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Zeros of Polynomials and Fractional Order Differences of Their Coefficients

G. T. CARGO*

Syracuse University, Syracuse, New York

and

National Bureau of Standards, Washington, D. C.

AND

O. SHISHA

National Bureau of Standards, Washington, D. C.

Submitted by Richard Bellman

I. INTRODUCTION

In 1893 Eneström [1] proved that, if $c_0, c_1, \cdots, c_n$ $(n \geq 1)$ are real numbers (not all zero) satisfying

$$c_0 \geq c_1 \geq c_2 \cdots \geq c_n \geq 0, \tag{1}$$

then no zero of the polynomial $E(z) \equiv \sum_{k=0}^{n} c_k z^k$ lies in the disk $|z| < 1$. The interested reader may consult Professor Marden's treatise [2, § 30] on this and related results. A generalization of Eneström's theorem for power series with complex coefficients was given by Krishnaiah [3].

Let ∇ denote the backward-difference operator defined by $\nabla a_k \equiv a_k - a_{k-1}$ (see, e.g., [4, pp. 207-208]). Then (1) is equivalent to the condition

$$\nabla c_k \leq 0 \qquad (k = 1, 2, \cdots, n+1) \tag{2}$$

where c_{n+1} is taken to be zero. Eneström's conclusion follows from the observation that

$$(1 - z)\, E(z) \equiv \sum_{k=1}^{n+1} \{\nabla c_k\}\, (z^k - 1)$$

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which implies that

$$\text{Re}\,\{(1-z)\,E(z)\} \equiv \sum_{k=1}^{n+1} \{\nabla c_k\}\,\text{Re}\,(z^k - 1) > 0$$

whenever $|z| < 1$.

In this paper we extend Eneström's theorem by replacing (2) by similar conditions involving fractional order differences.

Let $\mathbf{E}$ denote the displacement operator defined by $\mathbf{E}a_k \equiv a_{k+1}$, and let $\mathbf{I}$ be the identity operator: $\mathbf{I}a_k \equiv a_k$. Then, symbolically,

$$\nabla^\alpha = (\mathbf{I} - \mathbf{E}^{-1})^\alpha = \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} \mathbf{E}^{-m}$$

for every complex α.

Accordingly, we define ∇^α by means of the identity (see, e.g., [5, § 5.5])

$$\nabla^\alpha a_k \equiv \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} a_{k-m}\,. \tag{3}$$

If $a_k = 0$ for $k = -1, -2, \cdots$, then (3) yields, for $k = 0, 1, 2, \cdots$,

$$\nabla^\alpha a_k = \sum_{m=0}^{k} (-1)^m \binom{\alpha}{m} a_{k-m}\,. \tag{4}$$

Given complex numbers α, a_0, a_1, $\cdots$, a_k, we shall always mean by $\nabla^\alpha a_k$ the right-hand side of (4).

II. Polynomials with Positive Coefficients

Theorem 1. *Let $E(z) \equiv \Sigma_{k=0}^{n} c_k z^k$ ($\not\equiv 0$, $n \geq 1$) be a polynomial, and let $0 < \alpha \leq 1$. Assume that $c_k \geq 0$ ($k = 0, 1, \cdots, n$) and that $\nabla^\alpha c_k \leq 0$ ($k = 1, 2, \cdots, n$). Then no zero of $E(z)$ lies in $|z| < 1$.*

Proof. Consider the function

$$(1-z)^\alpha \equiv \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} z^m \qquad |z| \leq 1. \tag{5}$$

Setting $c_k = 0$ for $k = n+1, n+2, \cdots$, we have throughout $|z| \leq 1$

$$(1-z)^\alpha E(z) = \sum_{k=0}^{\infty} \left\{ \sum_{m=0}^{k} (-1)^m \binom{\alpha}{m} c_{k-m} \right\} z^k$$

$$= \sum_{k=0}^{\infty} z^k\, \nabla^\alpha c_k = \sum_{k=1}^{\infty} (z^k - 1)\, \nabla^\alpha c_k\,, \tag{6}$$

since

$$\sum_{k=0}^{\infty} \nabla^{\alpha} c_k = 0.$$

For $k = n + 1, n + 2, \cdots$, we have

$$\nabla^{\alpha} c_k = \sum_{m=k-n}^{k} (-1)^m \binom{\alpha}{m} c_{k-m} . \tag{7}$$

Since

$$(-1)^m \binom{\alpha}{m} \leq 0, \qquad m = 1, 2, \cdots, \tag{8}$$

$\nabla^{\alpha} c_k \leq 0$ for $k = n + 1, n + 2, \cdots$. Thus $\nabla^{\alpha} c_k \leq 0$ $(k = 1, 2, \cdots)$, and by (6), not all of these numbers $\nabla^{\alpha} c_k$ are zero. Hence, for $|z| < 1$,

$$\operatorname{Re} \{(1 - z)^{\alpha} E(z)\} = \sum_{k=1}^{\infty} \{\nabla^{\alpha} c_k\} \operatorname{Re} (z^k - 1) > 0,$$

and so $E(z) \neq 0$.

Remarks. (a) Theorem 1 with $\alpha = 1$ is just Eneström's theorem. If $n = 1$ and α is a given number satisfying $0 < \alpha < 1$, then the test of Theorem 1 is weaker than Eneström's test.

(b) Let $0 < \alpha < 1, c_0 = 16$, $c_1 = 2\alpha$, and $c_2 = \alpha(8 - 6\alpha)$. One can apply Theorem 1 to $E(z) \equiv c_0 + c_1 z + c_2 z^2$ and conclude that it has no zero in $|z| < 1$. Eneström's theorem, however, is not applicable to this $E(z)$.

(c) Let $0 < \alpha_1 < \alpha_2 \leq 1$, let n be an integer larger than 1, and set $E_1(z) \equiv (-1)^{n+1} \binom{\alpha_1}{n}^{-1} + z^n$, $E_2(z) \equiv \sum_{k=0}^{n} \alpha_2^k z^k$. We can conclude that no zero of $E_1(z)$ lies in $|z| < 1$ by means of Theorem 1 with $\alpha = \alpha_1$, but not by means of Eneström's theorem. On the other hand, one can apply to $E_2(z)$ Eneström's theorem, but not Theorem 1 with $\alpha = \alpha_1$, for the condition $\nabla^{\alpha_1} c_1 = \alpha_2 - \alpha_1 \leq 0$ is not fulfilled.

(d) Let $0 < \alpha_1 < \alpha_2 \leq 1$, and set $E(z) \equiv 4 + 2(\alpha_1 + \alpha_2) z + \alpha_2(1 + \alpha_1) z^2$. Then one can apply to this $E(z)$ Theorem 1 with $\alpha = \alpha_2$. One cannot, however, apply Theorem 1 with $\alpha = \alpha_1$.

(e) Let $\frac{1}{2} \leq \alpha_1 < \alpha_2 < 1$, and set

$$E(z) \equiv [\alpha_1(1 - \alpha_1)]^{-1} + [\alpha_2(1 - \alpha_2)]^{-1} + z^2.$$

Then one can apply to this $E(z)$ Theorem 1 with $\alpha = \alpha_1$, but one cannot apply Theorem 1 with $\alpha = \alpha_2$.

III. Complex Coefficients

We shall now consider polynomials with complex coefficients.

By a sector with vertex at the origin we mean a set of the form

$$\{\rho e^{i\varphi} : \rho \geq 0, \varphi_1 \leq \varphi \leq \varphi_2\}.$$

Theorem 2. *Let* $E(z) \equiv \sum_{k=0}^{n} c_k z^k$ $(\not\equiv 0, n \geq 1)$ *be a polynomial with complex coefficients, and let* $0 < \alpha \leq 1$. *Set* $c_k = 0$ $(k = n+1, n+2, \cdots)$, *and let* S *be a sector with vertex at the origin whose angular measure* 2θ *satisfies* $0 \leq 2\theta < \pi$. *Then each of the following three hypotheses implies that* $E(z)$ *has no zero in* $|z| < \cos\theta$: (I) $-c_k$ $(k = 0, 1, \cdots, n)$ *and* $\nabla^\alpha c_k$ $(k = 1, 2, \cdots, n)$ *belong to* S; (II) $\nabla^\alpha c_k \in S$ $(k = 1, 2, \cdots, n, n+1, n+2, \cdots)$; (III) $\nabla c_k \in S$ $(k = 1, 2, \cdots, n+1)$.

Theorem 2 with hypothesis I and with S taken as the negative real axis (including the origin) is our previous Theorem 1.

To establish Theorem 2, we prove first the following

Lemma. *Let* $E(z) \equiv \sum_{k=0}^{\infty} c_k z^k$ $(\not\equiv 0)$ *be a power series with complex coefficients converging at* $z = 1$, *and let* $0 < \alpha \leq 1$. *Let* γ *and* r $(0 < r \leq 1)$ *be real numbers such that, for* $k = 1, 2, \cdots$, $\nabla^\alpha c_k = |\nabla^\alpha c_k| e^{i(\varphi_k + \gamma)}$ *where* φ_k *is real and* $|\varphi_k| \leq \arccos r^k$. *Then* $E(z)$ *has no zero in* $|z| < r$.

(Whenever arccos, arcsin, or arg appears, its principal value is being used.)

Proof of the Lemma. Due to the absolute convergence of the right-hand member of (5) at $z = 1$, (6) holds in the present case throughout $|z| < 1$. Let κ be a positive integer for which $\nabla^\alpha c_\kappa \neq 0$. (Such a κ exists, for otherwise we would get from (6) that $E(z) \equiv 0$.) Then throughout $|z| < r$ we have

$$|\arg [e^{-i\gamma}(1 - z^\kappa)\nabla^\alpha c_\kappa]| \leq |\arg (e^{-i\gamma}\nabla^\alpha c_\kappa)| + |\arg (1 - z^\kappa)|$$

$$< \arccos r^\kappa + \arcsin r^\kappa = \pi/2,$$

and therefore

$$\mathrm{Re}\,\{e^{-i\gamma}(1 - z^\kappa)\nabla^\alpha c_\kappa\} > 0.$$

If $|z| < r$, then by (6),

$$\mathrm{Re}\,\{e^{i(\pi-\gamma)}(1 - z)^\alpha E(z)\} = \sum_{k=1}^{\infty} \mathrm{Re}\,\{e^{-i\gamma}(1 - z^k)\nabla^\alpha c_k\} > 0;$$

and, consequently, $E(z) \neq 0$. This proves the Lemma.

PROOF OF THEOREM 2. For a suitable real constant γ, every $z \in S$ can be written in the form $|z| e^{i(\varphi(z)+\gamma)}$ where $-\theta \leq \varphi(z) \leq \theta$. To prove Theorem 2 (with hypothesis II), observe that for every $k \geq 1$ we have

$$\nabla^\alpha c_k = |\nabla^\alpha c_k| e^{i(\varphi_k+\gamma)}$$

where ϕ_k is real and

$$|\varphi_k| \leq \theta = \arccos\cos\theta \leq \arccos(\cos\theta)^k.$$

By the lemma, $E(z)$ has no zero in $|z| < \cos\theta$. Next, we prove Theorem 2 with hypothesis I. It is sufficient to show that $\nabla^\alpha c_k \in S$ for every $k > n$. Now, for such a k, we see from (7) and (8) that $\nabla^\alpha c_k$ is a weighted sum of $-c_0$, $-c_1, \cdots, -c_n$ with real, nonnegative weights. Since $-c_m \in S$ for $m = 0, 1, \cdots, n$, it follows that $\nabla^\alpha c_k$ also belongs to S. Finally, hypothesis III obviously implies II (with $\alpha = 1$).

Let c_0 be a positive number, and let $c_1, c_2, \cdots, c_n$ be nonnegative real numbers. Let $0 < \alpha < 1$, and for every $r \geq 0$, let $\mu(r) = \max_{1 \leq k \leq n} \nabla^\alpha(r^k c_k)$. Since $\mu(0) = \max_{1 \leq k \leq n} [(-1)^k \binom{\alpha}{k} c_0] < 0$, there exists a positive r for which $\mu(r) \leq 0$. *Every such r has the property that all zeros of $E(z) \equiv \sum_{k=0}^n c_k z^k$ lie in $|z| \geq r$.* Indeed, by Theorem 1, no zero of $E(rz) \equiv \sum_{k=0}^n c_k r^k z^k$ lies in $|z| < 1$.

THEOREM 3. *Let $E(z) \equiv \sum_{k=0}^\infty c_k z^k$ ($\not\equiv 0$) converge at $z = 1$. Let $0 < \alpha \leq 1$, and assume that all numbers $\nabla^\alpha c_k$ ($k = 1, 2, \cdots$) lie in some sector with vertex at the origin whose angular measure 2θ satisfies $0 \leq 2\theta < \pi$. Then no zero of $E(z)$ lies in $|z| < \cos\theta$.*

The proof is the same as that of Theorem 2 (with hypothesis II).

IV. FURTHER RESULTS

Let S be a sector as in Theorem 2, let $0 < \alpha \leq 1$, and let $c_0, c_1, \cdots$ be complex numbers (not all zero) such that $-c_k$ ($k = 0, 1, \cdots$) and $\nabla^\alpha c_k$ ($k = 1, 2, \cdots$) lie in S. Then, as one easily concludes, $c_0 \neq 0$. For $n = 1, 2, \cdots$, let $E_n(z) \equiv \sum_{k=0}^n c_k z^k$. By Theorem 2 (with hypothesis I), no $E_n(z)$ can have a zero in $|z| < \cos\theta$. If $\overline{\lim} |c_n|^{1/n}$ were larger than $(\cos\theta)^{-1}$, then we could find an n such that $|c_n/c_0|^{1/n} > (\cos\theta)^{-1}$. For such an n, the geometric mean of the moduli of the zeros of $E_n(z)$ would be smaller than $\cos\theta$; and therefore $E_n(z)$ would have at least one zero in $|z| < \cos\theta$, contradicting our above observation. Thus, *the radius of convergence of $\sum_{k=0}^\infty c_k z^k$ is $\geq \cos\theta$*; and, by Hurwitz's theorem relating the zeros of a power series to those of its partial sums, $\sum_{k=0}^\infty c_k z^k \neq 0$ throughout $|z| < \cos\theta$.

Taking a more general consideration, let $0 < \alpha \leq 1$, let γ and r be real numbers $(0 < r \leq 1)$, and let $C_0, C_1, \cdots$ be complex numbers (not all zero) such that $-C_k = |C_k| e^{i(\psi_k+\gamma)}$, ψ_k real, $|\psi_k| \leq \arccos r^{k+1}$ $(k = 0, 1, \cdots)$, and such that $\nabla^\alpha C_k = |\nabla^\alpha C_k| e^{i(\Phi_k+\gamma)}$, Φ_k real, $|\Phi_k| \leq \arccos r^k$ $(k = 1, 2, \cdots)$. Again it follows that $C_0 \neq 0$. For $n = 1, 2, \cdots$, set $E_n(z) \equiv \Sigma_{k=0}^n C_k z^k$, and consider some arbitrary $E_n(z)$. Let $c_k = C_k$ and $\phi_k = \Phi_k$ for $k = 0, 1, \cdots, n$, and let $c_k = 0$ for $k = n + 1, n + 2, \cdots$. If k is larger than n, then all the numbers $|\psi_0|, |\psi_1|, \cdots, |\psi_n|$ are equal to or less than $\arccos r^k$, and therefore $-c_0, -c_1, \cdots, -c_n$ lie in the sector $\{\rho e^{i\varphi} : \rho \geq 0, \gamma - \arccos r^k \leq \varphi \leq \gamma + \arccos r^k\}$. Thus, in view of (7) and (8), $\nabla^\alpha c_k$ lies in that sector; and, consequently, we may write $\nabla^\alpha c_k = |\nabla^\alpha c_k| e^{i(\varphi_k+\gamma)}$, where φ_k is real and $|\varphi_k| \leq \arccos r^k$. Applying the lemma to $E_n(z)$ we conclude that $E_n(z)$ has no zero in $|z| < r$. Now, exactly as before we can show that the radius of convergence of $\Sigma_{\nu=0}^\infty C_\nu z^\nu$ is at least r and that this power series is different from zero throughout $|z| < r$.

Assume again the hypotheses of Eneström's theorem. Since $\Sigma_{k=0}^n c_k > 0$, $E(1) \neq 0$. Therefore, by considerations as presented in the Introduction, if $\nabla c_1 \neq 0$, $E(z)$ has no zero on $|z| = 1$. If $E(z) = 0$, $|z| = 1$, then $\nabla c_1 = 0$, and $z^k = 1$ for each k satisfying $2 \leq k \leq n + 1$, $\nabla c_k \neq 0$; in particular, $z^{n+1} = 1$ if $c_n \neq 0$. See [6]. Furthermore, if for some k with $1 \leq k \leq n + 1$, $\nabla c_k \neq 0$, a zero z of $E(z)$ satisfies $\mathrm{Re}\,(z^k) < 1$, then there exists a k' $(1 \leq k' \leq n + 1, \nabla c_{k'} \neq 0)$ such that $\mathrm{Re}\,(z^{k'}) > 1$.

Similarly, assume the hypothesis of Theorem 1 with $\alpha < 1$. Again $E(1) \neq 0$. However, now we have the result that $E(z)$ *has no zero on the unit circumference*. Indeed, by (7) and (8) it follows that $\nabla^\alpha c_k < 0$ for $k = n + 1$, $n + 2, \cdots$. Also, by hypothesis, $\nabla^\alpha c_k \leq 0$, $k = 1, 2, \cdots, n$. If $E(z) = 0$, $|z| = 1$, then since $\Sigma_{k=1}^\infty \{\nabla^\alpha c_k\} \mathrm{Re}\,(z^k - 1) = 0$, we have $z^k = 1$, $k = n + 1, n + 2, \cdots$. Therefore $z = 1$, contradicting our above remark that $E(1) \neq 0$.

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