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Coincidence and fixed points for hybrid strict contractions

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Abstract

We define the property $(E.A)$ for single-valued and multivalued mappings and introduce the notion of T-weak commutativity for a hybrid pair (f, T) of single-valued and multivalued maps. We obtain some coincidence and fixed point theorems for this class of maps and derive, as application, an approximation theorem.

Keywords: Fixed points; Compatible maps; Noncompatible maps; Weak commutativity

1. Introduction

Sessa [11] introduced the concept of weakly commuting maps. Jungck [3] defined the notion of compatible maps in order to generalize the concept of weak commutativity and showed that weakly commuting mappings are compatible but the converse is not true [3]. In recent years, a number of fixed point theorems have been obtained by various authors utilizing this notion. Jungck further weakens the notion of compatibility by introducing the notion of weak compatibility and in [4] Jungck and Rhoades further extended weak compatibility to the setting of single-valued and multivalued maps. Pant [6–9] initiated the study of noncompatible maps and introduced pointwise R-weak commutativity of map-

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pings. He also showed that for single-valued mappings pointwise R-weak commutativity is equivalent to weak compatibility at the coincidence points. In [12,13] Shahzad gave application of R-weakly commuting mappings in best approximation theory. Recently, the author and Shahzad [15] and Singh and Mishra [16] have independently extended the idea of R-weak commutativity to the setting of single- and multivalued mappings. In [16] Singh and Mishra also introduced the notion of (IT)-commutativity for a hybrid pair of single-valued and multivalued maps and showed that a pointwise R-weakly commuting hybrid pair need not be weakly compatible [16, Example 1]. However, at the coincidence points, pointwise R-weak commutativity for hybrid pairs is equivalent to (IT)-commutativity. More recently, Aamri and El Moutawakil [1] defined a property $(E.A)$ for self maps and obtained some fixed point theorems for such mappings under strict contractive conditions. The class of $(E.A)$ maps contains the class of noncompatible maps.

The aim of this paper is to extend the property $(E.A)$ for a hybrid pair of single- and multivalued maps and to generalize the notion of (IT)-commutativity for such pairs. We obtain some coincidence and fixed point theorems for hybrid pairs. Our results extend Theorem 2.6 in [1], to multivalued case. As an application, we derive an approximation result.

2. Preliminaries

Let X be a metric space with metric d. Then, for $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, y)\colon\ y \in A\}$. We denote by $\mathrm{CB}(X)$ the class of all nonempty bounded closed subsets of X. Let H be the Hausdorff metric with respect to d, that is,

$$H(A, B) = \max\Big\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\Big\}$$

for every $A, B \in \mathrm{CB}(X)$.

Definition 2.1 [5]. Maps $f : X \to X$ and $T : X \to \mathrm{CB}(X)$ are said to be compatible if $fTx \in \mathrm{CB}(X)$ for all $x \in X$ and $H(fTx_n, Tfx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to A \in \mathrm{CB}(X)$ and $fx_n \to t \in A$.

Therefore the maps $f : X \to X$ and $T : X \to \mathrm{CB}(X)$ are noncompatible if $fTx \in \mathrm{CB}(X)$ for all $x \in X$ and there exists at least one sequence $\{x_n\}$ in X such that $Tx_n \to A \in \mathrm{CB}(X)$ and $fx_n \to t \in A$ but $\lim_{n\to\infty} H(fTx_n, Tfx_n) \neq 0$ or nonexistent.

Definition 2.2 [4]. Maps $f : X \to X$ and $T : X \to \mathrm{CB}(X)$ are weakly compatible if they commute at their coincidence points, i.e., if $fTx = Tfx$ whenever $fx \in Tx$.

Definition 2.3 [2,16]. Maps $f : X \to X$ and $T : X \to \mathrm{CB}(X)$ are said to be (IT)-commuting at $x \in X$ if $fTx \subseteq Tfx$.

Definition 2.4 [15]. Maps $f : X \to X$ and $T : X \to \mathrm{CB}(X)$ are said to be R-weakly commuting if, for given $x \in X$, $fTx \in \mathrm{CB}(X)$ and there exists some positive real number R such that $H(fTx, Tfx) \leqslant Rd(fx, Tx)$.

Definition 2.5 [1]. Maps $f : X \to X$ and $g : X \to X$ are said to the satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X$.

We now state the following theorem due to [1] for convenience.

Theorem 2.6 [1]. *Let g and f be two weakly compatible mappings of a metric space (X, d) such that*

(i) *g and f satisfy the property $(E.A)$,*
(ii) *for all $x \neq y \in X$*

$$d(gx, gy) < \max\big\{d(fx, fy), \big[d(gx, fx) + d(gy, fy)\big]/2, \big[d(gy, fx) + d(gx, fy)\big]/2\big\},$$

(iii) *$gX \subset fX$.*

If gX or fX is a complete subspace of X, then g and f have a unique common fixed point.

3. Main results

We begin with the following definition.

Definition 3.1. Maps $f : X \to X$ and $T : X \to \mathrm{CB}(X)$ are said to satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ in X, some $t \in X$ and $A \in \mathrm{CB}(X)$ such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n. \tag{1}$$

Example 3.2. Let $X = [1, \infty)$ with the usual metric. Define $f : X \to X$, $T : X \to \mathrm{CB}(X)$ by $fx = x + 1$ and $Tx = [1, x + 2]$ for all $x \in X$. Consider the sequence $\{x_n\} = \{1/n\}$. Clearly

$$\lim_{n\to\infty} fx_n = 1 \in [1, 2] = \lim_{n\to\infty} Tx_n.$$

Therefore f and T satisfy property $(E.A)$.

Example 3.3. Let $X = [2, \infty)$ with the usual metric. Define $f : X \to X$, $T : X \to \mathrm{CB}(X)$ by $fx = x$ and $Tx = \{2x\}$ for all $x \in X$. Suppose that the property $(E.A)$ holds; then there exists in X a sequence $\{x_n\}$ such that for some $t \in X$ and $A \in \mathrm{CB}(X)$

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n.$$

Then $\lim_{n\to\infty} x_n = t$, $A = \{2t\}$ and obviously $t \notin A$. Thus f and T do not satisfy the property $(E.A)$.

Theorem 3.4. *Let f be a self map of the metric space (X, d) and T be a map from X into $\mathrm{CB}(X)$ such that*

(i) *f and T satisfy the property $(E.A)$,*
(ii) *for all $x \neq y \in X$*

$$H(Tx, Ty) < \max\big\{d(fx, fy), \big[d(fx, Tx) + d(fy, Ty)\big]/2, \big[d(fy, Tx) + d(fx, Ty)\big]/2\big\}. \quad (2)$$

If fX be a closed subset of X, then f and T have a coincidence point.

Proof. By virtue of (1), there exist a sequence $\{x_n\}$ in X, $t \in X$ and $A \in \mathrm{CB}(X)$ such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n.$$

Since fX is closed, we have $\lim_{n\to\infty} fx_n = fa$ for some $a \in X$. Thus $t = fa \in A$. We claim $fa \in Ta$. If not, then condition (2) implies

$$H(Tx_n, Ta) < \max\big\{d(fx_n, fa), \big[d(fx_n, Tx_n) + d(fa, Ta)\big]/2, \big[d(fa, Tx_n) + d(fx_n, Ta)\big]/2\big\}.$$

Taking the limit as $n \to \infty$, we obtain

$$H(A, Ta) \leqslant \max\big\{d(fa, fa), \big[d(fa, A) + d(fa, Ta)\big]/2, \big[d(fa, Ta) + d(fa, A)\big]/2\big\} \leqslant d(fa, Ta)/2.$$

Since $fa \in A$, it follows from the definition of Hausdorff metric that

$$d(fa, Ta) \leqslant d(fa, Ta)/2,$$

which is a contradiction. Hence $fa \in Ta$. □

Example 3.5. Let $X = [1, \infty)$ with the usual metric. Define $f : X \to X$, $T : X \to \mathrm{CB}(X)$ by $fx = x^2$ and $Tx = [1, x + 1]$ for all $x \in X$. Then f and T satisfy the property $(E.A)$ for the sequence $\{1 + 1/n\}_{n\in\mathbb{N}}$ and

$$H(Tx, Ty) < d(fx, fy) \leqslant \max\big\{d(fx, fy), \big[d(fx, Tx) + d(fy, Ty)\big]/2, \big[d(fy, Tx) + d(fx, Ty)\big]/2\big\}.$$

Thus all conditions of Theorem 3.4 are satisfied and $1 = f1 \in T1$.

Since a noncompatible hybrid pair (f, T) satisfy property $(E.A)$, we have the following.

Corollary 3.6. *Let f be a self map of the metric space (X, d) and T be a map from X into $\mathrm{CB}(X)$ such that*

(i) *f and T are noncompatible,*
(ii) *for all $x \neq y \in X$*

$$H(Tx, Ty) < \max\big\{d(fx, fy), \big[d(fx, Tx) + d(fy, Ty)\big]/2, \big[d(fy, Tx) + d(fx, Ty)\big]/2\big\}.$$

If fX be a closed subset of X, then f and T have a coincidence point.

We now introduce the following notion.

Definition 3.7. Let $T : X \to \mathrm{CB}(X)$. The map $f : X \to X$ is said to be T-weakly commuting at $x \in X$ if $ffx \in Tfx$.

Here we remark that for hybrid pairs (f, T), (IT)-commuting at the coincidence points implies that f is T-weakly commuting but the following example shows that the converse is not true in general.

Example 3.8. Let $X = [1, \infty)$ with the usual metric. Define $f : X \to X$, $T : X \to \mathrm{CB}(X)$ by $fx = 2x$ and $Tx = [1, 2x + 1]$ for all $x \in X$. Then for all $x \in X$, $fx \in Tx$, $ffx = 4x \in [1, 4x + 1] = Tfx$, $fTx = [2, 4x + 2] \nsubseteq Tfx$. Therefore f is T-weakly commuting but not IT-commuting also note that f and T are not weakly compatible. Moreover, if $\{x_n\}$ is a sequence in X such that $x_n \to 1$. Then $\lim_{n\to\infty} fx_n = 2 \in [1, 3] = \lim_{n\to\infty} Tx_n$ and $\lim_{n\to\infty} H(fTx_n, Tfx_n) = 1$. Therefore the mappings f and T are not compatible. Furthermore, f and T satisfy property $(E.A)$.

Remark 3.9. (i) If T is a single-valued mapping, then T-weak commutativity at the coincidence points is equivalent to the weak compatibility.

(ii) It is known [7] that pointwise weak commutativity is a minimal condition for the existence of fixed points.

Theorem 3.10. *Let f be a self map of the metric space (X, d) and T be a map from X into $\mathrm{CB}(X)$ such that*

(i) *f and T satisfy the property $(E.A)$,*
(ii) *for all $x \neq y \in X$*

$$H(Tx, Ty) < \max\bigl\{d(fx, fy), \bigl[d(fx, Tx) + d(fy, Ty)\bigr]/2, \bigl[d(fy, Tx) + d(fx, Ty)\bigr]/2\bigr\},$$

(iii) *f is T-weakly commuting at v and $ffv = fv$ for $v \in C(f, T) :=$ set of coincidence points of f and T.*

If fX be a closed subset of X, then f and T have a common fixed point.

Proof. By Theorem 3.4, there exist $t, a \in X$ such that $t = fa \in Ta$. From this, and $ffv = fv$ for $v \in C(f, T)$, we have $t = ft$ and T commutativity of f at a further implies that $t = ft \in Tt$. □

Example 3.11 [10, p. 266]. Let $X = [0, \infty)$ with the usual metric. Define $f : X \to X$, $T : X \to \mathrm{CB}(X)$ by $fx = x$ and $Tx = [0, x^2/(x + 1)]$ for all $x \in X$. Then

(i) condition (2) is satisfied since $H(Tx, Ty) < d(x, y)$ for all $x \neq y \in X$,
(ii) f and T satisfy property $(E.A)$ for the sequence $\{x_n\} = \{1/n\}$ in X,
(iii) f is T weakly commuting at the coincidence point.

Thus all conditions of Theorem 3.10 are satisfied and 0 is the common fixed point of f and T.

Remark 3.12. The conclusions of Theorem 3.4, Corollary 3.6 and Theorem 3.10 remain valid if we assume that $T(X)$ is closed instead of $f(X)$ provided that $TX \subseteq f(X)$.

The problem of obtaining invariant approximations for non-commuting maps was considered first time by Shahzad [12,13]. Recently, Shahzad [14] introduced the class of R-subweakly commuting multimaps. It is worth mentioning that the concept of R-subweak commutativity is a useful tool for obtaining the existence of invariant approximations for a hybrid pair of maps. Our next result complements the work of Shahzad [12,13]. Let S be a subset of a normed space X. Then S is called p-star-shaped if there exists $p \in S$ such that for each $x \in S$, the segment joining x to p is contained in S, that is $(1-k)p + kx \in S$ for all $x \in S$ and all real k with $0 \leqslant k \leqslant 1$. Suppose $\hat{x} \in X$. The set $P_S(\hat{x}) = \{y \in S\colon \|y - \hat{x}\| = d(\hat{x}, S)\}$ is called the set of best approximants to $\hat{x} \in X$ out of S. The set of fixed points of $f : X \to X$ (respectively $T : X \to \mathrm{CB}(X)$) is denoted by $F(f)$ (respectively $F(T)$). The set of coincidence points of f and T is represented by $C(f, T)$.

Definition 3.13 [14]. Let $f : S \to S$ and $T : S \to \mathrm{CB}(S)$. Suppose S is p-star-shaped with $p \in F(f)$. Then the pair $\{f, T\}$ is called R-subweakly commuting if for all $x \in S$, $fTx \in \mathrm{CB}(S)$ and there exists $R > 0$ such that

$$H(Tfx, fTx) \leqslant Rd(fx, A_\lambda x)$$

for every $\lambda \in [0, 1]$, where $A_\lambda x = (1-\lambda)p + \lambda Tx$.

Theorem 3.14. *Suppose S be subset of a normed space X and let $f : X \to X$ and $T : X \to \mathrm{CB}(X)$ be such that $\hat{x} \in F(f) \cap F(T)$. Suppose that*

(i) *f and T are R-subweakly commuting on $P_S(\hat{x})$,*
(ii) *$H(Tx, Ty) \leqslant \|fx - fy\|$ for all $x, y \in P_S(\hat{x}) \cup \{\hat{x}\}$,*
(iii) *f is affine continuous on $P_S(\hat{x})$ and $\|ffx - fx\| \leqslant d(fx, Tx)$ for all $x \in P_S(\hat{x})$,*
(iv) *f and A_λ satisfy the property $(E.A)$ for each $0 \leqslant \lambda \leqslant 1$.*

If $P_S(\hat{x})$ is nonempty, compact, p-star-shaped with $p \in F(f)$, T-invariant and $f(P_S(\hat{x})) = P_S(\hat{x})$, then $P_S(\hat{x}) \cap F(f) \cap F(T) \neq \emptyset$.

Proof. Choose a sequence of real numbers $\{k_n\}$ $(0 \leqslant k_n < 1)$ converging to 1. For each n, define a sequence of maps A_n by

$$A_n x = (1-k_n)p + k_n Tx = \bigcup_{y \in Tx} (1-k_n)p + k_n y \quad \text{for each } x \in P_S(\hat{x}).$$

Since $P_S(\hat{x})$ is p-star-shaped, for each n, $A_n : P_S(\hat{x}) \to \mathrm{CB}(P_S(\hat{x}))$. Also $A_n(P_S(\hat{x})) \subset P_S(\hat{x}) = f(P_S(\hat{x}))$ for each n. Since f is affine on $P_S(\hat{x})$, it follows from the R-subweak commutativity of f and T that

$$H(A_n fx, fA_n x) \leqslant k_n H(Tfx, fTx) \leqslant k_n R d(fx, A_n x)$$

for all $x \in P_S(\hat{x})$. This implies that for each n, A_n and f commute at their coincidence points and so f is A_n-weakly commuting at v and $ffv = fv$ for $v \in C(f,T) \subset P_S(\hat{x})$. Also

$$H(A_n x, A_n y) \leqslant k_n H(Tx, Ty) \leqslant k_n \|fx - fy\| < \|fx - fy\|$$

for all $x \neq y \in P_S(\hat{x})$.

In view of (iv), we have f and A_n satisfy $(E.A)$ for each n. By Theorem 3.10, there exists $x_n \in P_S(\hat{x})$ such that

$$x_n = fx_n \in A_n x_n, \quad \text{for each } n.$$

Since $P_S(\hat{x})$ is compact, $\{x_n\}$ has a convergent subsequence $\{x_m\}$ with $x_m \to z \in P_S(\hat{x})$ as $m \to \infty$. Since f is continuous, $z = fz$. Since T is continuous, $k_m \to 1$ as $m \to \infty$ and $x_m \in A_m x_m = (1 - k_m)p + k_m T x_m$, it follows that $z \in Tz$. As a result $P_S(\hat{x}) \cap F(f) \cap F(T) \neq \emptyset$. □

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