–(III), respectively.

Before we shall do this let us define the (n, Δ)-characteristic of elements and subsets. Let A be an arbitrary well-ordering and $a_1 < \cdots < a_m \in A$. By Proposition 5.2 there is some $s \in C^\Delta_n(c_1, \ldots, c_m)$ such that

$$\langle a_1, \ldots, a_m, A \rangle \overset{n}{\sim} \langle c^0_1, \ldots, c^0_m, s^0 \rangle \mod G^2_\Delta.$$

In this case the Δ-polynomial s is called an (n, Δ)-characteristic of $a_1 < \cdots < a_m$ (in A). Note that there may be different Δ-polynomials in $C^\Delta_n(c_1, \ldots, c_m)$ satisfying the equivalence above. We consider them as defining one and the same (n, Δ)-characteristic. According to property (I), $f^n(t)$ contains exactly those $(n-1, \Delta)$-characteristics of elements (up to $\overset{n-1}{\sim}$-equivalence) which are realized in t^0. Now this concept is extended to subsets $X \subseteq A$ as follows. X is said to be (n, Δ)-uniform iff all of its elements have one and the same (n, Δ)-characteristic which is then called the (n, Δ)-characteristic of X. Thus $g^n_\alpha(t)$ characterizes the different $(n-1, \Delta)$-uniform subsets of t^0. Furthermore, by Proposition 5.2 we may conclude that a subset $X \subseteq A$ is (n, Δ)-order-homogeneous iff there is some Δ-polynomial $s \in C^\Delta_n(c, d)$ such that all pairs $a < b \in X$ have the (n, Δ)-characteristic s. Suppose X has the (n, Δ)-characteristic $s = s_1 + c + s_2 + d + s_3$ then the elements of X have the (n, Δ)-characteristic $s_1 + c + s_3$. In case that ω_α is regular, $h^n_\alpha(t)$ describes the possible $(n-1, \Delta)$-order-homogeneous subsets of t^0. Now let ω_α be singular and $X = \{X_i : i \in J\}$ an ordered family of subsets of A. Again by Proposition 5.2, the ordered family X is (n, Δ)-quasi-homogeneous in A iff there are Δ-polynomials $r, s \in C^\Delta_n(c, d)$ such that each pair $a < b \in X$ has either the (n, Δ)-characteristic r if $a < b \in X_i$ for some $i \in J$, or s otherwise. The Δ-polynomial r (s) is called the inner (outer) (n, Δ)-characteristic of X. Note that we may assume $q_1(r) = q_1(s)$ and $q_3(r) = q_3(s)$. In the singular case, $h^n_\alpha(t)$ consists of the (n, Δ)-characteristics (r, s) of the possible (n, Δ)-quasi-homogeneous ordered families in t^0.

Lemma 5.6. *For every* $n \geq 1$ *and* $\alpha \in \Delta$:
(A) f^n *has property* (I),
(B) g^n_α *has property* (II), *and*
(C) h^n_α *has property* (III).

Proof. The lemma is proved by induction on the complexity of Δ-polynomials. The cases $t = 1$ and $t = \omega$ can be easily verified and are, therefore, omitted. The induction step for sums is straightforward and is left to the reader.

Proof of (A)

Case A.1: $t = \omega_\beta$ *for* $\beta \in \Delta$. Let $a \in \omega_\beta$. Then $\sigma^\Delta_{n-1}(\omega_\beta^{<a}) = s \in R^{\Delta \cap \beta}_{n-1}$. Thus $f^n(\omega_\beta)$ has the form given in the definition.

Case A.2: $t = \omega_\beta \cdot t_1$ *for* $\beta \in \Delta$ *or* $\beta = 0$. Using the induction hypothesis for t_1, we may conclude that every element of t_1^0 has an $(n-1, \Delta)$-characteristic contained in $f^n(t_1)$. Then the desired result,

$$f^n(t) = \sigma^\Delta_{n-1}\{\omega_\beta \cdot s_1 + r + \omega_\beta \cdot s_2 : r \in f^n(\omega_\beta) \text{ and } s_1 + c + s_2 = s \in f^n(t_1)\},$$

is derived by arguing in the same way as in the first part of the proof of Lemma 4.4.

Proof of (B)

Case B.1: $t = \omega_\beta$ *for* $\beta \in \Delta$. The trivial case $\alpha > \beta$ is omitted. Now let $X \subseteq \omega_\beta$ be a subset of cardinality ω_α. Clearly, there is an $(n-1, \Delta)$-uniform subset $X' \subseteq X$ of the same cardinality. In case $\alpha = \beta$, X' has to be cofinal in ω_α. Hence the $(n-1, \Delta)$-characteristic of X' has the form $s + c + \omega_\alpha$, $s \in Q^{\Delta \cap \alpha}_{n-1}$. If $\alpha < \beta$, then by taking subsets we may assume either $X' \subseteq \omega^{n-1}_\mu$ or $X' \cap \omega^{n-1}_\mu = \emptyset$. As a consequence we get the recursion given in the definition of g^n_α. For details we refer to the proof of Lemma 4.3.

Case B.2: $t = \omega_\beta \cdot t_1$ *for* $\beta \in \Delta$ *or* $\beta = 0$. To overcome this case, we use the following two properties of (n, Δ)-uniform subsets:

(P1) Assume $Y \subseteq \omega_\beta$ and $Z \subseteq t_1^0$ are non-empty (n, Δ)-uniform subsets; then $X = Y \times Z$ is (n, Δ)-uniform in t^0.

(P2) Let $X \subseteq t^0$ be a non-empty (n, Δ)-uniform subset. Then there are non-empty (n, Δ)-uniform subsets $Y \subseteq \omega_\beta$ and $Z \subseteq t_1^0$, respectively, such that $Y \times Z$ has the same (n, Δ)-characteristic as X. If in addition $|X| \geq \omega_\alpha$, then Y and Z can be chosen so that either $|Y| \geq \omega_\alpha$ or $|Z| \geq \omega_\alpha$.

All the necessary arguments to derive (P1) and (P2) are contained in the proofs of the Lemmata 4.3 and 4.4. For the second part of (P2) we require in addition Lemma 3.1. Using (P1), (P2), and the induction hypothesis for t_1, we get the desired recursion for $g^n_\alpha(t)$.

Proof of (C)

Case C.1: $t = \omega_\beta$ *for* $\beta \in \Delta$. Assume $\alpha \leq \beta$. Let $X \subseteq \omega_\beta$ be a subset of cardinality ω_α. In case $\alpha = \beta$ we proceed as in case B.1. Now let $\alpha < \beta$ and

$\mu = \max \Delta \cap \beta$. For the elements of t^0 we define the following equivalence relations:

$$a \approx b \bmod \omega_\mu^{n-1} \quad \text{iff} \quad (a < b \text{ and } b < a + \omega_\mu^{n-1})$$
$$\text{or} \quad (b \leqslant a \text{ and } a < b + \omega_\mu^{n-1}).$$

The equivalence classes can be ordered canonically, because the equivalence relations are convex. Note that every equivalence class (except possibly the greatest) is isomorphic to ω_μ^{n-1}.

Case C.1.1: ω_α *is regular.* By Lemma 3.3 there is an $(n-1, \Delta)$-order-homogeneous subset $X' \subseteq X$ of cardinality ω_α. Since ω_α is regular, we may choose X' so that either (1) any two elements of X' are equivalent to each other (with respect to $\approx$-mod ω_μ^{n-1}), or (2) X' has with each equivalence class of $\approx$-mod ω_μ^{n-1} at most one element in common. Suppose X' satisfies (1). Then the $(n-1, \Delta)$-characteristic of X' is either $s + \omega_\beta$ or $\omega_\mu^{n-1} + s + \omega_\beta$ where $s \in h_\alpha^n(\omega_\mu^{n-1})$. In case that X' satisfies (2), the $(n-1, \Delta)$-characteristic of X' has the form $s + c + s + d + \omega_\beta$ for some $s \in Q_{n-1}^{\Delta \cap \beta}$. This implies the desired recursion.

Case C.1.2: ω_α *is singular.* Suppose $\omega_\delta = \operatorname{cf} \omega_\alpha$. By Lemma 3.3 there is an $(n-1, \Delta)$-quasi-homogeneous ordered family $\{X_i : i < \omega_\delta\}$ of subsets of X which has u-power ω_α. Now we proceed as in the previous case with $X' = \bigcup \{X_i : i < \omega_\delta\}$. Note that there arises an additional case (3). If X' does not satisfy (1) or (2) of C.1.1, then it can be chosen so that for all $a < b \in X'$: $a \approx b \bmod \omega_\mu^{n-1}$ iff there is some $i < \omega_\delta$ such that $a < b \in X_i$. The third case is reflected in the recursion for $h_\alpha^n(\omega_\beta)$ by the set Z_4. The details are left to the reader.

Case C.2: $t = \omega_\beta \cdot t_1$ *for* $\beta \in \Delta$ *or* $\beta = 0$. We proceed as in Case C.1. However, instead of $\approx$-mod ω_μ^{n-1} we use the equivalence relation $\approx$-mod ω_β. The details can be obtained from the proof of Lemma 4.4. This completes the proof of the lemma. □

For any Δ-polynomials $s, t \in C^\Delta(c_1, \ldots, c_m)$ we set $s \overset{n}{\sim} t$ iff $q_i(s)^0 \overset{n}{\sim} q_i(t)^0 \bmod G_\Delta^2$ for each i, $1 \leqslant i \leqslant m+1$. If we want to emphasize that t is considered as a representative of an equivalence class of $\overset{n}{\sim}$, then we write more precisely t/n instead of only t. Similarly, Z/n denotes the set of equivalence classes of Δ-polynomials arising from Z, $Z/n = \{t/n : t \in Z\}$. In the next lemma the relations $\overset{n}{\sim}$ are characterized by means of the functions f^n, g_α^n, and h_α^n.

Lemma 5.7. *Let* $s, t \in C^\Delta(c_1, \ldots, c_m)$ *and* $n \geqslant 1$. *Then* $s \overset{n}{\sim} t$ *iff* $f^n(q_i(s))/n-1 = f^n(q_i(t))/n-1$, $g_\alpha^n(q_i(s))/n-1 = g_\alpha^n(q_i(t))/n-1$, *and* $h_\alpha^n(q_i(s))/n-1 = h_\alpha^n(q_i(t))/n-1$ *for every* i, $1 \leqslant i \leqslant m+1$, *and every ordinal* $\alpha \in \Delta$.

Proof. By definition $s \overset{n}{\sim} t$ iff $q_i(s) \overset{n}{\sim} q_i(t)$ for all i, $1 \leqslant i \leqslant m+1$. Hence, it is sufficient to prove the lemma for Δ-polynomials without constants. We may, therefore, assume $s, t \in R^\Delta$. Now suppose $s \overset{n}{\sim} t$.

Claim 1. $f^n(s)/n-1 \subseteq f^n(t)/n-1$.

Let $s' = s_1 + c + s_2 \in f^n(s)$. By Property (I) of f^n there is some $a \in s^0$ so that $\langle a, s^0\rangle \overset{n-1}{\sim} \langle c^0, s'^0\rangle$. By the hypothesis there is some $b \in t^0$ satisfying $\langle a, s^0\rangle \overset{n-1}{\sim} \langle b, t^0\rangle$. By Property (I) there is some $t' = t_1 + c + t_2 \in f^n(t)$ such that $\langle b, t^0\rangle \overset{n-1}{\sim} \langle c^0, t'^0\rangle$. Thus $t_1 \overset{n-1}{\sim} s_1$ and $t_2 \overset{n-1}{\sim} s_2$, and finally $s' \overset{n-1}{\sim} t'$. This means that $s'/n-1 \in f^n(t)/n-1$.

Claim 2. *$h^n_\alpha(s)/n-1 \subseteq h^n_\alpha(t)/n-1$ for each ordinal $\alpha \in \Delta$ with ω_α regular.*

Let $s' = s_1 + c + s_2 + d + s_3 \in h^n_\alpha(s)$. By Property (IIIa) of h^n_α there is some subset $X' \subseteq s^0$ of cardinality ω_α so that for all $a_1 < a_2 \in X'$, $\langle a_1, a_2, t^0\rangle \overset{n-1}{\sim} \langle c^0, d^0, s'^0\rangle$. Since $s \overset{n}{\sim} t$ there is some subset $Y \subseteq t^0$ of cardinality ω_α with property $\langle Y, t^0\rangle \lesssim^{n-1} \langle X', s^0\rangle$. Again by Property (IIIa) of h^n_α there is some $t' = t_1 + c + t_2 + d + t_3$ in $h^n_\alpha(t)$ which satisfies (3). Then we can conclude $t_1 \overset{n-1}{\sim} s_1$, $t_2 \overset{n-1}{\sim} s_2$, and $t_3 \overset{n-1}{\sim} s_3$. Hence $s'/n-1 = t'/n-1 \in h^n_\alpha(t)/n-1$.

By symmetry we get the inverse inclusions, hence equality. The remaining cases are verified in the same way. Now, on the other hand, suppose $f^n(s)/n-1 = f^n(t)/n-1$, $g^n_\alpha(s)/n-1 = g^n_\alpha(t)/n-1$, and $h^n_\alpha(s)/n-1 = h^n_\alpha(t)/n-1$ for each $\alpha \in \Delta$. We show $s \lesssim^{n-1} t$.

(i) Let $a \in s^0$. By Property (I) of f^n there is some $s' = s_1 + c + s_2 \in f^n(s)$ such that $\langle a, s^0\rangle \overset{n-1}{\sim} \langle c^0, s'^0\rangle$. Since $f^n(s)/n-1 = f^n(t)/n-1$ there is some $t' = t_1 + c + t_2 \in f^n(t)$ $(n-1)$-equivalent to s'. Again by Property (I) there is some $b \in t^0$ satisfying $\langle b, t^0\rangle \overset{n-1}{\sim} \{c^0, t'^0\rangle$. Thus $\langle a, s^0\rangle \overset{n-1}{\sim} \langle b, t^0\rangle$.

(ii) Let $X \subseteq s^0$ with $|X| = \omega_\alpha$, $\alpha \in \Delta$. Suppose ω_α is regular. By Property (IIIa) of h^n_α there is some $s' = s_1 + c + s_2 + d + s_3 \in h^n_\alpha(s)$ and a subset $X' \subseteq X$ of cardinality ω_α such that for all $a_1 < a_2 \in X'$ it holds

$$\langle a_1, a_2, s^0\rangle \overset{n-1}{\sim} \langle c^0, d^0, t'^0\rangle.$$

Since $h^n_\alpha(s)/n-1 = h^n_\alpha(t)/n-1$ there is some $t' \in h^n_\alpha(t)$ $(n-1)$-equivalent to s'. By Property (IIIa) we find some subset $Y \subseteq t^0$ of cardinality ω_α so that for all $b_1 < b_2 \in Y$

$$\langle b_1, b_2, t^0\rangle \overset{n-1}{\sim} \langle c^0, d^0, t'^0\rangle \quad \text{is valid.}$$

Hence $\langle Y, t^0\rangle \lesssim^{n-1} \langle X, s^0\rangle$. The singular case is treated similarly. This proves $s \lesssim^{n-1} t$. In the same way we can derive $t \lesssim^{n-1} s$. This completes the proof of the lemma. □

For Δ-polynomials t and s we can define the substitution of the ith part of t by s. The resulting Δ-polynomial is denoted by $t(q_i:s)$. More precisely, let $t \in C^\Delta(c_1, \ldots, c_m)$ and $s \in C^\Delta(d_1, \ldots, d_k)$ where t can be represented as $t_1 + c_1 + \cdots + c_m + t_{m+1}$. Then $t(q_i:s)$ denotes the Δ-polynomial $t_1 + \cdots + c_i +$

$s + c_{i+1} + \cdots + t_{m+1}$ provided that the constants $d_1, \ldots, d_k$ are different from each of the constants $c_1, \ldots, c_m$.

Lemma 5.8. *Let $t \in C_n^{\Delta}(c_1, \ldots, c_m)$, $n \geq 1$. Suppose $\varphi(x)$ and $\psi(x, y)$ are formulas of $L_{\Delta}^2(c_1, \ldots, c_m)$ with quantifier ranks at most $n-1$. Then:*

(I) $t \models \exists x\, \varphi(x)$ iff $t \models \varphi(c_j)$ for some $j \leq m$, or there are some $i \leq m+1$ and $s(c) \in f^n(q_i(t))$ such that $t(q_i : s(c)) \models \varphi(c)$.

(II) $t \models Q_\alpha x\, \varphi(x)$ iff there are some $i \leq m+1$ and $s(c) \in g_\alpha^n(q_i(t))$ such that $t(q_i : s(c)) \models \varphi(c)$.

(III) For ω_α regular, $t \models Q_\alpha^2 xy\, \psi(x, y)$ iff there are some $i \leq m+1$ and $s(c, d) \in h_\alpha^n(q_i(t))$ such that $t(q_i : s(c, d)) \models \psi(c, d)$.

(IV) For ω_α singular, $t \models Q_\alpha^2 xy\, \psi(x, y)$ iff there are some $i \leq m+1$ and $(r, s) \in h_\alpha^n(q_i(t))$ such that $t(q_i : r(c, d)) \models \psi(c, d)$ and $t(q_i : s(c, d)) \models \psi(c, d)$.

Proof. Suppose $t \models \exists x\, \varphi(x)$. Then there is some $a \in t$ with (1) $t \models \varphi(a)$. Assume $a \in q_i(t)$ for some $i \leq m+1$. By Property (I) of f^n there is some $s(c) \in (q_i(t))$ such that $\langle a, q_i(t) \rangle \overset{n-1}{\sim} \langle c, s(c) \rangle$. Since then

$$\langle c_1, \ldots, c_i, a, c_{i+1}, \ldots, c_m, t \rangle \overset{n-1}{\sim} \langle c_1, \ldots, c_i, c, c_{i+1}, \ldots, c_m, t(q_i : s(c)) \rangle,$$

we can conclude $t(q_i : s(c)) \models \varphi(c)$ using Property 1.3 and (1). The converse implication follows similarly. This proves (I). The remaining parts (II)–(IV) are derived in the same way using the properties of g_α^n and h_α^n. □

In the following let $K = \{c_1, c_2, \ldots\}$ be a countable set of new constants. Then $C^{\Delta}(K)$ denotes the set of Δ-polynomials with constants from K (in their natural ordering given by the enumeration). Now we can state the main theorem.

Theorem 5.9. *There is a decision procedure which effectively decides "$t \models \varphi$" for any Δ-polynomial t from $C^{\Delta}(K)$ and any sentence φ of $L_{\Delta}^2(K)$.*

Proof. We may assume that φ is defined in t. If φ is atomic, then "$t \models \varphi$" can be easily decided. Now by induction on the complexity of sentences: in case that φ is a conjunction $\varphi_1 \wedge \varphi_2$ or a negation $\neg\psi$, respectively, then "$t \models \varphi$" can be decided using the induction hypotheses. There remain the following two cases (we suppose $t \in C^{\Delta}(c_1, \ldots, c_m)$):

Case 1: $\varphi = \exists x\, \psi(x, c_1, \ldots, c_m)$ *and* $q(\varphi) = n$. By the preceding lemma, $t \models \varphi$ iff (1) $t \models \psi(c_j, c_1, \ldots, c_m)$ for some $j \leq m$ or (2) there are some $i \leq m+1$ and some $s(c) \in f^n(q_i(t))$, respectively, so that $t(q_i : s(c)) \models \psi(c, c_1, \ldots, c_m)$. For given t, $f^n(q_i(t))$ can be effectively calculated, and $t \models \psi(c_j, c_1, \ldots, c_m)$ and $t(q_i : s(c)) \models \psi(c, c_1, \ldots, c_m)$ can be effectively decided by the induction hypothesis (after renaming). Hence $t \models \varphi$ is also effectively decidable.

Case 2: $\varphi = Q_\alpha^2 xy\, \psi(x, y, c_1, \ldots, c_m)$ *for some* $\alpha \in \Delta$ *and* $q(\varphi) = n$. Again by Lemma 5.8, "$t \models \varphi$" has an equivalent representation which is decidable by the induction hypothesis. □

As an easy consequence we get:

Corollary. *Every* Δ*-polynomial has a decidable theory in* L_Δ^2.

Moreover, we can state our desired result.

Theorem 5.10. *Let* Δ *be a finite set of ordinals. Then*:

(a) *For every natural number* m, $\mathrm{Th}_\Delta^m(\mathrm{WO})$ *is decidable.*

(b) $\mathrm{Th}_\Delta^{<\omega}(\mathrm{WO})$ *is decidable.*

Proof. (b) is a consequence of (a). Part (a) is proved only for $m = 2$. In case $m > 2$ we have to generalize all the necessary facts to $\overset{n}{\sim}$-mod G_Δ^m (this requires much more notation but no new ideas). Alternatively we can prove that the elimination of Q_α^m with respect to L_Δ^2, $m > 2$, can be carried out effectively.

Now let φ be a sentence of L_Δ^2 with quantifier rank n. According to Lemma 5.2 and Proposition 1.3, the sentence φ is valid in all well-orderings iff "$t \models \varphi$" is true for all $t \in R_n^\Delta$. However, this is effectively decidable by Theorem 5.9 and the finiteness of R_n^Δ .Thus, for every sentence φ we can effectively decide after finitely many steps whether it is true in all well-orderings or not. □

Note that we get further decidable theories of well-orderings by taking finite extensions of $\mathrm{Th}_\Delta^m(\mathrm{WO})$, for instance the theories of the classes of infinite, countable, or uncountable well-orderings, respectively, are such finite extensions.

6. Further decidability results

From the preceding section the question arises whether the theory of all linear orderings in a language with Malitz quantifiers is also decidable. The question is answered affirmatively for languages which only contain quantifiers with interpretations in regular cardinalities. In the following we assume that Δ is a finite set of ordinals such that ω_α is regular for each $\alpha \in \Delta$. Furthermore let $0 \in \Delta$. Contrary to the case of well-orderings additional set-theoretical axioms are required to make sure that there are no independent sentences in $L_\Delta^{<\omega}$. For certain extensions of ZFC the decidability of the theory of all linear orderings in the language L_Δ^1 was proved in [18]. For simplicity in this section we adopt the GCH although weaker assumptions would be sufficient. The main result will be the decidability of the theory of all linear orderings in the language $L_\Delta^{<\omega}$. In fact we will only prove it for L^2_Δ, but everything can be easily generalized to L^m_Δ. We restrict us to L^2_Δ for notational convenience. The proof of the decidability is as follows. We will single out a set P^Δ of 'simple' linear orderings. It will turn out that every sentence, which has an ordered set as a model, has already a model in P^Δ, i.e. topologically speaking, P^Δ is dense in the model space of linear orderings. In this sense P^Δ corresponds to R^Δ of the preceding section. The orderings in P^Δ carry additional informations which are used to form a decision procedure.

The main defect of R^Δ (with respect to the class of all linear orderings) is that it does not contain dense linear orderings. To get P^Δ we require some well-behaved dense linear orderings which we will introduce now. Let A be an arbitrary linear ordering and $W_1, \ldots, W_k$ be a partition of A, i.e., they are pairwise disjoint non-empty subsets and their union is A. The partition is said to be minimal if $|W_i \cap (a, b)_A| = |W_i|$ for every $a < b \in A$ and every i, $1 \leq i \leq k$. Clearly, in this case all W_i have to be dense in A, thus $|A| \leq 2^{|W_i|}$. by the GCH, $2^{|W_i|} = |W_i|^+$, hence either $|W_i| = |A|$ or $|W_i|^+ = |A|$. Remark: this is the main point where we use the GCH.

To have a better reference we state the property above in a proposition.

Proposition 6.1 (GCH). *Let A be a linear ordering and $W_1, \ldots, W_k$ be a minimal partition of A. Then for every i, $1 \leq i \leq k$, either $|W_i| = |A|$ or $|W_i|^+ = |A|$.*

For every regular cardinal ω_α and for all $k, l \geq 0$ with $l + k > 0$ let $S^{k,l}_\alpha$ be a fixed linear ordering without endpoints which possesses a minimal partition $U_1, \ldots, U_k, V_1, \ldots, V_l$ so that

(i) $|U_i| = \omega_{\alpha-1}$ for each i, $1 \leq i \leq k$, and

(ii) $|V_j| = \omega_\alpha$ for each j, $1 \leq j \leq l$

(if α is a limit ordinal condition (i) is omitted).

A proof of the existence of $S^{k,l}_\alpha$ can be found in [2].

Partitions of linear orderings enable us to form various sorts of products. Let $W_1, \ldots, W_m$ be a partition of A and $H = \langle C_1, \ldots, C_m \rangle$ a finite sequence of linear orderings. Then $A(H)$ denotes the product ordering arising from A by substituting for every element of W_i, $1 \leq i \leq m$, a copy of C_i. In the particular case that A is $S^{k,l}_\alpha$ the above product is denoted by $S_\alpha(F, G)$ where $F = \langle A_1, \ldots, A_k \rangle$ and $G = \langle B_1, \ldots, B_l \rangle$ are two sequences of linear orderings (k and l are omitted because they are uniquely determined by the sequences F and G). The product $S_\alpha(F, G)$ is called the shuffling of (F, G).

Now we define the set P^Δ of so-called Δ-terms that can be considered as canonical names of certain 'simple' linear orderings.

Definition 6.1. (1) **1** is a Δ-term.

(2) If t is a Δ-term, then $t \cdot w_\alpha$ and $t \cdot w^*_\alpha$ are Δ-terms for each $\alpha \in \Delta$.

(3) If t_1 and t_2 are Δ-terms, then $t_1 + t_2$ is a Δ-term.

(4) If F and G are finite sequences of Δ-terms (not both are the empty sequence), then $S_\alpha(F, G)$ is a Δ-term for each $\alpha \in \Delta$.

(5) Nothing else is a Δ-term.

For every Δ-term t there is a linear ordering t^0 which is defined inductively as follows.

(1) $\mathbf{1}^0$ denotes an ordering with a one-element domain, say the ordinal 1.

(2) If $t = s \cdot w_\alpha$ or $t = s \cdot w_\alpha^*$, then $t^0 := s^0 \cdot \omega_\alpha$ or $t^0 := s^0 \cdot \omega_\alpha^*$, respectively.

(3) If $t = t_1 + t_2$, then $t^0 := t_1^0 + t_2^0$.

(4) If $t = S_\alpha(F, G)$ where $F = \langle s_1, \ldots, s_k \rangle$ and $G = \langle t_1, \ldots, t_l \rangle$ are sequences of Δ-terms, then $t^0 := S_\alpha(F^0, G^0)$.

Remark. $+$, $\cdot$, and S_α are used as symbols in the definition of P^Δ, and at the same time they denote operations on the class of linear orderings. In the following we shall not distinctly distinguish between t and t^0. Sometimes, to avoid additional considerations, we require a term $\mathbf{0}$ representing the ordering with empty domain. In such cases we assume that $\mathbf{0}$ belongs to P^Δ. In this paper we are not interested in the 'word problem' of Δ-terms, i.e., in the various different representations of linear orderings as Δ-terms.

To reduce the notation we make the following simplifications: Since Δ is fixed throughout this section, it is always omitted if possible. Furthermore, we restrict our considerations to L^2_Δ and the relations $\overset{n}{\sim}$-mod G^2_Δ, thus mod G^2_Δ can be also omitted.

In the following many proofs will be proved by induction on n. To avoid endless repetitions of trivial facts the cases $n = 0$ will be regarded as solved. Assuming the theorems proved for $n - 1$ the induction steps for n will be shown.

Usually we have to prove equivalences of the form $A \overset{n}{\sim} B$. This will be done showing $A \lesssim^n B$. Then by symmetry $B \lesssim^n A$, hence $A \overset{n}{\sim} B$.

To show $A \lesssim^n B$ we have to verify the two conditions (i) and (ii) of the definition of $\lesssim^n$. For short we call (i) the element condition (E-condition) and (ii) the quantifier condition (Q-condition). For the Q-conditions we make further simplifications.

In most cases we have situations like the following: we are given a linear ordering A, a subset $X \subseteq A$, $|X| = \omega_\alpha$ for some $\alpha \in \Delta$, and a monotone mapping $f: A \to C$. By the regularity of ω_α either (1) $|X \cap f^{-1}(c)| = \omega_\alpha$ for some $c \in C$ or (2) $|f(X)| = \omega_\alpha$. In the first case we may additionally assume that $X \subseteq f^{-1}(c)$ and that X is (n, Δ)-order-homogeneous in the subordering $f^{-1}(c) \subseteq A$. In the second case we may assume that $|f^{-1}(c) \cap X| \leq 1$ for all $c \in C$ and that for all $a < b \in X$

$$\langle a, f^{-1}(f(a)) \rangle \overset{n}{\sim} \langle b, f^{-1}(f(b)) \rangle.$$

Subsets X with the properties above are called n-sections over C. Let $X \subseteq A$,

$Y \subseteq B$, $f: A \to C$, and $g: B \to D$ be given. Suppose X and Y are n-sections over C and D, respectively, such that for all $a \in X$ and all $b \in Y$

$$\langle a, f^{-1}(f(a)) \rangle \overset{n}{\sim} \langle b, g^{-1}(g(b)) \rangle.$$

Then X and Y are said to be n-equivalent.

To show that P^{Δ} is dense in the model space of linear orderings we need some basic properties of $\overset{n}{\sim}$ which we are going to prove now. To clarify the proofs of the following lemmata the reader is advised to draw pictures for himself.

Lemma 6.2. *Let A, B, and C be linear orderings. Suppose $f: A \to C$ and $g: B \to C$ are monotone mappings such that for each $c \in C$ the equivalence $f^{-1}(c) \overset{n}{\sim} g^{-1}(c)$ holds. Then $A \overset{n}{\sim} B$.*

Proof. By induction on n. We show $A \lesssim^n B$.

E-condition. Let $a \in A$ and $c = f(a)$. Then there is some $b \in g^{-1}(c)$ with $\langle a, f^{-1}(c) \rangle \overset{n-1}{\sim} \langle b, g^{-1}(c) \rangle$. Hence $f \restriction A^{<a}$ and $g \restriction B^{<b}$ satisfy the hypothesis of the lemma for $n-1$. By the induction hypothesis $A^{<a} \overset{n-1}{\sim} B^{<b}$. For the same reason $A^{>a} \overset{n-1}{\sim} B^{>b}$, thus $\langle a, A \rangle \overset{n-1}{\sim} \langle b, B \rangle$.

Q-condition. Let $X \subseteq A$. $|X| = \omega_\alpha$ for some $\alpha \in \Delta$.

Q1: *$X \subseteq f^{-1}(c)$ for some $c \in C$.* Choose $Y \subseteq g^{-1}(c)$, $|Y| = \omega_\alpha$ so that

$$\langle Y, g^{-1}(c) \rangle \lesssim^{n-1} \langle X, f^{-1}(c) \rangle.$$

This is possible because $f^{-1}(c) \overset{n}{\sim} g^{-1}(c)$ by the hypothesis. By the induction hypothesis $\langle Y, B \rangle \lesssim^{n-1} \langle X, A \rangle$ immediately follows.

Q2: *X is an n-section over C.* For every $a \in X$ we can choose an element $b \in B$ so that

$$g(b) = f(a) = c \quad \text{and} \quad \langle a, f^{-1}(c) \rangle \overset{n-1}{\sim} \langle b, g^{-1}(c) \rangle.$$

Thus there is an $(n-1)$-section $Y \subseteq B$ over C, $|Y| = \omega_\alpha$, which is $(n-1)$-equivalent to X, hence $\langle Y, B \rangle \lesssim^{n-1} \langle X, A \rangle$ by the induction hypothesis. □

Corollary. *The relation $\overset{n}{\sim}$ is compatible with the operations $+$, $\cdot\, \omega_\alpha$, and $\cdot\, \omega_\alpha^*$ for any ordinal α.*

Proof. Use the canonical mappings $A + B \to \mathbf{2}$, $A \cdot \omega_\alpha \to \omega_\alpha$, and $A \cdot \omega_\alpha^* \to \omega_\alpha^*$ and apply the preceding lemma. □

In the same way we can prove the following two lemmata.

Lemma 6.3. *Let A, B, C, and D be linear orderings with $C \overset{n}{\sim} D$. Suppose there are monotone onto mappings $f: A \to C$ and $g: B \to D$, respectively, so that $f^{-1}(c) \overset{n}{\sim} g^{-1}(d)$ for all $c \in C$ and all $d \in D$. Then $A \overset{n}{\sim} B$.*

Lemma 6.4. *Let A, B, C, and D be linear orderings without endpoints. Suppose there are monotone onto mappings $f:A\to C$ and $g:B\to D$, respectively, and minimal partitions $U_1,\ldots,U_k$ and $V_1,\ldots,V_k$ of C and D, respectively, so that*
(i) $|U_i|\geqslant\omega_\alpha$ *iff* $|V_i|\geqslant\omega_\alpha$ *for every* $\alpha\in\Delta$ *and every i,* $1\leqslant i\leqslant k$,
(ii) $f^{-1}(c)\overset{n}{\sim}g^{-1}(d)$ *for all* $c\in U_i$ *and all* $d\in V_i$*, for each i,* $1\leqslant i\leqslant k$.
Then $A\overset{n}{\sim}B$.

The next lemma is a crucial step in the proof that P^{Δ} is dense in the model space of linear orderings. To state the lemma we require the following notions. Let C be any linear ordering and $C'=C+1$. The greatest element of C' is denoted by c. For every ordinal λ define

$$C_\lambda=(C'\cdot(\omega^*+\omega))\cdot\lambda.$$

The ordering C_λ can be identified with the cross product $\lambda\times(\mathbb{Z}\times C')$ ordered lexicographically (where $\mathbb{Z}$ is the set of integers).

$v_{\alpha,k}$ and v_α denote the elements $\langle\alpha,k,c\rangle$ and $\langle\alpha,0,c\rangle$, respectively, of C_λ. Furthermore, set $V=\{v_\alpha:\alpha<\lambda\}$. Obviously V is a strictly increasing cofinal sequence of length λ in C_λ. Now let A be a given linear ordering with a strictly increasing cofinal λ-sequence $U=\{u_\alpha:\alpha<\lambda\}$ which is in addition (n,Δ)-order-homogeneous. Suppose C was chosen so that $C\overset{n}{\sim}(u_0,u_1)_A$. Define $B=A^{\leqslant u_0}+C'\cdot\omega+C_\lambda$. We are mainly interested in the elements of B which belong to the part C_λ. For simplicity we keep on the same notation for the elements of C_λ if they occur as elements of the sum B. We assume that λ is a regular cardinal.

Lemma 6.5. (i) *V is (n,Δ)-order-homogeneous and*
(ii) $A\overset{n}{\sim}B$.

The proof breaks into several cases. Most of them are similar to ones occurring in the preceding lemmata. For each case, therefore, we only give some hints and then refer to the corresponding case already proved.

Proof. By induction on n we prove $(v_\alpha,v_\beta)_B\overset{n}{\sim}(u_0,u_1)_A$ for all $\alpha<\beta<\lambda$. In the following we will omit the subscripts A and B at the intervals.

Claim 1. $(u_0,u_3)\lesssim^n(v_\alpha,v_\beta)$ *for all* $\alpha<\beta<\lambda$.

E-condition. Let $a\in(u_0,u_3)$. Then the following cases arise: (1) $a\in(u_0,u_1)$, (2) $a\in(u_1,u_2)$, (3) $a\in(u_2,u_3)$, (4) $a=u_1$, or (5) $a=u_2$. Then choose $b\in(v_\alpha,v_{\alpha,1})$ (1), $b\in(v_{\alpha,1},v_{\alpha,2})$ (2), $b\in(v_{\beta,-1},v_\beta)$ (3), $b=v_{\alpha,1}$ (4), or $b=v_{\beta,-1}$, respectively, and then argue as in the preceding lemmata using the induction hypothesis.

Q-condition. Let $X\subseteq(u_0,u_3)$, $|X|=\omega_\delta$ for some $\delta\in\Delta$. We may distinguish the following cases: (1) $X\subseteq(u_0,u_1)$, (2) $X\subseteq(u_1,u_2)$, or (3) $X\subseteq(u_2,u_3)$. All cases are like the Q1-cases of the preceding lemmata.

Claim 2. $(v_0, v_1) \lesssim^n (u_0, u_{\omega+2})$.

E-condition. Let $a \in (v_0, v_1)$. We have the following cases: (1) $a \in (v_{0,k}, v_{0,k+1}]$ for some $k \geq 0$ or (2) $a \in [v_{1,k-1}, v_{1,k})$ for some $k \leq 0$.

Both cases are treated as in Claim 1. As an example we discuss the case $a \in (v_{1,-5}, v_{1,-4})$. Since $(v_{1,-5}, v_{1,-4}) \overset{n}{\sim} (u_0, u_1)$ and U is (n, Δ)-order-homogeneous, we find some $b \in (u_\omega, u_{\omega+1})$ satisfying

$$\langle a, (v_{1,-5}, v_{1,-4})\rangle \overset{n-1}{\sim} \langle b, (u_\omega, u_{\omega+1})\rangle.$$

By the induction hypothesis $(v_0, v_{1,-5}) \overset{n-1}{\sim} (u_0, u_\omega)$ and $(v_{1,-4}, v_1) \overset{n-1}{\sim}$ $(u_{\omega+1}, u_{\omega+2})$, thus

$$\langle a, (v_0, v_1)\rangle \overset{n-1}{\sim} \langle b, (u_0, u_{\omega+2})\rangle.$$

Q-condition. Let $X \subseteq (v_0, v_1)$, $|X| = \omega_\delta$ for some $\delta \in \Delta$. We may restrict us to the following cases (other cases are reduced to them by taking subsets): (1) $X \subseteq (v_{0,k}, v_{0,k+1})$ for some $k \geq 0$, (2) $X \subseteq (v_{1,k-1}, v_{1,k})$ for some $k \leq 0$, (3) X is countable, i.e. $\omega_\delta = \omega$, and for all $k \geq 0$ $X \cap (v_{0,k}, v_{0,k+1})$ contains at most one element whereas $X \cap (v_{1,-k-1}, v_{1,-k})$ is empty, or (4) X is countable, i.e. $\omega_\delta = \omega$, and for all $k \geq 0$ $(v_{0,k}, v_{0,k+1})$ is empty whereas $X \cap (v_{1,-k-1}, v_{1,-k})$ contains at most one element.

The cases (1) and (2) are like the Q1-cases of the preceding lemmata. On the other hand, (3) and (4) are treated like Q2-cases of the preceding lemmata: we choose $Y \subseteq (u_0, u_\omega)$ so that for each $k \geq 0$, $Y \cap (u_{2k+1}, u_{2k+2})$ contains exactly one element and $Y \cap (u_{2k}, u_{2k+1})$ is empty, i.e., Y is something like an $(n-1)$-section which is $(n-1)$-equivalent to X.

Claim 3. $(v_\alpha, v_\beta) \lesssim^n (u_\alpha, u_{\beta+2})$ for every $\alpha < \beta < \lambda$.

We prove the claim by transfinite induction on β. If β is a successor ordinal, then the claim easily follows from the induction hypothesis on $\beta - 1$ (in case that $\beta > 1$) or from Claim 2 (for $\beta = 1$) using the (n, Δ)-order-homogeneity of U. There remains the case that β is a limit.

E-condition. Let $a \in (v_\alpha, v_\beta)$. Then either (1) $a \in (v_\alpha, v_{\gamma,k})$ for some $\alpha \leq \gamma < \beta$ and some $k \geq 0$ or (2) $a \in (v_{\beta,k-1}, v_{\beta,k})$ for some $k \leq 0$.

In Case (1) the induction hypothesis on γ is used. Case (2) is similar to Case (2) of the E-condition of Claim 2.

Q-condition. Let $X \subseteq (v_\alpha, v_\beta)$, $|X| = \omega_\delta$ for some $\delta \in \Delta$. We may restict us to the following cases:

Case 1. $X \subseteq (v_\alpha, v_\gamma)$ for some $\alpha < \gamma < \beta$.

Case 2. $|X \cap (v_\alpha, v_\gamma)| < \omega_\delta$ for all $\gamma < \beta$ but $|\bigcup_{\gamma<\beta} (X \cap (v_\alpha, v_\gamma))| = \omega_\delta$.

Case 3. $X \subseteq (v_{\beta,k-1}, v_{\beta,k})$ for some $k \leq 0$.

Case 4. X is countable, i.e. $\omega_\delta = \omega$, and for each $k \leq 0$ $X \cap (v_{\beta,k-1}, v_{\beta,k})$ contains at most one element, and X is a subset of $\bigcup_{k\leq 0} (v_{\beta,k-1}, v_{\beta,k})$.

The first case is solved using the induction hypothesis for γ. In Case 2 we define a monotone mapping $f:(v_\alpha, v_\beta) \to (\alpha, \beta)$ where (α, β) denotes the interval between α and β in the class of ordinals. Then we may assume that X is an n-section over (α, β) and proceed as in the Q2-cases of the preceding lemmata. The cases 3 and 4 are similar to the cases (2) and (3), respectively, of the Q-condition of Claim 2.

This proves the claim for limit β.

Since U is (n, Δ)-order-homogeneous the third claim implies that $(v_\alpha, v_\beta) \lesssim^n (u_0, u_1)$. From Claim 1 we get the relation $(u_0, u_1) \lesssim^n (v_\alpha, v_\beta)$, hence $(v_\alpha, v_\beta) \overset{n}{\sim} (u_0, u_1)$ as stated.

In the same way we can prove that $A^{>u_\alpha} \overset{n}{\sim} B^{>v_\alpha}$. The initial part of B is isomorphic to $A^{\leq u_0}$. Let u_0' be the greatest element of this initial part. Clearly, $(u_0', v_0) \overset{n}{\sim} (u_0, u_1)$. Thus for all $\alpha < \lambda$, $A^{<u_\alpha} \overset{n}{\sim} B^{<v_\alpha}$. We can conclude that V is (n, Δ)-order-homogeneous. Moreover it follows $A \overset{n}{\sim} B$. This completes the proof. □

Definition 6.2. A linear ordering A is said to be (n, Δ)-term-like iff there is some Δ-term $t \in P^\Delta$ so that $A \overset{n}{\sim} t$.

The (n, Δ)-term-like linear orderings form a subclass of the class of all linear orderings. As we shall see below it is not a proper subclass.

Lemma 6.6. *The class of (n, Δ)-term-like linear orderings is closed under $+$, multiplication with ω_α and ω_α^*, respectively, and under the shuffling operation $S_\alpha(F, G)$ for every $\alpha \in \Delta$.*

Proof. By the corollary to Lemma 6.2 and Lemma 6.4. □

Lemma 6.7. *Let A be any linear ordering. If every bounded segment of A is (n, Δ)-term-like, then A itself is (n, Δ)-term-like.*

Proof. We may assume that A has a least element. The proof is similar in the other cases: if A has a greatest element, then we use the corresponding results for the inverse orderings; in case that A has neither least nor greatest element we partition $A = B + C + D$ where B has a greatest element, D has a least element, and C is bounded. If A has a greatest element, then we are done. Otherwise there is a strictly increasing cofinal sequence $U \subseteq A$. Let U be of shortest length, say λ. Clearly, λ is a regular cardinal. By Theorem 3.3(i) we may assume that U is (n, Δ)-order-homogeneous. Furthermore set $C = (u_0, u_1)_A$. Then

$$A \overset{n}{\sim} A^{\leq u_0} + C' \cdot \omega + C' \cdot (\omega^* + \omega) \cdot \lambda$$

by the preceding lemma. Since every bounded segment of A is (n, Δ)-term-like there are terms t_0 and t_1 so that $A^{\leq u_0} \overset{n}{\sim} t_0$ and $C' \overset{n}{\sim} t_1$, respectively. Thus

$$A \overset{n}{\sim} (t_0 + t_1 \cdot w + t_1 \cdot (w^* + w) \cdot \lambda$$

by Lemma 6.2. From the preceding section we know that there is some $s \in R^{\Delta}$ such that $s \overset{n}{\sim} \lambda$, hence using Lemma 6.3 we can conclude

$$A \overset{n}{\sim} t_0 + t_1 \cdot w + t_1 \cdot (w^* + w) \cdot s.$$

Thus A is (n, Δ)-term-like. □

Lemma 6.8. *Every linear ordering is (n, Δ)-term-like.*

Proof. We prove the lemma in a series of steps. Let A be a given linear ordering. We define the following equivalence relation $\approx$ on A: $x \approx y$ iff (i) $x = y$, (ii) $x < y$ and every segment of the closed interval $[x, y]_A$ is (n, Δ)-term-like , or (iii) $y < x$ and every segment of the closed interval $[y, x]_A$ is (n, Δ)-term-like.

We are going to prove that all elements of A are equivalent to each other.

1. $\approx$ is an equivalence relation, and each of its equivalence classes is convex.

Proof. By the definition of $\approx$ and Lemma 6.6. □

2. Every equivalence class C has the property that each segment of it is (n, Δ)-term-like.

Proof. By the definition of $\approx$ each bounded segment of C is (n, Δ)-term-like, hence any segment of C is (n, Δ)-term-like by Lemma 6.7. □

3. Let $M = A/{\approx}$ be the quotient ordering, i.e., the set of equivalence classes with the canonical ordering. Then M has order type $\mathbf{1}$, or equivalently, all elements of A are equivalent to each other.

Proof. Assume on the contrary that M has at least two elements $C < D \in M$.

Case 1: *There are no elements between C and D.* Let $a \in C$ and $b \in D$ be any elements. Furthermore, let U be a segment of $[a, b]_A$. We show that U is (n, Δ)-term-like. If U is contained in C or D, respectively, then U is (n, Δ)-term-like by (2) above. Otherwise, C and D induce a partition on U with the parts $U \cap C$ and $U \cap D$ which are (n, Δ)-term-like by (2) above. Then U is also (n, Δ)-term-like by Lemma 6.6. Thus $a \approx b$, what contradicts $C < D$.

Case 2: *M is dense and $C < D \in M$ are chosen so that the cardinality of $|(C, D)_M|$ is minimal, say $|(C, D)_M| = \omega_\beta$.* Let $\Gamma_\Delta^{2,n} = \{E_1, \ldots, E_p\}$ be a set of linear orderings which is a set of representatives for $\overset{n}{\sim}$ (cf. Section 3). The set of those parts $B \in (C, D)_M$ satisfying $B \overset{n}{\sim} E_i$ is denoted by W_i, $1 \leqslant i \leqslant p$. Without loss of generality we assume that $W_1, \ldots, W_k$ are non-empty and $W_{k+1} = \cdots = W_p = \emptyset$ for some $k \leqslant p$. Now suppose $C < D \in M$ were chosen so that $W_1, \ldots, W_k$ form a partition of $(C, D)_M$ which is minimal. From (2) above it follows that there are Δ-terms $t_1, \ldots, t_k$ such that $t_i \overset{n}{\sim} E_i$ for each i, $1 \leqslant i \leqslant k$. By Proposition 6.1 for each i, $1 \leqslant i \leqslant k$, either $|W_i| = \omega_\beta$ or $|W_i| = \omega_{\beta-1}$ (if β is not a limit ordinal). We may assume that $|W_1| = \cdots = |W_l| = \omega_{\beta-1}$ and $|W_{l+1}| = \cdots = |W_k| = \omega_\beta$ for some $l \leq k$ (if there is no W_i with $|W_i| < \omega_\beta$, then set $l = 0$).

Now we distinguish the following cases:

Case 2.1: $\beta \in \Delta$. Let U be a segment of A so that $U_1 = U/\approx$ has no endpoints and belongs to $(C, D)_M$. By Lemma 6.4 we get

$$U \overset{n}{\sim} S_\beta(F, G)$$

where $F = \langle t_1, \ldots, t_l \rangle$ and $G = \langle t_{l+1}, \ldots, t_k \rangle$ (if $l = 0$ set $F = \emptyset$).

Case 2.2: $\beta \notin \Delta$. Let $\alpha \in \Delta$ be the greatest ordinal with $\alpha < \beta$. As in the preceding case let U be a segment of A so that $U_1 = U/\approx$ has no endpoints and $U_1 \subseteq (C, D)_M$. Again by Lemma 6.4 we can conclude that $U \overset{n}{\sim} S_\alpha(F, G)$ where $F = \emptyset$ and $G = \langle t_1, \ldots, t_k \rangle$, respectively.

Now let $a \in C$ and $b \in D$ be any elements. Furthermore, let U be a segment of $[a, b]_A$. $U_1 = U/\approx$ denotes the segment of $[C, D]_M$ induced by U. Using the results of the cases 2.1 and 2.2, together with (2) above, we infer U is (n, Δ)-term-like. Thus $a \approx b$ contradicting the hypothesis $C < D$.

Thus M cannot possess more than one element as stated. Thus A is (n, Δ)-term-like by (2) again. □

An application of Lemma 6.8 is given in the next theorem.

Theorem 6.9. *Each sentence φ of L^2_Δ, which has an ordered set as a model, has a model in P^Δ.*

Proof. Let A be a linear ordering which is a model of φ. Suppose $q(\varphi) = n$. By Lemma 6.8 there is some Δ-term $t \in P^\Delta$ so that $A \overset{n}{\sim} t$, thus $t \models \varphi$ by Proposition 1.3. □

We have reduced the decision problem for the class of all linear orderings to the set P^Δ of Δ-terms. This has been the first step in our solution of the decision problem. Now we are going to study the set P^Δ in detail. The set of equivalence classes of P^Δ with respect to the relation $\overset{n}{\sim}$ is denoted by P^Δ/n.

Lemma 6.10. *P^Δ/n is a finite algebraic structure with respect to the functions $+$, $\cdot\, w_\alpha$, $\cdot\, w^*_\alpha$, and $S^{k,l}_\alpha$ for each $\alpha \in \Delta$ and all natural numbers k and l with $k + l > 0$.*

Proof. By Lemma 3.1, P^Δ/n is finite. the functions are well-defined by Lemma 6.4 and the corollary of Lemma 6.2. □

For simplicity the elements of P^Δ/n are represented by Δ-terms instead of writing classes. Moreover, from now $\mathbf{0}$ is always included in the set of Δ-terms. For our decision procedure we require Δ-terms with constants which are defined by recursion.

Definition 6.3. The set $P^{\Delta}(c_1, \ldots, c_m)$ of Δ-terms with the constants $c_1, \ldots, c_m$ is the set of formal expressions described as follows:

$$P^{\Delta}(c_1, \ldots, c_m) = \{s + c_m + t : s \in P^{\Delta}(c_1, \ldots, c_{m-1}) \text{ and } t \in P^{\Delta}\}.$$

It is convenient to think of the elements of $P^{\Delta}(c_1, \ldots, c_m)$ as sums

$$t_1 + c_1 + t_2 + \cdots + t_m + c_m + t_{m+1}$$

where $t_1, \ldots, t_{m+1}$ are elements of P^{Δ}.

The terms $t(c_1, \ldots, c_m)$ are interpreted as linear orderings with distinguished constants as follows:

$$t^0(c_1, \ldots, c_m) := t_1^0 + c_1^0 + t_2^0 + \cdots + t_m^0 + c_m^0 + t_{m+1}^0$$

where c_i^0 are one-element orderings.

If Z is any set, then $P_\omega(Z)$ denotes the set of all finite subsets of Z. For every natural number n and every ordinal $\alpha \in \Delta$ we introduce the mappings $f^n : P^{\Delta} \to P_\omega(P^{\Delta}(c))$, $g_\alpha^n : P^{\Delta} \to P_\omega(P^{\Delta}(c))$, and $h_\alpha^n : P^{\Delta} \to P_\omega(P^{\Delta}(c, d))$ with the following properties:

(i) For each Δ-term t and every $a \in t^0$ there is some $s(c) \in f^n(t)$ so that

$$\langle a, t^0 \rangle \overset{n-1}{\simeq} \langle c^0, s^0 \rangle, \tag{1}$$

and for every $s(c) \in f^n(t)$ there is some $a \in t^0$ which satisfies (1).

(ii) For each Δ-term t and every subset $X \subseteq t^0$ of cardinality ω_α, $\alpha \in \Delta$, there is some $s(c) \in g_\alpha^n(t)$ and a subset $X' \subseteq X$ of cardinality ω_α so that

$$\text{for all } a \in X' \quad \langle a, t^0 \rangle \overset{n-1}{\simeq} \langle c^0, s^0 \rangle, \tag{2}$$

and for every $s(c) \in g_\alpha^n(t)$ there is a subset $X' \subseteq t^0$ of cardinality ω_α which satisfies (2).

(iii) For each Δ-term t and every subset $X \subseteq t^0$ of cardinality ω_α, $\alpha \in \Delta$, there is some $s(c, d) \in h_\alpha^n(t)$ and a subset $X' \subseteq X$ of cardinality ω_α so that

$$\text{for all } a < b \in X' \quad \langle a, b, t^0 \rangle \overset{n-1}{\simeq} \langle c^0, d^0, s^0 \rangle, \tag{3}$$

and for every $s(c, d) \in h_\alpha^n(t)$ there is a subset $X' \subseteq t^0$ of cardinality ω_α which satisfies (3).

The existence of f^n, g_α^n, and h_α^n is easily shown using Lemma 6.8. However, we will give a definition of f^n, g_α^n, and h_α^n by recursion on terms. As an immediate consequence we get the recursiveness of the mappings f^n, g_α^n, and h_α^n. The proof that they have in fact the properties (i)–(iii) uses arguments similar to those of the previous lemmata. It is similar to the proof of Lemma 5.6. The details are left to the reader.

1. We already have determined the values for **1**:

$$f^n(\mathbf{1}) = \{c\}, \qquad g_\alpha^n(\mathbf{1}) = \emptyset, \quad \text{and} \quad h_\alpha^n(\mathbf{1}) = \emptyset.$$

2. The values $f^n(w_\beta)$, $g^n_\alpha(w_\beta)$, and $h^n_\alpha(w_\beta)$ can be determined effectively using the results of the preceding section.

3. $f^n(w^*_\beta)$, $g^n_\alpha(w^*_\beta)$, and $h^n_\alpha(w^*_\beta)$ are determined in the same way as for w_β using the corresponding results for the inverse orderings.

4.
$$f^n(t_1+t_2)=\{t_1+s:s\in f^n(t_2)\}\cup\{s+t_2:s\in f^n(t_1)\},$$
$$g^n_\alpha(t_1+t_2)=\{t_1+s:s\in g^n_\alpha(t_2)\}\cup\{s+t_2:s\in g^n_\alpha(t_1)\},$$
$$h^n_\alpha(t_1+t_2)=\{t_1+s:s\in h^n_\alpha(t_2)\}\cup\{s+t_2:s\in h^n_\alpha(t_1)\}.$$

5.
$$f^n(t\cdot w_\beta)=\{t\cdot s_1+s(c)+t\cdot w_\beta:s(c)\in f^n(t) \text{ and } s_1+c+w_\beta\in f^n(w_\beta)\},$$
$$g^n_\alpha(t\cdot w_\beta)=\{t\cdot s_1+s(c)+t\cdot w_\beta:[s(c)\in g^n_\alpha(t) \text{ and } s_1+c+w_\beta\in f^n(w_\beta)] \text{ or } [s(c)\in f^n(t) \text{ and } s_1+c+w_\beta\in g^n_\alpha(w_\beta)]\},$$
$$h^n_\alpha(t\cdot w_\beta)=\{t\cdot s_1+s(c,d)+t\cdot w_\beta:s(c,d)\in h^n_\alpha(t) \text{ and } s_1+c+w_\beta\in f^n(w_\beta)\} \cup\{t\cdot s_1+t_1(c)+t\cdot s_2+t_1(d)+t\cdot w_\beta: s_1+c+s_2+d+w_\beta\in h^n_\alpha(w_\beta) \text{ and } t_1(c)\in f^n(t)\},$$

6. Let $F=\langle s_1,\ldots,s_k\rangle$ and $G=\langle t_1,\ldots,t_l\rangle$ be two sequences of Δ-terms so that $k+l>0$.

$$f^n(S_\beta(F,G))=\{S_\beta(F,G)+s(c)+S_\beta(F,G):s(c)\in f^n(s_i) \text{ for some } i\leqslant k \text{ or } s(c)\in f^n(t_j) \text{ for some } j\leqslant l\}$$
$$g^n_\alpha(S_\beta(F,G))=\{S_\beta(F,G)+s(c)+S_\beta(F,G):s(c)\in g^n_\alpha(s_i) \text{ for some } i\leqslant k \text{ or } s(c)\in g^n_\alpha(t_j) \text{ for some } j\leqslant l \text{ or } s(c)\in T_{\alpha,\beta}\}$$

where $T_{\alpha,\beta}$ is defined as follows:

$$T_{\alpha,\beta}=\emptyset \quad \text{if } \alpha>\beta,$$
$$T_{\alpha,\alpha}=\{s(c):s(c)\in f^n(t_j) \text{ for some } j\leqslant l\}, \quad \text{and}$$
$$T_{\alpha,\beta}=\{s(c):s(c)\in f^n(t_j) \text{ for some } j\leqslant l \text{ or } s(c)\in f^n(s_i) \text{ for some } i\leqslant k\} \quad \text{if } \alpha<\beta.$$

$$h^n_\alpha(S_\beta(F,G))=\{S_\beta(F,G)+s(c,d)+S_\beta(F,G):s(c,d)\in h^n_\alpha(s_i) \text{ for some } i\leqslant k \text{ or } s(c,d)\in h^n_\alpha(t_j) \text{ for some } j\leqslant l\} \cup\{S_\beta(F,G)+s(c)+S_\beta(F,G)+s(d)+S_\beta(F,G):s(c)\in T_{\alpha,\beta}\}$$

where $T_{\alpha,\beta}$ is the set defined above.

Now we continue as in the previous section. The part-functions q_i are defined for terms with constants in the same way as for composite Δ-polynomials. For any terms $s,t\in P^\Delta(c_1,\ldots,c_m)$ define $s\overset{n}{\sim}t$ iff $q_i(s)\overset{n}{\sim}q_i(t)$ for each i, $1\leqslant i\leqslant m+1$. The corresponding quotient sets are denoted by $P^\Delta/n(c_1,\ldots,c_m)$. By modifying the definitions of t/n and Z/n we get an analogue of Lemma 5.7.

Lemma 6.11. *Let $s, t \in P^{\Delta}(c_1, \ldots, c_m)$ and $n \geq 1$. Then $s \overset{n}{\sim} t$ iff $f^n(q_i(s))/n-1 = f^n(q_i(t))/n-1$, $g^n_\alpha(q_i(s))/n-1) = g^n_\alpha(q_i(t))/n-1$, and $h^n_\alpha(q_i(s))/n-1 = h^n_\alpha(q_i(t))/n-1$ for every i, $1 \leq i \leq m+1$, and every $\alpha \in \Delta$.*

Proof. See Lemma 5.7. □

Lemma 6.12. *There is a decision procedure which decides whether "$s \overset{n}{\sim} t$" is true or not where s, t, and n vary over Δ-terms (with constants) and natural numbers, respectively.*

Proof. We may assume that s and t contain exactly the same constants, say $c_1, \ldots, c_m$ (otherwise the relation in question is false). Now we proceed by induction on n. In case $n=0$ the equivalence $s \overset{n}{\sim} t$ is true for any $s,t \in P^{\Delta}(c_1, \ldots, c_m)$. Suppose $n \geq 1$. By the preceding lemma we have $s \overset{n}{\sim} t$ iff $f^n(q_i(s))/n-1 = f^n(q^i(t))/n-1$, $g^n_\alpha(q_i(s))/n-1 = g^n_\alpha(q_i(t))/n-1$, and $h^n_\alpha(q_i(s))/n-1 = h^n_\alpha(q_i(t))/n-1$ for all $i \leq m+1$ and $\alpha \in \Delta$, respectively. The values of $f^n(q_i(x))$, $g^n_\alpha(q_i(x))$, and $h^n_\alpha(q_i(x))$ for $x = s, t$ can be calculated effectively. By the induction hypothesis the equations above can be verified effectively, hence the truth of "$s \overset{n}{\sim} t$" can be decided after finitely many steps. □

The structure $P^{\Delta}(c_1, \ldots, c_m)/n$ is said to be presented effectively iff it has a set $\{t_0, \ldots, t_{r_n}\} \subseteq P^{\Delta}(c_1, \ldots, c_m)$ of representatives for $\overset{n}{\sim}$ which can be effectively specified.

Lemma 6.13. *For each n, $P^{\Delta}(c_1, \ldots, c_m)/n$* is effectively presented.

Proof. Clearly, it suffices to prove the lemma for p^{Δ}/n. P^{Δ} is finitely generated by $\{\mathbf{0}, \mathbf{1}\}$ using the operations $+$, $\cdot w_\alpha$, $\cdot w^*_\alpha$, and $S^{k,l}_\alpha$ for $\alpha \in \Delta$. Hence P^{Δ}/n can be generated from $\{\mathbf{0}/n, \mathbf{1}/n\}$ by successive applications of the operations mentioned above. Since P^{Δ}/n is finite, the process generating it stops after finitely many steps. By the preceding lemma we can effectively determine when the process generating P^{Δ}/n will stop. Then a set of representatives can be specified effectively using again Lemma 6.12. □

For any Δ-terms t and s we define the substitution of the i-th part of t by s as for composite Δ-polynomials. Again the resulting term is denoted by $t(q_i : s)$. The following propositions are proved in the same way as the corresponding theorems in the previous section.

Lemma 6.14. *Let $t \in P^{\Delta}(c_1, \ldots, c_m)$, $n \geq 1$. Suppose $\varphi(x)$ and $\psi(x, y)$ are formulas of $L^2_\Delta(c_1, \ldots, c_m)$ with quantifier ranks at most $n-1$. Then:*

(I) *$t \models \exists x\, \varphi(x)$ iff $t \models \varphi(c_j)$ for some $j \leq m$, or there are some $i \leq m+1$ and $s(c) \in f^n(q_i(t))$ such that $t(q_i : s(c)) \models \varphi(c)$.*

(II) $t \vDash Q_\alpha x\, \varphi(x)$ *iff there are some* $i \leqslant m+1$ *and* $s(c) \in g^n_\alpha(q_i(t))$ *such that* $t(q_i : s(c)) \vDash \varphi(c)$.

(III) $t \vDash Q^2_\alpha xy\, \psi(x, y)$ *iff there are some* $i \leqslant m+1$ *and* $s(c, d) \in h^n_\alpha(q_i(t))$ *such that* $t(q_i : s(c, d)) \vDash \psi(c, d)$.

Proof. See Lemma 5.8. □

Theorem 6.15. *There is a decision procedure which effectively decides "$t \vDash \varphi$" for any Δ-term t (with constants) and any sentence φ of L^2_Δ (with constants).*

Proof. See Theorem 5.9. □

Corollary. *Every Δ-term has decidable theory in L^2_Δ.*

Theorem 6.16. *Let Δ be a finite set of ordinals such that* cf $\Delta = \Delta$. *Then*:

(a) *For every natural number m,* $\mathrm{Th}^m_\Delta(\mathrm{LO})$ *is decidable.*

(b) $\mathrm{Th}^{<\omega}_\Delta(\mathrm{LO})$ *is decidable.*

Proof. (b) is an immediate consequence of (a). Let us prove (a) for $m = 2$. By Lemma 6.9, $\mathrm{Th}^2_\Delta(\mathrm{LO})$ is equal to the set of all sentences of L^2_Δ which are valid in all Δ-terms t ($\neq \mathbf{0}$). Let φ be an arbitrary sentence of L^2_Δ. Assume $q(\varphi) = n$. According to Lemma 6.13 we effectively find a set $\{t_0 = \mathbf{0}, t_1, \ldots, t_{r_n}\}$ of representatives for $\overset{n}{\sim}$. By Proposition 1.3 the sentence φ holds in all non-zero Δ-terms iff $t_i \vDash \varphi$ for all i, $1 \leqslant i \leqslant r_n$.

By Theorem 6.15 it is effectively decidable whether $t_i \vDash \varphi$ is true or not. Thus we have a method to decide effectively whether some sentence φ holds in all linear orderings or not. To prove (a) for $m > 2$, all the results of this section have to be generalized to $\overset{n}{\sim}$-mod G^m_Δ. Alternatively we can prove (a) by showing that the elimination of Q^m_α with respect to L^2_Δ can be carried out effectively. □

Since every finite extension of $\mathrm{Th}^{<\omega}_\Delta(\mathrm{LO})$ is also decidable we get further decidable theories in languages with various Malitz quantifiers.

For example the theories of the following classes are all decidable (provided cf $\Delta = \Delta$): ILO (infinite linear orderings), ULO (uncountable linear orderings), DLO (dense linear orderings without endpoints), and UDLO (uncountable dense linear orderings without endpoints).

We close the paper with some questions.

1. Is the theory $\mathrm{Th}^2_\Delta(\mathrm{LO})$ decidable for arbitrary finite Δ? An affirmative answer would imply the decidability for singular Malitz quantifiers.

2. How does $\mathrm{Th}^2_\Delta(\mathrm{LO})$ depends on Δ? In particular, what conditions on Δ and Γ are equivalent to $\mathrm{Th}^2_\Delta(\mathrm{LO}) = \mathrm{Th}^2_\Gamma(\mathrm{LO})$?

3. How does $\mathrm{Th}^2_\Delta(\mathrm{LO})$ depends on the various set-theoretical assumptions?

4. Suppose $\Delta = \{\omega\}$. Is Q_ω^m eliminable with the help of Q_ω^2 with respect to LO? Or, more generally, is the assumption cf $\Delta \subseteq \Delta$ really necessary?

5. Are the theories $\mathrm{Th}^2_{\{\omega\}}(\mathrm{LO})$ and $\mathrm{Th}^2_{\{\omega_1\}}(\mathrm{LO})$ equal to each other? In case they are not, find some simple sentence which is valid in one interpretation but not in the other one!

References

[1] L. Badger, The Malitz Quantifier meets its Ehrenfeucht game, Ph.D. Thesis, Colorado, 1975.
[2] A. Baudisch, D. Seese, H.P. Tuschik and M. Weese, Decidability and Generalized Quantifiers (Akademie-Verlag, Berlin, 1980).
[3] A. Ehrenfeucht, Decidability of the theory of one linear ordering relation (abstract), Notices AMS 6 (1959) 556–38.
[4] A. Ehrenfeucht, An application of games to the completeness problem for formalized theories, Fund. Math. 49 (1961) 129–141.
[5] Y. Gurevich, M. Magidor and S. Shelah, The monadic theory of ω_2, J. Symbolic Logic 48 (1983) 387–398.
[6] Y. Gurevich and S. Shelah, Monadic theory of order and topology in ZFC, Report No. 30/80, The Inst. for Advanced Studies.
[7] H. Herre and H. Wolter, Entscheidbarkeit von Theorien mit verallgemeinerten Quantoren, Z. Math. Logik Grundlag. Math. 21 (1975) 229–246.
[8] H. Herre and H. Wolter, Entscheidbarkeit der Theorie der linearen Ordnung in Logiken mit Mächtigkeitsquantoren bzw. mit Chang-Quantor, Z. Math. Logik Grundlag Math. 25 (1979) 345–358.
[9] H. Läuchli and J. Leonhard, On the elementary theory of linear order, Fund. Math. 49 (1966) 109–116.
[10] L. Lipner, Some aspects of generalized quantifiers, Ph.D. Thesis, Univ. of California, Berkeley, 1968.
[11] M. Magidor and J. Malitz, Compact extensions of $L(Q)$, Ann. Math. Logic 12 (1977) 217–261.
[12] A. Mostowski and A. Tarski, Arithmetical classes and types of well-ordered systems, Preliminary report, Bull. AMS 55 (1949) 65.
[13] S. Shelah, The monadic theory of order, Ann. of Math. 102 (1975) 379–419.
[14] A.B. Slomson, Generalized quantifiers and well-orderings, Arch. Math. Logik 15 (1972) 57–73.
[15] A.B. Slomson, Decision problems for generalized quantifiers — a survey, Lecture Notes in Math. 537 (Springer, Berlin, 1976) 249–258.
[16] D. Seese, H.P. Tuschik and M. Weese, Undecidable theories in stationary logic, Bull. AMS 84 (1982) 563–567.
[17] H.P. Tuschik, On the decidability of the theory of linear orderings in the language $L(Q_1)$, Lecture Notes in Math. 619 (Springer, Berlin, 1977) 291–305.
[18] H.P. Tuschik, On the decidability of the theory of linear orderings with generalized quantifiers, Fund. Math. 107 (1980) 21–32.
[19] H.P. Tuschik, Untersuchung linearer Ordnungen in der Logik mit Malitzquantoren, Dissertation B (1981), Humboldt-Universität zu Berlin.
[20] S. Vinner, A generalization of Ehrenfeucht's game and some applications, Israel J. Math. 12 (1972) 279–298.
[21] H. Wolter, Untersuchungen zu nicht-elementaren Logiken, Dissertation B (1975), Humboldt-Universität zu Berlin.