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# THE REDUCTION OF SOLUTIONS OF SOME INTEGRABLE PARTIAL DIFFERENTIAL EQUATIONS

MIN HO LEE
Department of Mathematics, University of Northern Iowa
Cedar Falls, Iowa 50614, U.S.A.

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**Abstract**—We reduce the finite-zone solutions of the Korteweg-de Vries equation, sine-Gordon equation and nonlinear Schrödinger equation associated to a hyperelliptic curve of genus four.

## 1. INTRODUCTION

Let $\Omega$ be a $g \times g$ Riemann matrix, that is, a complex symmetric matrix with positive definite imaginary part. Then $\Omega$ determines a Riemann theta function $\theta(\Omega, z)$ of dimension $g$. Among the examples of $g \times g$ Riemann matrices are the period matrices of hyperelliptic curves of genus $g$, and theta functions associated to such matrices describe solutions of certain nonlinear partial differential equations. In this paper we consider three such partial differential equations, i.e., Korteweg-de Vries (KdV) equation, sine-Gordon equation, and nonlinear Schrödinger equation. The so-called finite-zone solutions of these equations associated to hyperelliptic curves of genus $g$ can be expressed in terms of Riemann theta functions of dimension $g$.

Theta functions of dimension $g$ are essentially $g$-dimensional Fourier series, so such finite-zone solutions are not convenient for actual computations if $g$ is large. For this reason several methods of reducing theta functions to lower dimensional theta functions have been developed recently (see e.g., [1–4]). One of these methods was proposed by Babich, Bobenko and Matveev [2]. They modified a result that was obtained by Appell [5] in the late nineteenth century, and applied this to reduce the dimensions of the theta functions that appear in the solutions of the above partial differential equations associated hyperelliptic curves mostly of genus two or three. In this paper we apply this method to such solutions associated to a certain class of hyperelliptic curves of genus four and express the solutions in terms of the Riemann theta functions of dimension one and two only. We also show that these solutions can further be reduced to the ones involving only theta functions of dimension one.

## 2. FINITE-ZONE SOLUTIONS

In this section we describe the finite-zone solutions of the KdV equation $u_{xxx} - 6uu_x + u_t = 0$, the sine-Gordon equation $v_{tt} - v_{xx} = \sin v$, and the nonlinear Schrödinger equation $i\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0$ associated to hyperelliptic curves. Let $S$ be a hyperelliptic curve of genus $g$ given by $w^2 = P_{2g+1}(z)$ or $w^2 = P_{2g+2}(z)$, where $P_{2g+1}$ and $P_{2g+2}$ are polynomials of degree $2g+1$ and $2g+2$, respectively, without multiple roots. Let $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ be a canonical basis of cycles in

$$H_1(S, \mathbf{Z}) = \mathbf{Z} + \cdots + \mathbf{Z} \qquad (2g \text{ terms})$$

such that $a_i \cdot a_j = b_i \cdot b_j = 0$ and $a_i \cdot b_j = \delta_{ij}$ for $1 \le i,\ j \le g$, where $\delta_{ij}$ is the Kronecker delta and $(\,\cdot\,)$ denotes the intersection number. The dimension of the space of holomorphic 1-forms $H^{1,0}(S)$ of $S$ is $g$. Let $\{du_1, \dots, du_g\}$ be a basis of $H^{1,0}(S)$ such that

$$du_j = \sum_{k=1}^{g} c_{jk} \frac{z^{k-1}\,dz}{w} \qquad \text{and} \qquad \int_{c_j} du_k = \delta_{jk}$$

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for $1 \le j,\ k \le g$. Then the matrix $B = (b_{jk})$ with

$$b_{jk} = \int_{b_j} du_k, \qquad 1 \le j,\ k \le g$$

is called the period matrix of the Riemann surface $S$ and it determines the Riemann theta function

$$\theta(\mathbf{x} \mid B) = \sum_{m \in \mathbf{Z}^g} \exp\{\pi i \langle Bm, m\rangle + 2\pi i \langle m, x\rangle\},$$

where $\mathbf{x}$ is a vector in $\mathbf{C}^g$ and $\langle\ ,\ \rangle$ denotes the standard inner product on $\mathbf{C}^g$ (see e.g., [6–8]). More generally, if $\alpha, \beta \in \mathbf{R}^g$, then the Riemann theta function associated to $B$ with characteristic $[\alpha, \beta]$ is defined by

$$\theta\begin{pmatrix}\alpha \\ \beta\end{pmatrix}(\mathbf{x} \mid B) = \sum_{m \in \mathbf{Z}^g} \exp\{\pi i \langle B(m+\alpha), m+\alpha\rangle + 2\pi i \langle m+\alpha, x+\beta\rangle\}.$$

The finite-zone solutions of the KdV equation, sine-Gordon equation and nonlinear Schrödinger equation associated to the above hyperelliptic curve can be given in the following forms (see [2] for details):

(a) KdV equation: $\quad u(x,t) = -2\dfrac{\partial^2}{\partial x^2} \ln \theta_g(\mathbf{V}x + \mathbf{W}t + \mathbf{D} \mid B) + \text{constant},$

(b) sine-Gordon equation: $\quad v(x,t) = \dfrac{2}{i} \ln \dfrac{\theta\binom{0}{M/2}(2\pi^{-1}(\mathbf{V}x + \mathbf{W}t) + \zeta + \Delta)}{\theta(2\pi^{-1}(\mathbf{V}x + \mathbf{W}t) + \zeta)},$

(c) nonlinear Schrödinger equation: $\quad \psi(x,t) = S\dfrac{\theta(2\pi^{-1}(\mathbf{V}x + \mathbf{W}t - \mathbf{E} \mid B)}{\theta(2\pi^{-1}(\mathbf{V}x + \mathbf{W}t \mid B)}.$

## 3. PERIOD MATRICES OF CURVES WITH NONTRIVIAL AUTOMORPHISM

Let $S$ be a hyperelliptic curve of genus $g$ and let $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ be a canonical basis of cycles in $H_1(S, \mathbf{Z})$ as in Section 1. Suppose that $S$ has a nontrivial automorphism $\tau$ such that the cycles $\tau a_i$ are expressed only in terms of $a$-cycles and the cycles $\tau b_i$ are expressed only in terms of $b$-cycles. Then we have

$$a_i = \sum_{k=1}^{g} Q_{ik}(\tau a_k), \qquad b_i = \sum_{k=1}^{g} T_{ik}(\tau b_k)$$

for $1 \le i \le g$, where $Q = (Q_{ik})$ and $T = (T_{ik})$ are integral matrices such that $Q^{-1}$ and $T^{-1}$ are also integral. Since $\{\tau a_1, \dots, \tau a_g, \tau b_1, \dots, \tau b_g\}$ is also a canonical basis, the bases $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ and $\{\tau a_1, \dots, \tau a_g, \tau b_1, \dots, \tau b_g\}$ are related by a symplectic matrix. Thus, if $a = (a_1, \dots, a_g)^t$ and $b = (b_1, \dots, b_g)^t$, we obtain

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} Q & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} \tau a \\ \tau b \end{pmatrix}, \qquad Q = (T^t)^{-1}.$$

Let $du_1, \dots, du_g$ be holomorphic differentials on $S$ normalized relative to the basis $\{a_1, \dots, a_g, b_1, \dots, b_g\}$. Then the differentials $\tau^* du_j$ $(1 \le j \le g)$ are normalized relative to the basis $\{\tau a_1, \dots, \tau a_g, \tau b_1, \dots, \tau b_g\}$. Since we have

$$\int_{a_j} \tau^* du_k = \int_{\tau^{-1} a_j} du_k \qquad \text{and} \qquad \int_{b_j} \tau^* du_k = \int_{\tau^{-1} b_j} du_k$$

for $1 \le j,\ k \le g$, the $a$-period and the $b$-period of $\tau^* du_j$ with respect to the original basis $\{a_1, \dots, b_1, \dots, b_g\}$ are given by the matrices

$$Q = (T^t)^{-1} \qquad \text{and} \qquad TB,$$

respectively. Hence the differentials $dv_1, \dots, dv_g$ with

$$dv_j = \sum_{k=1}^{g} T_{kj}(\tau^* du_k), \qquad 1 \le j \le g$$

are normalized relative to the basis $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ with period matrix $TBT^t$. From the uniqueness of the normalized holomorphic differentials, it follows that $dv_j = du_j$ for $1 \le j \le g$ and $B = TBT^t$. This matrix equation gives linear relations among the entries of the period matrix $B$ of the hyperelliptic curve $S$.

## 4. THE REDUCTION METHOD

In this section we describe the method of reducing the dimensions of Riemann theta functions associated to matrices of certain type proposed by Babich, Bobenko and Matveev [2]. Let $g$ be an integer greater than one, and let $B = (b_{ij})$ be a complex symmetric $g \times g$ matrix whose imaginary part is positive definite. We assume that the last column of the matrix $B$ satisfies the condition $n_j b_{jg} = 0$ for $1 \leq j \leq \nu$ and $n_j b_{jg} = n_g b_{gg}$ for $\nu + 1 \leq j \leq g - 1$. Here $n_k$ is a positive integer chosen as small as possible for each $k$. Thus $n_j = 1$ for $1 \leq j \leq \nu$. Let $\mathbf{n} = (n_1, \ldots, n_g)$ be an integral vector with entries $n_j$ chosen in such a way, and let $\mathbf{x} = (x_1, \ldots, x_g)$ be a vector in $\mathbf{C}^g$. Let $\mathbf{f}(\mathbf{x}) = (f_1, \ldots, f_{g-1})$ be the vector in $\mathbf{C}^{g-1}$ defined by $f_j = n_j x_j$ for $1 \leq j \leq \nu$ and $f_j = n_j x_j - n_g x_g$ for $\nu + 1 \leq j \leq g - 1$, and let $A = (a_{ij})$ be the $(g-1) \times (g-1)$ matrix defined by

$$\begin{aligned} a_{ii} &= n_i^2 b_{ii} \quad \text{for} \quad 1 \leq i \leq \nu, & a_{ii} &= n_i^2 b_{ii} - n_g^2 b_{gg} \quad \text{for} \quad \nu + 1 \leq i \leq g - 1, \\ a_{ij} &= n_i n_j b_{ij} \quad \text{for} \quad 1 \leq i \text{ or } j \leq \nu, & a_{ij} &= n_i n_j b_{ij} - n_g^2 b_{gg} \quad \text{for} \quad \nu + 1 \leq i,\ j \leq g - 1. \end{aligned}$$

For each $\mathbf{t} = (t_1, \ldots, t_g)$ satisfying $0 \leq t_i \leq n_i - 1$ for $1 \leq i \leq g$, we set $q = \sum_{i=\nu+1}^{g} t_i/n_i \in \mathbf{R}$ and define $\mathbf{p} \in \mathbf{R}^{g-1}$ by $\mathbf{p} = (p_1, \ldots, p_{g-1})$, where $p_i = t_i/n_i$ for $1 \leq i \leq g - 1$. Then the reduction formula of Babich, Bobenko and Matveev for the Riemann theta function of dimension $g$ associated to the matrix $B$ is given by the following theorem:

THEOREM 1. *If* $\mathbf{x} = (x_1, \ldots, x_g) \in \mathbf{C}^g$, *we have*

$$\theta_g(\mathbf{x} \mid B) = \sum_{\mathbf{t} \in T} \theta_{g-1}\begin{pmatrix}\mathbf{p}\\0\end{pmatrix}(\mathbf{f}(\mathbf{x}) \mid A)\, \theta_1\begin{pmatrix}q\\0\end{pmatrix}(n_g x_g \mid n_g^2 b_{gg}).$$

PROOF. See [2, Theorem 1a].

In order to reduce the dimensions of theta functions further, we also need a reduction formula for theta functions with characteristics of the form $[\mathbf{p}, 0]$, which is given in the following theorem:

THEOREM 2. *Let* $\mathbf{p} = (p_1, \ldots, p_g)$ *and* $\mathbf{r} = (r_1, \ldots, r_{g-1})$ *be vectors with* $r_i = (t_i + p_i)/n_i$ *for* $1 \leq i \leq g - 1$, *and let* $s = \sum_{j=\nu+1}^{g} (t_j + p_j)/n_j$. *Then we have*

$$\theta_g\begin{pmatrix}\mathbf{p}\\0\end{pmatrix}(\mathbf{x} \mid B) = \sum_{\mathbf{t} \in T} \theta_{g-1}\begin{pmatrix}\mathbf{r}\\0\end{pmatrix}(\mathbf{f}(\mathbf{x}) \mid A)\, \theta_1\begin{pmatrix}s\\0\end{pmatrix}(n_g x_g \mid n_g^2 b_{gg})\ \exp(-2\pi i \langle B\mathbf{p}, \mathbf{t}\rangle).$$

PROOF. See [2, p. 487].

## 5. THE COMPUTATION OF THE PERIOD MATRIX

In the rest of the paper we shall apply the reduction method described in Section 4 to the theta function of a hyperelliptic curve of genus four given by an equation of the form

$$w^2 = z(z^4 - \alpha^4)(z^4 - \beta^4),$$

where $\alpha$ and $\beta$ are complex numbers. Then the hyperelliptic Riemann surface $S$ has a nontrivial automorphism $\tau$ sending $z$ to $iz$. In this section we determine the period matrix of $S$ by using the method described in Section 3.

The Riemann surface $S$ has ten branch points $e_1, \ldots, e_{10}$, where $e_1 = \alpha$, $e_2 = \beta$, $e_3 = i\alpha$, $e_4 = i\beta$, $e_5 = -\alpha$, $e_6 = -\beta$, $e_7 = -i\alpha$, $e_8 = -i\beta$, $e_9 = 0$, $e_{10} = \infty$. We choose a basis of cycles $\{a_1, \ldots, a_4, b_1, \ldots, b_4\}$ in $H_1(S, \mathbf{Z})$ as follows: consider $S$ as two copies of the Riemann sphere glued using cuts between $e_{2k-1}$ and $e_{2k}$ for $1 \leq k \leq 5$. For $1 \leq k \leq 4$, $a_k$ is a simple smooth closed curve winding once around the cut between $e_{2k-1}$ and $e_{2k}$ in one sheet in the clockwise orientation, and $b_k$ is a closed curve starting from a point on the cut between $e_{2k-1}$ and $e_{2k}$ going on the first sheet to a point on the cut between $e_9$ and $e_{10}$ and returning on the second sheet (see [8, p. 97]). Then these cycles satisfy the relations

$$a_1 = \tau a_3, \quad a_2 = -\tau a_1, \quad a_3 = -\tau a_4, \quad a_4 = \tau a_2;$$

hence we have $\mathbf{a} = Q(\tau\mathbf{a})$ and $\mathbf{b} = T(\tau\mathbf{b})$, where

$$Q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$\mathbf{a} = (a_1, a_2, a_3, a_4)^t$, $\mathbf{b} = (b_1, b_2, b_3, b_4)^t$, and $T = (Q^t)^{-1} = Q$. However, the last column of the matrix $B$ obtained by solving $B = TBT^t$ for this $t$ does not satisfy the conditions described in Section 4. In order to find a matrix that works, we consider the basis $\{a'_1, \dots, a'_4, b'_1, \dots, b'_4\}$ of $H_1(S, \mathbf{Z})$ obtained from the previous basis by the change of basis given by $\mathbf{b}' = \Phi\mathbf{b}$ and $\mathbf{a}' = (\Phi^t)^{-1}\mathbf{a}$ with

$$\Phi = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Then we have $\mathbf{b}' = T'(\tau\mathbf{b}')$ and $\mathbf{a}' = Q'(\tau\mathbf{a}')$, where

$$T' = \Phi T \Phi^{-1} = \begin{pmatrix} 0 & 1 & -2 & 2 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Applying the method described in Section 3, the period matrix $B$ of $S$ can be computed by solving the matrix equation $B = T'BT'^t$. Thus let $B$ be a symmetric matrix of the form

$$\begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix}.$$

Then, from the condition $B = T'BT'^t$, we obtain a system of equations

$$a = e - 4f + 4g + 4h - 8i + 4j, \quad b = b - 2c + 2d - f + g + 2h - 4i + 2j, \quad c = -f + 2g + 2h - 6i + 4j,$$

$$d = -f + g + 2h - 4i + 2j, \quad e = a - 2c + 2d + h - 2i + j, \quad f = -c + 2d + h - 3i + 2j,$$

$$g = -c + d + h - 2i + j, \quad h = h - 4i + 4j, \quad i = h - 3i + 2j, \quad j = h - 2i + j.$$

Solving these equations we obtain the period matrix of $S$ of the form

$$B = \begin{pmatrix} a & b & 3c & 2c \\ b & a & 3c & c \\ 3c & 3c & 4c & 2c \\ 2c & c & 2c & 2c \end{pmatrix},$$

where $a$, $b$, and $c$ are arbitrary constants in $\mathbf{C}$. Now the last column of $B$ certainly satisfies the condition described in Section 4.

## 6. THE REDUCTION OF THE THETA FUNCTION

In this section we apply the reduction method described in Section 4 to the theta function $\theta_4(\mathbf{x} \mid B)$ of dimension four associated to the period matrix $B$ obtained in Section 5 and express $\theta_4(\mathbf{x} \mid B)$ in terms of Riemann theta functions of dimension one and two only. The entries of the last column of $B$ satisfies

$$1 \cdot B_{14} = 1 \cdot B_{44}, \quad 2 \cdot B_{24} = 1 \cdot B_{44}, \quad 1 \cdot B_{34} = 1 \cdot B_{44}.$$

Hence we have $n_1 = n_3 = n_4 = 1$, $n_2 = 2$,

$$A = \begin{pmatrix} a-2c & 2b-2c & c \\ 2b-2c & 4a-2c & 4c \\ c & 4c & 2c \end{pmatrix}, \qquad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1 - x_4 \\ 2x_2 - x_4 \\ x_3 - x_4 \end{pmatrix},$$

and $T = \{(0,0,0,0)^t, (0,1,0,0)^t\}$. We also have $\mathbf{p} = (0,0,0)^t$, $q = 0$ for $\mathbf{t} = (0,0,0,0)^t$, and $\mathbf{p} = (0,1/2,0)^t$, $q = 1/2$ for $\mathbf{t} = (0,1,0,0)^t$. Therefore we obtain

$$\theta_4(\mathbf{x} \mid B) = \theta_3(\mathbf{f}(\mathbf{x}) \mid A)\, \theta_1(x_4 \mid 2c) + \theta_3\begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}(\mathbf{f}(\mathbf{x}) \mid A)\, \theta_1\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}(x_4 \mid 2c).$$

Since the last column of the matrix $A$ also satisfies the condition (4), we can apply the method of Appell-Babich-Bobenko-Matveev once again to the theta functions

$$\theta_3(\mathbf{f}(\mathbf{x}) \mid A) \quad \text{and} \quad \theta_3\begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}(\mathbf{f}(\mathbf{x}) \mid A).$$

By using Theorem 1 we obtain

$$\begin{aligned}
\theta_3(\mathbf{f}(\mathbf{x}) \mid A) &= \theta_2(\mathbf{y} \mid A')\, \theta_1(y \mid 8c) + \theta_2\begin{pmatrix} 1/4 & 0 \\ 0 & 0 \end{pmatrix}(\mathbf{y} \mid A')\, \theta_1\begin{pmatrix} 1/4 \\ 0 \end{pmatrix}(y \mid 8c) \\
&+ \theta_2\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}(\mathbf{y} \mid A')\, \theta_1\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}(y \mid 8c) + \theta_2\begin{pmatrix} 3/4 & 0 \\ 0 & 0 \end{pmatrix}(\mathbf{y} \mid A')\, \theta_1\begin{pmatrix} 3/4 \\ 0 \end{pmatrix}(y \mid 8c) \\
&+ \theta_2(\mathbf{y} \mid A')\, \theta_1\begin{pmatrix} 1/4 \\ 0 \end{pmatrix}(y \mid 8c) + \theta_2\begin{pmatrix} 1/4 & 0 \\ 0 & 0 \end{pmatrix}(\mathbf{y} \mid A')\, \theta_1\begin{pmatrix} 3/4 \\ 0 \end{pmatrix}(y \mid 8c) \\
&+ \theta_2\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}(\mathbf{y} \mid A')\, \theta_1(y \mid 8c) + \theta_2\begin{pmatrix} 3/4 & 0 \\ 0 & 0 \end{pmatrix}(\mathbf{y} \mid A')\, \theta_1\begin{pmatrix} 1/4 \\ 0 \end{pmatrix}(y \mid 8c),
\end{aligned}$$

and by using Theorem 2 we obtain

$$\begin{aligned}
\theta_3\begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}(\mathbf{f}(\mathbf{x}) \mid A) &= \theta_2\begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}(\mathbf{y} \mid A')\, \theta_1\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}(y \mid 8c) \\
&+ \theta_2\begin{pmatrix} 1/4 & 1/2 \\ 0 & 0 \end{pmatrix}(\mathbf{y} \mid A')\, \theta_1\begin{pmatrix} 3/4 \\ 0 \end{pmatrix}(y \mid 8c)\, \exp(-2\pi i b) \\
&+ \theta_2\begin{pmatrix} 1/2 & 1/2 \\ 0 & 0 \end{pmatrix}(\mathbf{y} \mid A')\, \theta_1(y \mid 8c)\, \exp(-4\pi i b) \\
&+ \theta_2\begin{pmatrix} 3/4 & 1/2 \\ 0 & 0 \end{pmatrix}(\mathbf{y} \mid A')\, \theta_1\begin{pmatrix} 1/4 \\ 0 \end{pmatrix}(y \mid 8c)\, \exp(-6\pi i b) \\
&+ \theta_2\begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}(\mathbf{y} \mid A')\, \theta_1(y \mid 8c)\, \exp(-4\pi i c) \\
&+ \theta_2\begin{pmatrix} 1/4 & 1/2 \\ 0 & 0 \end{pmatrix}(\mathbf{y} \mid A')\, \theta_1\begin{pmatrix} 1/4 \\ 0 \end{pmatrix}(y \mid 8c)\, \exp(-2\pi i(b+2c)) \\
&+ \theta_2\begin{pmatrix} 1/2 & 1/2 \\ 0 & 0 \end{pmatrix}(\mathbf{y} \mid A')\, \theta_1\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}(y \mid 8c)\, \exp(-\pi i(2b+2c)) \\
&+ \theta_2\begin{pmatrix} 3/4 & 1/2 \\ 0 & 0 \end{pmatrix}(\mathbf{y} \mid A')\, \theta_1\begin{pmatrix} 3/4 \\ 0 \end{pmatrix}(y \mid 8c),
\end{aligned}$$

where $y = 2x_3 - 2x_4$,

$$\mathbf{y} = \begin{pmatrix} 4x_1 - 2x_3 - 2x_4 \\ 2x_2 - 2x_3 + x_4 \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} 16a - 40c & 8b - 16c \\ 8b - 16c & 4a - 10c \end{pmatrix}.$$

From the computations described above, it follows that the theta function $\theta_4(\mathbf{x} \mid B)$, and therefore the solutions of the KdV equation, sine-Gordon equation, and nonlinear Schrödinger equation associated to the hyperelliptic curve $S$, can be expressed in terms of theta functions of dimension one and two only.

## 7. FURTHER REDUCTIONS

In this section we shall show that the two-dimensional theta functions associated to the matrix

$$A' = \begin{pmatrix} 16a - 40c & 8b - 16c \\ 8b - 16c & 4a - 10c \end{pmatrix}$$

can be expressed in terms of one-dimensional theta functions. First we state the following theorem that characterizes reducible Riemann theta functions of dimension two:

THEOREM 3. *Two dimensional Riemann theta functions associated to a $2 \times 2$ matrix $B = (b_{ij})$ can be expressed in terms of one-dimensional theta functions if and only if the entries of $B$ satisfy the condition*

$$\nu_1 + \nu_2 b_{11} + \nu_3 b_{12} + \nu_4 b_{22} + \nu_5 (b_{11} b_{22} - b_{12}^2) = 0,$$

*where $\nu_1 \dots, \nu_5$ are integers such that the number $\nu_3^2 + 4(\nu_1\nu_5 - \nu_2\nu_4)$ is the square of an integer.*

PROOF. See [3, Corollary 2.2].

Now we can easily show that the entries of $A'$ satisfy the condition in Theorem 3 by choosing $\nu_1 = \nu_3 = \nu_5 = 0$, $\nu_2 = 1$, and $\nu_4 = -4$. Thus the finite-zone solutions of the three partial differential equations associated to the hyperelliptic curve $S$ can be expressed in terms of theta functions of dimension one.

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