Principal Indecomposable Representations
for the Group \( SL(2, q) \)

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INTRODUCTION

The study of the \( p \)-modular absolutely irreducible representations of the finite Chevalley groups has been made fruitfully by various authors. These representations for the group \( SL(2, p^n) \) have been constructed by Brauer and Nesbitt as early as 1941 (see [2]). Later, based on the “lifting process” used by Curtis [3], Steinberg [11] showed that the irreducible representations of a semisimple linear algebraic group over an algebraically closed field of characteristic \( p > 0 \) can be obtained by means of the twisted tensor products, and in particular this yields all the irreducible representations of a finite Chevalley group.

Thus one has a fairly satisfactory construction of the irreducible modules of a finite Chevalley group. However, no such construction seems to have been achieved for the principal indecomposable modules (PIM's, in short). Even the dimensions of the PIM's are not known, except in the case of \( SL(2, p^n) \), where we have the table of principal characters as given by Bhama Srinivasan in [10] (it may be mentioned that the dimensions of the PIM's of the \( u \)-algebra of the corresponding Lie algebra have been determined recently by Humphreys [7]). In this paper an attempt is made to construct the PIM's in the simplest case, namely the group of type \( A_1 \). We construct certain modules \( R_K \), where \( K \) is an algebraically closed field of characteristic \( p > 0 \) for the group algebra \( K[G_K] \) of the group \( G_K = SL(2, K) \) and prove that these modules when considered as modules over the group algebra \( K[G_p](G_p = SL(2, p)) \) are projective. Then we prove that the “twisted tensor product” of these modules (with one exception, where we take a suitable quotient of a tensor product module) can be taken as a full set of nonisomorphic PIM's for the group \( G_q = SL(2, q) \), \( (q = p^n) \).
1. Principal Characters of the Group $SL(2, q)$

We will use $G_K$ to denote the group $SL(2, K)$ of all nonsingular $2 \times 2$ matrices of determinant 1 with entries in a field $K$ of fixed characteristic $p > 0$. Writing $K(q)$ for the subfield of $K$ of order $q$, for each power $q = p^n (n = 1, 2, ...)$, we identify $G_q$ with the subgroup $G_K(q)$ of $G_K$. All modules (representations) considered will be unitary left $K[G]$-modules (where $G = G_K$ or $G_q$) taking the field to be algebraically closed.

It is an elementary and classical fact that for $0 \leq r \leq p - 1$ the symmetric $r$th power of the natural 2-dimensional representation of $G_q$ has dimension $r + 1$ and is irreducible; we will denote it by $V_{r,q}$, and these exhaust all the irreducible modules of $K[G_q]$.

Now let $r$ denote the vector $(r_0, r_1, ..., r_{n-1})$ where the $r_i$ are nonnegative integers ranging between 0 and $p - 1$. Then the $q$ nonisomorphic irreducible $K[G_q]$ modules are given by the tensor product

$$V_{r,q} = V_{r_0,q} \otimes V_{r_1,q} \otimes ... \otimes V_{r_{n-1},q},$$

where $V_{r_i,q}$ is the $K[G_q]$-module obtained from $V_{r_i,q}$ corresponding to the automorphism $\theta^i$ of $K(q)$ given by $\theta^i: t \mapsto t^{p^i} (t \in K(q))$ (see Brauer and Nesbitt [2] and Steinberg [11]).

By a PIM of $G_q$ we shall mean a principal indecomposable module of $K[G_q]$ (i.e., an indecomposable direct summand of the left regular module $K[G_q]$). It is well known that each PIM has a unique irreducible quotient, and that this sets up a one-to-one correspondence between the isomorphism classes of the PIM's and those of the irreducibles. Let $U_{r,q}$ be the PIM corresponding to $V_{r,q}$. From the table of principal characters (i.e., the characters of the principal indecomposable representations) for $G_q$ given by Bhama Srinivasan in [10], we get the dimensions of the $U_{r,q}$ as follows:

$$\dim U_{0,q} = (2^n - 1)q, \quad (0 = (0, 0, ..., 0));$$

$$\dim U_{r,q} = 2^m q, \quad (r \neq 0),$$

where $m$ is the number of $r_i \neq p - 1$.

(We remark that the explicit construction of these $U_{r,q}$'s owes much to the knowledge of their dimensions, and in fact to the principal characters themselves. Later, we are able to arrive at the desired results independently by a simple dimension counting argument pointed out by D.N. Verma.)
Let $\mathfrak{g}$ be the 3-dimensional simple Lie algebra over the complex field $\mathbb{C}$ with the standard basis $e, f, h$ and with the multiplications given by $[ef] = h$, $[he] = 2e$, $[hf] = -2f$. The fact that the structure constants are all integers shows that the $\mathbb{Z}$-module generated by $e, f, h$ is closed to multiplication, and so is a $\mathbb{Z}$-form of $\mathfrak{g}$ (see [1, p. A-3]).

Let $\mathcal{U}$ be the universal enveloping algebra of $\mathfrak{g}$. Then ([12], Theorem 2 or [1], Section 1) the subring $\mathcal{U}_Z$ of $\mathcal{U}$ which is generated by the elements

$$\frac{e^r}{r!}, \frac{f^r}{r!} \quad (r \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\})$$

is a $\mathbb{Z}$-form of $\mathcal{U}$, i.e., $\mathcal{U}_Z$ is a subring of $\mathcal{U}$ which has an additive basis which is also a basis of $\mathcal{U}$ as a $\mathbb{C}$-space. Moreover if we identify $\mathfrak{g}$ in the natural way as a subspace of $\mathcal{U}$, then $\mathcal{L}_Z \subseteq \mathcal{U}_Z$.

Let $(V, \pi)$ be a finitely generated (left) $\mathcal{L}$-module where $\pi$ is the corresponding representation. Then $(V, \pi)(or V, in short)$ can be regarded naturally as a $\mathcal{U}$-module. An additive subgroup $V_Z$ of $V$ is called an admissible $\mathbb{Z}$-form of $V$ if the following two conditions are satisfied

1. $V_Z$ has an additive basis $\{m_i \mid i \in I\}(I$ is a suitable index set) which is also a basis of $V$ as $\mathbb{C}$-space, and
2. $V_Z$ is stable under the action of $\mathcal{U}_Z$.

Condition (2) is clearly equivalent to

$$\left(\frac{e^r}{r!}\right) v \in V_Z \quad and \quad \left(\frac{f^r}{r!}\right) v \in V_Z \quad for \ all \ r \in \mathbb{Z}^+, v \in V_Z. \quad (2.1)$$

Steinberg has shown ([12, p. 17] or [1, Proposition 2.4]) that every finite-dimensional $\mathcal{L}$-module $V$ has at least one admissible $\mathbb{Z}$-form $V_Z$.

Now let $K$ be any field, and write $V_K = V_Z \otimes K$. $V_K$ can be regarded as a $K$-space in the natural way, and has as basis the set of elements $m_i, K = m_i \otimes 1_K(i \in I)$, where $1_K$ is the identity element of $K$. We shall show how $V_K$ can be made into a $K[G_K]$-module where $G_K = SL(2, K)$. This construction is due to Chevalley; see [12, p. 21] or [1, Section 3].

We construct two linear automorphisms $\alpha(t), \gamma(t)(t \in K)$ of $V_K$ by defining

$$\begin{align*}
\alpha(t) : v \otimes 1_K &\mapsto \sum_{r=0}^{\infty} t^r \left( \frac{e^r}{r!} v \otimes 1_K \right), \\
\gamma(t) : v \otimes 1_K &\mapsto \sum_{r=0}^{\infty} t^r \left( \frac{f^r}{r!} v \otimes 1_K \right).
\end{align*} \quad (2.2)$$
Notice that the right hand expressions are well defined, by (2.1), and by the fact that $e^r, f^r$ both annihilate $V$, for sufficiently large $r$ (see the references above). One may use the notation, which is suggested by these expressions,

$$\bar{x}(t) = \exp(t \pi(e)), \quad \bar{y}(t) = \exp(t \pi(f)).$$

The subgroup $G_{V,K}$ of $GL(V_K)$ which is generated by all the $\bar{x}(t), y(t)(t \in K)$, is the Chevalley group associated with $V$ and $K$. There is a unique homomorphism from $G_K = SL(2, K)$ onto $G_{V,K}$ which takes $x(t) \mapsto \bar{x}(t), y(t) \mapsto \bar{y}(t)$ for all $t \in K$, where

$$x(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad y(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

We use this homomorphism to define a $K$-linear action of $G_K$ on $V_K$, hence to make $V_K$ into a $K[G_K]$-module,

$$\begin{cases}
    x(t)(v \otimes 1_K) = \sum_{r=0}^{\infty} t^r \left( \frac{e^r}{r!} v \otimes 1_K \right), \\ y(t)(v \otimes 1_K) = \sum_{r=0}^{\infty} t^r \left( \frac{f^r}{r!} v \otimes 1_K \right).
\end{cases}$$

(2.3)

for all $v \in V_K, t \in K$.

We can also describe the action on $V_K$ of the subgroup $H_K$ of $G_K$, which consists of all diagonal matrices $h(t) = (t \ 0 \ -1)(t \in K^\times)$. Namely if $\lambda$ is a weight of $\mathcal{L}$ then

$$h(t)(v \otimes 1_K) = \pi(h)(v \otimes 1_K),$$

(2.4)

for all $v \in V_{\lambda,K} = (V_\lambda \cap V_Z) \otimes K$ where $V_\lambda$ is the weight space of $V$ corresponding to $\lambda$ (see [12, p. 27] or [1, Section 3.2]).

3. IRREDUCIBLE REPRESENTATIONS OF $SL(2, K)$

Let $s \in \mathbb{Z}^+$, and let $V_s$ be an irreducible $\mathcal{L}$-module of dimension $s + 1$, with highest vector $m_0$ and highest weight $\lambda(h) = s$. Then the vectors $m_i = (f^i/i!) m_0 (0 \leq i \leq s)$ form a basis of $V_s$ (see [11], p. 41). The action of $\mathcal{H}$ on $V_s$ is given by

$$\begin{align*}
    em_0 &= 0, \quad em_i = (s - i + 1) m_{i-1} \quad (1 \leq i \leq s), \\
    fm_s &= 0, \quad fm_i = (i + 1) m_{i+1} \quad (0 \leq i \leq s - 1), \\
    hm_i &= (s - 2i) m_i \quad (0 \leq i \leq s).
\end{align*}$$

If we make the convention that $m_i$ is defined for all $i \in \mathbb{Z}$, and that $m_i = 0$ if either $i < 0$ or if $i > s$, then we can give the action of $u$ on $V_s$ by

$$em_i = (s - i + 1)m_i, fm_i = (i + 1)m_{i+1}$$

for all $i \in \mathbb{Z}$.

It follows at once that for all $r \in \mathbb{Z}^+$ and all $i \in \mathbb{Z}$

$$\begin{align*}
\frac{e^r}{r!} m_i &= \binom{s - i + 1}{s - i} m_{i-r}, \\
\frac{f^r}{r!} m_i &= \binom{i + r}{i} m_{i+r}.
\end{align*}$$

Hence the additive subgroup $V_{s,\mathbb{Z}}$ generated by $m_0, \ldots, m_s$ is an admissible $\mathbb{Z}$-form of $V_s$. By the formulae (2.3), (2.4) we have, for the action defining the $K[G_K]$-module $V_{s,K} = V_{s,\mathbb{Z}} \otimes K$,

$$\begin{align*}
\chi(t) m_{i,K} &= \sum_{j=0}^s \binom{s - j}{s - i} t^{s-j} m_{j,K}, \\
\gamma(t) m_{i,K} &= \sum_{j=i}^0 \binom{i - j}{i} t^{j-i} m_{j,K},
\end{align*}$$

for all $t \in K$, $i \in \mathbb{Z}$,

$$h(t) m_{i,K} = t^{s-i} m_{i,K}$$

for all $t \in K^*$, $i \in \mathbb{Z}$.

It is easy to see, as we mentioned earlier, that the representation afforded by $V_{s,K}$ is equivalent to the $s$th symmetric power of the natural representation of $G_K$. From now on we take $K$ to be an algebraically closed field of characteristic $p > 0$, and write $K(q)$ for the subfield of $K$ of order $q$, for each power $q = p^n(n = 1, 2, \ldots)$. We identify $G_q$ with the subgroup $G_{K(q)}$ of $G_K$. In particular, the restrictions to $G_p$ of the $K[G_K]$-modules

$$V_{0,K}, V_{1,K}, \ldots, V_{p-1,K}$$

give a full set of irreducible $K[G_p]$-modules; see Section 2.


Choose $r \in \mathbb{Z}$ with $1 \leq r \leq p$. Let $\{v_i \mid 0 \leq i \leq r - 1\}, \{w_j \mid 0 \leq j \leq p - 1\}$ be the basis of the irreducible $\mathcal{W}$-modules $V_{r-1}, V_{p-1}$ which correspond to the basis $\{m_i\}$ of $V_s$ described above.
Now consider the tensor product \( V_{r-1} \otimes V_{p-1} \) (\( \otimes \) means \( \otimes_\mathbb{C} \)). This is a \( \mathbb{C} \)-space of dimension \( rp \), and as a basis of \( V_{r-1} \otimes V_{p-1} \) can be taken
\[ v_i \otimes w_j (i = 0, 1, \ldots, r - 1; j = 0, 1, \ldots, p - 1). \tag{4.1} \]
\( V_{r-1} \otimes V_{p-1} \) can be made into an \( \mathcal{L} \)-module by the action
\[ l(v \otimes w) = lv \otimes w + v \otimes lw \quad (l \in \mathcal{L}, v \in V_{r-1}, w \in V_{p-1}). \]
It is clear that (4.1) form an additive basis of \( V_{r-1,Z} \otimes V_{p-1,Z} \) and that this is an admissible \( \mathbb{Z} \)-form of \( V_{r-1} \otimes V_{p-1} \). The corresponding \( K[G_K] \)-module \( (V_{r-1,Z} \otimes V_{p-1,Z}) \otimes K \) can be identified with
\[ V_{r-1,K} \otimes V_{p-1,K} (\otimes \) means \( \otimes_K \),
\( G_K \) acting by the usual "diagonal action"
\[ g(v \otimes w) = gv \otimes gw \quad (g \in G_K, v \in V_{r-1,K}, w \in V_{p-1,K}). \]
Write \( y_{ij} = v_i \otimes w_j \), for all \( i, j \in \mathbb{Z} \), with the convention (as in Section 3) that \( v_i = 0 \) if \( i < 0 \) or \( i > r - 1 \) and \( w_j = 0 \) if \( j < 0 \) or \( j > p - 1 \). Consider the \( \mathbb{C} \)-space \( P = \mathcal{P} \) in \( V_{r-1} \otimes V_{p-1} \) generated by the \( p + r - 1 \) vectors \( E_s \), where
\[ E_s = \sum_{i+j=s} y_{ij} (0 \leq s \leq p + r - 2). \]
For convenience we shall write \( P \) instead of \( \mathcal{P} \) throughout the rest of this section.

**Lemma 4.2.** (i) \( P \) is an \( \mathcal{L} \)-submodule of \( V_{r-1} \otimes V_{p-1} \), and is isomorphic to \( V_{p+r-2} \).

(ii) The additive subgroup \( P_Z \) generated by \( E_0, \ldots, E_{p+r-2} \) is an admissible \( \mathbb{Z} \)-form of \( P \), and \( P_Z \) is isomorphic as \( \mathbb{Z} \)-module to \( V_{p+r-2,Z} \).

(iii) \( P_K = P_Z \otimes K \) is isomorphic, as \( K[G_K] \)-module, to \( V_{p+r-2,K} \).

**Proof.** By direct calculation
\[ e \cdot E_s = (p + r - s - 1) E_{s-1}, fE_s = (s + 1) E_{s+1} \]
for all \( s \in \mathbb{Z} \) (notice \( E_s = 0 \), if \( s < 0 \) or \( s > p + r - 2 \)); (i) and (ii) follow at once by comparison with (3.1)(replacing \( s \) in (3.1) by \( p + r - 2 \)). Then (iii) is also immediate.

**Remark 4.3.** One can check that \( P_Z = P \cap (V_{r-1,Z} \otimes V_{p-1,Z}) \) and hence \( P_K \) can be identified with a submodule of \( V_{r-1,K} \otimes V_{p-1,K} \). However we do not need this in the sequel. (See Remark 1 at the end.)
For the rest of this section, we assume $1 < r \leq p$. Our next step is to construct a subspace $R_r$ in $V_{r-1} \otimes V_{p-1}$ containing $P$, which is an $\mathcal{L}$-module, and whose dimension is $2p$. Again we drop the suffix $r$ in $R_r$ throughout this section.

First, we would like to find a vector

$$Z = b_0 y_{0,r-1} + b_1 y_{1,r-2} + \cdots + b_{r-1} y_{r-1,0} (b_i \in \mathbb{Z})$$

in $V_{r-1} \otimes V_{p-1}$ with the following properties

(i) $Z \notin P$,

(ii) $eZ = c \cdot E_{r-2}$, where $c = +(r - 1)!$, and

(iii) $hZ = (p - r)Z$. (4.4)

Condition (iii) is trivially satisfied for any $Z$ in the form as above, since the vectors $y_{0,r-1}, y_{1,r-2}, \ldots, y_{r-1,0}$ have the same weight. We will show that we can choose scalars $b_0, b_1, \ldots, b_{r-1}$ such that the conditions (i) and (ii) are satisfied for the corresponding element $Z$. In fact,

$$eZ = y_{0,r-2}(b_0(p - r + 1) + b_1(r - 1))$$

$$+ y_{1,r-3}(b_1(p - r + 2) + b_2(r - 2))$$

$$+ \cdots$$

$$+ y_{r-3,1}(b_{r-3}(p - 2) + b_{r-2} \cdot 2)$$

$$+ y_{r-2,0}(b_{r-2}(p - 1) + b_{r-1}).$$

If condition (ii) is to be satisfied for any value of $c$, the coefficients of all the terms must be the same. This gives that

$$pb_0 - (r - 1)(b_0 - b_1) = pb_1 - (r - 2)(b_1 - b_2) = \cdots$$

$$= pb_{r-2} - 1 \cdot (b_{r-2} - b_{r-1}).$$ (4.5)

Noticing that we have assumed that $r > 1$, we verify that (4.5) has a solution, for arbitrary $b_0, b_1$, with the remaining $b_i (i = 2, 3, \ldots, r - 1)$ uniquely determined by

$$b_i - b_{i+1} = \frac{(-1)^{i}(p - r + 1) \cdots (p - r + i)}{(r - 2) \cdots (r - i - 1)} (b_0 - b_1)(i = 0, 1, \ldots, r - 2).$$ (4.6)

So if we take $b_0 = 0$, and $b_0 - b_1 = -(r - 2)!$, we shall get a solution of (4.5) with all the $b_i \in \mathbb{Z}$, and satisfying (ii). Since $b_0 - b_1 \neq 0$, we also have (i). From now on, we shall take $b_0, b_1, \ldots, b_{r-1}$ as above.

Next consider the $\mathcal{L}$-submodule $R$ of $V_{r-1} \otimes V_{p-1}$ generated by the $\mathcal{L}$-module $P$ and the vector $Z$. We proceed to prove the following
Theorem 4.7. (i) Let \( Z_i = \sum_{k=0}^{i} Z_k \) (\( i = 0, 1, \ldots \)). Then the set of vectors \( Z_0, Z_1, \ldots, Z_p, Z_{p+1} \) together with \( E_0, E_1, \ldots, E_{p+1} \) forms a basis for \( R \) as \( \mathbb{C} \)-space. Hence \( \dim R = 2p \).

(ii) The additive subgroup \( R_z \) generated by the same vectors is an admissible \( \mathbb{Z} \)-form of \( R \).

(iii) \( Z_{p-r+1} = (f^p \cdot (p-r+1)! Z = d \cdot E_p \) where

\[
d = (p-1)(p-2) \cdots (p-r+2).
\]

(iv) \( P_K = P_z \otimes K \) can be identified with a \( K[G] \)-submodule of \( R_K = R_z \otimes K \). With this identification, \( R_K/P_K \cong V_{p-r,K} \).

Proof. By an easy induction argument, which uses only property (4.4) (iii) of \( Z \), we find, for all \( i > 1 \),

\begin{align*}
HZ_i &= (p - r - 2i) Z_i, \quad (4.8) \\
eZ_i &= (p - r - i + 1) Z_{i-1} + (f^i/i!)(eZ). \quad (4.9)
\end{align*}

Putting \( i = p - r + 1 \) in (4.9), and using (4.4) (ii), we have

\[
eZ_{p-r+1} = \frac{f^p \cdot (p-r+1)!}{(p-r+1)!} \cdot cE_{r-2}, \quad \text{where} \quad c = (r-1)!,
\]

\[
= (p-1)(p-2) \cdots (p-r+2) E_{p-1}. \quad (4.10)
\]

On the other hand, it follows from the definition of \( Z \) that

\[
Z_{p-r+1} = c_1 y_{1, p-1} + \cdots + c_{r-1} y_{r-1, p-r+1}
\]

for some \( c_1, \ldots, c_{r-1} \in \mathbb{C} \) (note \( y_{0,p} = 0 \)). From this we find

\[
Z_{p-r+1} = c_1[(r - 1) y_{0,p-1} + y_{1,p-1}] + c_2[(r - 2) y_{1,p-2} + 2 \cdot y_{2,p-3}] + \cdots + c_{r-1}[y_{r-2,p-r+1} + (r - 1) y_{r-1,p-1}].
\]

Comparing this with (4.10) we soon find that

\[
c_1 = c_2 = \cdots = c_{r-1} = (p-1)(p-2) \cdots (p-r+2),
\]

which proves part (iii) of the theorem.
Since $eZ \in PZ$, and since $PZ$ admits $UZ$, (4.9) gives

$$e \cdot Z_i \equiv (p - r - i + 1) Z_{i-1} \mod PZ \ (i = 1, 2, \ldots). \tag{4.11}$$

We have also just proved

$$Z_{p-r+1} \equiv 0 \mod PZ.$$

Hence $R/P$ is an $L$-module, generated by a nonzero element $Z \mid P$ of weight $p - r$, such that $(f^{p-r+1}/(p - r + 1)) (Z + P) = 0 + P$ and $e(Z + P) = 0 + P$. It follows that $R/P$ is isomorphic to the irreducible $L$-module $V_{p-r}$, and that the cosets $Z_0 + P, \ldots, Z_{p-r} + P$ form a basis of $R/P$. This proves (i).

To prove (ii), it is enough, in view of Lemma 4.2, to prove

$$(e' r! Z_i \equiv \binom{p - r}{r} Z_{i-r} \mod PZ,$$

and

$$(f' r! Z_i \equiv \binom{i + r + 1}{r} Z_{i+1} \mod PZ,$$

for all $r \in \mathbb{Z}^+$, and $i = 0, 1, \ldots, p - 1$ (we take $Z_i = 0$ if $i < 0$). This follows directly from (4.11) and the equations $fZ_i = (i + 1) Z_{i+1}$, in the same way as (3.2) follows from (3.1).

Finally it is now clear that there is an exact sequence of $UZ$-modules

$$0 \to PZ \overset{i}{\to} RZ \overset{\phi}{\to} V_{p-r, Z} \to 0,$$

where $i$ is the inclusion and $\phi$ is the map which takes $PZ$ to 0 and $Z_i$ to $m_i (i = 0, 1, \ldots, p - r)$, using the basis $\{m_i\}$ of $V_{p-r}$, as given in Section 3, (with $s = p - r$). It is immediate that if we tensor this with $K$, we get an exact sequence of $K[G_K]$-modules

$$0 \to P_K \to R_K \to V_{p-r,K} \to 0;$$

this proves (iv) and completes the proof of Theorem 4.7.

### 5. Projectivity of the $K[G_K]$-Module $R_K$

At this stage, the author originally proved, by a rather lengthy process, that $P_K$ is the unique maximal $K[G_K]$-submodule of $R_K$. But the following elegant theorem, which is due to J.A. Green, avoids these calculations and helps us to land directly at the crucial Theorem 6.4 in the next section.
**Theorem 5.1.** Take $1 < r < p$ and $R = R_r$ as in Section 4. Then $R_K$, regarded as $K[G_r]$-module, is projective.

We prove this by means of two lemmas (both well known).

**Lemma 5.2.** (see [5] or [6]). Let $A$ be a finite group and $B$ any $p$-Sylow subgroup of $A$. Let $M$ be a $K[A]$-module such that $M_B$ is a free $K[B]$-module. Then $M$ is projective.

**Lemma 5.3.** (see [9]). Let $B$ be a finite $p$-group, and let $\sigma = \sum_{b \in B} b$, an element of $K[B]$. Let $m_1, \ldots, m_r$ be elements of a $K[B]$-module $M$, such that $\sigma m_1, \ldots, \sigma m_r$ are linearly independent over $K$. Then $m_1, \ldots, m_r$ are linearly independent over $K[B]$. In particular if $\dim M = rp^b$, where $p^b = |B|$, then $M$ is a free $K[B]$-module.

So to prove Theorem 5.1, it is enough to prove that when we regard $R_K$ as $K[B]$-module, $B$ being any given Sylow $p$-subgroup of $G_r$, we can find elements $m_1, m_2$ of $R_K$ such that $\sigma m_1, \sigma m_2$ are linearly independent over $K$.

Take $B = \{ y(t) \mid t \in K(p) \}$, so that for any $m \in R_K$,

$$\sigma(m \otimes 1_K) = \sum_{t \in K(p)} \sum_{r \geq 0} \eta_r \left( \frac{r!}{r!} m \otimes 1_K \right)$$

where $\eta_r = \sum_{t \in K(p)} t^r$.

By elementary algebra, one verifies $\eta_0 = 0$, and if $r > 0$ then

$$\eta_r = \begin{cases} 0 & \text{if } p - 1 \not| r, \\ 1 & \text{if } p - 1 \mid r. \end{cases}$$

Hence

$$\sigma(m \otimes 1_K) = - \sum_{n>0} \frac{j^{n(p-1)}}{n(p-1)!} m \otimes 1_K. \quad (5.4)$$

Now take $m_1 = E_0 \otimes 1_K$ and $m_2 = Z \otimes 1_K$. Using (4.2), (4.7), (5.4), we get

$$\sigma m_1 = - E_{p-1} \otimes 1_K - E_{2p-2} \otimes 1_K,$$

$$\sigma m_2 = -(p+1)(p+2) \cdots (p+r-2) E_{p+r-2} \otimes 1_K.$$  

These are independent over $K$, and this proves the theorem.

We make an observation here for later use. If $z$ is any nonzero element of $K$, write $B_z = \{ y(tz) \mid t \in K(p) \}$. Then by a trivial modification of the argument above, we find $R_K$ is free, regarded as a $K[B_z]$-module.
6. Principal Indecomposable Modules for $K[G_q]$

From this point on, we do not need the complex $\mathcal{L}$-modules $V_\gamma, P_\gamma, R_\gamma$ any more, but just the $K[G_k]$-modules which were constructed from them. So to simplify notation, we shall write henceforth

$$V_s \text{ for } V_{s,K}, \quad P_\gamma \text{ for } P_{\gamma,K}, \quad R_\gamma \text{ for } R_{\gamma,K},$$

and understand that these are the $K[G_k]$-modules constructed above.

Let $q = p^n (n \geq 1)$, $K(q)$ is the unique subfield of $K$ of order $q$, and $G_q = G_{K(q)}$. Then the modules $V_\gamma, P_\gamma, R_\gamma$ can be regarded as $K[G_k]$-modules by restricting from $G_k$ to its subgroup $G_q$.

We have defined $R_\gamma$ only for $\gamma$ in the range $1 < \gamma \leq p$. We now define $R_\gamma$ to be the projective, irreducible $K[G_k]$-module $V_{p-1}$, of dimension $p$.

Our next step is to construct certain other $K[G_k]$-modules starting from the $p$ basic modules $R_{(1 \leq r \leq p)}$ by using the "twisted tensor products".

Let $R_{r_k}$ stand for the $K[G_k]$-module $R_{r_k}(1 \leq r \leq p)$ that we have described above. The automorphism $\theta^k$ of $K$ given by $\theta^k : t \mapsto t^{p^k} (k = 0, 1, 2, ...)$ gives rise to an automorphism of $G_k$ taking $x(t)$ to $x(t^{p^k})$ and $y(t)$ to $y(t^{p^k})$. Thus given a $K[G_k]$-module $R_{r_k}$ we get another $K[G_k]$-module denoted by $R_{r_k}$, the action being given by

$$g \cdot s = \theta^k(g \in G_k, s \in R_{r_k}),$$

and extending this to $K[G_k]$ by linearity. Likewise, let $P_{r_k}$ be the submodule of $R_{r_k}$ obtained from $P_{r_k}$ by this "twisted action."

Now consider the tensor product

$$R_r = R_{r_0} \otimes R_{r_1} \otimes \cdots \otimes R_{r_{n-1}},$$

for each $r \in \mathcal{R} = \{(r_0, r_1, ..., r_{n-1}) | r_k \in \mathbb{Z}, 1 \leq r_k \leq p, k = 0, 1, ..., n - 1\}$. $R_r$ is clearly a $K[G_k]$-module of dimension $2^m p^n$, where $m$ is the number of $r_k \neq 1$.

**Lemma 6.1.** $R_r$ is a projective $K[G_q]$-module, for all $r \in \mathcal{R}$.

**Proof.** By Lemma 5.2, it is enough to prove that $R_r$ is free, as a $K[B]$-module, where $B$ is any given Sylow $p$-subgroup of $G_q$. Take

$$B = \{ y(t) | t \in K(q) \}.$$

It is well known that $K(q)$ has a "normal basis" over $K(p)$, i.e., there exists
$z \in K(q)$ such that $z, z^q, \ldots, z^{q^{n-1}}$ form a basis of $K(q)$ over $K(p)$. Now if we put $B_z = \{ y(tz) | t \in K(p) \}$, we find at once

$$B = B_z \times B_{z^q} \times \cdots \times B_{z^{q^{n-1}}}.$$  

By (5.5), each $R_{r_k}$ is free, as $K[B_z]$-module, (it can easily be verified that $R_1$ is a free $K[B]$-module) and hence $R_{r_0} \otimes R_{r_1}^q \otimes \cdots \otimes R_{r_{n-1}}^{q^{n-1}}$ is a free $K[B]$-module, as required.

Now let $P_t$ be the $K[G_K]$-submodule of $R_t$ given by

$$P_t = \sum_{k=0}^{n-1} R_{r_0} \otimes R_{r_1}^q \otimes \cdots \otimes R_{r_{k-1}}^{q^{k-1}} \otimes R_{r_k}^q \otimes R_{r_{k+1}}^{q^{k+1}} \otimes \cdots \otimes R_{r_{n-1}}^{q^{n-1}}.$$  

Then $R_t/P_t$, as a $K[G_K]$-module, is isomorphic to

$$R_{r_0}/P_{r_0} \otimes R_{r_1}/P_{r_1}^q \otimes \cdots \otimes R_{r_{n-1}}^{q^{n-1}}/P_{r_{n-1}}^{q^{n-1}},$$  

which, by Theorem 4.7 (iv) is isomorphic to

$$V_{p-r_0} \otimes V_{p-r_1}^q \otimes \cdots \otimes V_{p-r_{n-1}}^{q^{n-1}}.$$  

Since $V_{p-r_k}$ is irreducible, it is clear that $V_{p-r_k}^q (0 \leq k \leq n-1)$ is also irreducible; hence by Steinberg’s tensor product theorem (see [11], p. 42), the $K[G_K]$-module $V_{p-r_0} \otimes V_{p-r_1}^q \otimes \cdots \otimes V_{p-r_{n-1}}^{q^{n-1}}$ is irreducible, and is denoted by $V_{p-r}$.  

Next let $R_t$ and $R_t'$ be two $K[G_K]$-modules given by the ordered sets $r = (r_0, r_1, \ldots, r_{n-1})$ and $r' = (r'_0, r'_1, \ldots, r'_{n-1})$ in $\mathcal{R}$ respectively and let $V_t$ and $V_{t'}$ be the corresponding irreducible $K[G_K]$-modules. Then by the same theorem of Steinberg, $V_t$ and $V_{t'}$ will be $K[G_K]$-isomorphic if and only if $r_i = r'_i$ for each $i$. Hence $R_t$ and $R_{t'}$ will be $K[G_K]$-isomorphic if and only if $r_i = r'_i$ for each $i$.

For $r \in \mathcal{R}$ let $U_r$ denote the PIM of $G_q$ with unique irreducible quotient $V_r$.

**Theorem 6.2.** Suppose $r \in \mathcal{R}$ and $r \neq (p, p, \ldots, p)$. Then there exists a surjective $K[G_q]$-module homomorphism $\Psi_t$ from $R_t$ onto $U_{p-r}$.

**Proof.**

Let $\beta, \gamma$ be surjective homomorphisms from $U_{p-r}$, $R_t$, respectively, onto $V_{p-r}$. Since $R_t$ is projective, there exists $\Psi : R_t \rightarrow U_{p-r}$ such that $\beta \Psi = \gamma$.  

$$\begin{array}{c}
R_t \\
\downarrow \Psi \\
U_{p-r} \rightarrow V_{p-r}
\end{array}$$  

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U_{p-r} \rightarrow V_{p-r}
\end{array}$$  

Let $\beta, \gamma$ be surjective homomorphisms from $U_{p-r}$, $R_t$, respectively, onto $V_{p-r}$. Since $R_t$ is projective, there exists $\Psi : R_t \rightarrow U_{p-r}$ such that $\beta \Psi = \gamma$.
Since $\beta$ maps $\Psi(R_r)$ onto $V_{p-r} = \beta(U_{p-r})$, it follows that $\Psi(R_r) + \text{Ker } \beta = U_{p-r}$. Since $\text{Ker } \beta$ is the unique maximal submodule of $U_{p-r}$, this implies $\Psi(R_r) = U_{p-r}$, i.e., $\Psi = \Psi_r$ is surjective.

**Theorem 6.3.** Write $q = (p, p, \ldots, p)$. Then $R_q$ has a $K[G_q]$-submodule $Q$ of dimension $q$, which is projective. There exists a surjective $K[G_q]$-module homomorphism $\Psi_q$ from $R_q/Q$ onto $U_o$.

**Proof.** For $k = 0, 1, \ldots, n - 1$, let $E_i^{(k)}$ stand for the elements $E_i$ in the $K[G_q]$-module $R_{p^k}$ (i.e., $g \cdot E_i^{(k)} = g^{p^k}E_i$, $g \in G_q$). Then the element

$$E^* = E_0^{(0)} \otimes \cdots \otimes E_{n-1}^{(n-1)} = \sum_{m=0}^{q-1} t^m E^{*}_m(t \in K(q), E^{*}_m \in R_q).$$

Let $Q$ denote the $K[G_q]$-submodule generated by $E^*$. We claim that $\dim Q = q$.

In fact, write

$$y(t) E^* = \sum_{m=0}^{q-1} t^m E^{*}_m(t \in K(q), E^{*}_m \in R_q),$$

which belongs to $Q$. Multiplying the above by $t^{-m}$, substituting for $t$ the $q - 1$ nonzero elements of $K(q)$ and adding up, we get that $E^{*}_m \in Q$ for each $m$. It is easy to see that the $E^{*}_m$ are all non-zero, distinct and linearly independent. Next we prove that the $K(q)$-space $S$ spanned by the $E^{*}_m$ is actually a $K[G_q]$-module, and since $E^{*}_m = E^* \in S$, it follows that $S = Q$.

Finally we prove that the module $Q_B$, where $B = \{ y(t) | t \in K(q) \}$ is a free $K[B]$-module. In fact, $Q_B = K[B] \cdot E^* = K[B]/\mathcal{A}$ as a $K[B]$-module, where $\mathcal{A}$ is the annihilator of $E^*$. But since $\dim Q_B = q = |B|$, it follows that $\mathcal{A} = 0$, and hence $Q_B$ is free.

The proof of the last statement is carried over verbatim as in Theorem 6.2, replacing $R_r$, $V_{p-r}$ and $U_{p-r}$ by $R_q/Q$, $V_o$, and $U_o$, respectively.

**Theorem 6.4.** The modules $R_r(r \in \mathcal{R}, r \neq q)$ and $R_q/Q$ are all the $q$ nonisomorphic PIM's for the group $G_q = SL(2, q)$.

(As the dimensions of $R_r(r \neq q)$ and $R_q/Q$ and the dimensions of $U_{p-r}(r \neq 0)$ and $U_o$ as given in [10] respectively are the same, it is immediately seen that the surjective module homomorphisms $\Psi_r$ and $\Psi_q$ obtained in Theorems 6.2 and 6.3 respectively are isomorphisms. The proof given below is, on the other hand, based on an interesting dimension counting argument suggested by D.N. Verma which avoids the use of the dimensions of the PIM's.)
Proof. Let \( u_r \) denote the dimensions of the PIM
\[ U_{n-r} = U_{(p-r_0, \ldots, p-r_{n-1})} (1 \leq r_i \leq p) \]
and \( v_r \), the dimensions of the corresponding irreducible modules \( V_{n-r} \).
(It is known that \( v_r = \prod_{i=0}^{n-1} (p - r_i + 1) \)). Since the module homorphisms \( \Psi_r \) from \( R_i \) to \( U_{n-r} \) in Theorem 6.2, and \( \Psi_q \) from \( R_q \) to \( U_0 \) in Theorem 6.3 are surjective, it follows that \( u_r \leq 2^m q (1 \leq r_i \leq p) \) \( r = (p, p, \ldots, p) \), where \( m \) is the number of \( r_i \neq 1 \), and thus \( u_q \leq (2^n - 1)q \). Then by a well known theorem on the multiplicities of the PIM's in the regular representation of \( G_q \) (see e.g., [4, p. 591]), we get that
\[
q(q^2 - 1) = \sum_{r_{s=1}}^{n} u_r \sum_{i=0}^{n-1} (p - r_i + 1)
\leq (2^n - 1) q \cdot 1 + \sum_{(r_{s=1}, p)} 2^m q \cdot \Pi (q - r_i + 1) + q \cdot q
= q(q^2 - 1) \text{ (by actual computation).}
\]
Hence \( u_q = (2^n - 1)q \) and \( u_r (r \neq (p, \ldots, p)) = 2^m q \), and hence \( \Psi_q \) and \( \Psi_r \) are isomorphisms. Hence the theorem.

Remark 1. The author originally obtained the above construction of the PIM's of \( K[G_K] \) using the representations of the restricted Lie algebra \( L = \mathcal{L} \otimes K \) of \( \mathcal{L} \), equivalently, of the representations of the restricted universal enveloping algebra (\( u \)-algebra) \( U \) of \( L \). One can show that the modules \( V_{i,K} (0 \leq i \leq p - 1) \), can be made into restricted \( U \)-modules and that these are irreducible left \( K[G_K]\)-modules as well (see e.g., Curtis [3]). A \( U \)-submodule \( \bar{R}_K \) of \( V_{r-1,K} \otimes V_{p-1,K} \) was constructed in the same way as the module \( R \) of \( V_{r-1} \otimes V_{p-1} \) was obtained in Section 4. Then it was proved that \( R_K \) is a (suitable) PIM for \( K[G_K] \). The only difference was that the author proved straightaway that the \( K[G_K] \)-submodule \( \bar{P}_K \) (the analogue of \( P_K \) in our construction) is the unique maximal submodule of \( \bar{R}_K \); this proof was rather laborious, particularly in the general tensor product form.

However, we get back the module \( \bar{R}_K \) in the following sense. The module \( R_K \) can actually be identified with a submodule of \( V_{r-1,K} \otimes V_{p-1,K} \). For this it is enough to verify that
\[
R = R \cap (V_{r-1,z} \otimes V_{p-1,z}).
\]
This can be verified directly by using the fact that for each \( i, 0 \leq i \leq p + r - 2 \), and for each \( j, 0 \leq j \leq p - r \), the sets of integers involved in the expressions of \( E_i \) and \( Z_j \) have g.c.d. 1 (see 4.3 and 4.6).
Remark 2. Recently J. E. Humphreys used some of his general theory in his paper [7], to show that the \( K[G,K] \), or \( U \)-modules \( V_{r-1} \otimes V_{p-1} \) (which are projective) contain direct summands which must be equivalent to the \( R_K \) which we have constructed. This is of course natural to expect, after we have identified \( R_K \) with a submodule of \( V_{r-1,K} \otimes V_{p-1,K} \) and remembering that \( R_K \) is a projective (and hence injective) \( K[G,K] \)-module.

Remark 3. Another interesting fact is that the \( 2p \)-dimensional modules \( R_K, 1 < r < p \), are actually PIM’s for the \( u \)-algebra \( U \) too; one can verify this directly or see J. E. Humphreys [8], where he proves that, the \( U \)-modules \( V_{r-1} \otimes V_{p-1} \), as in the case of the group algebra module, are projective.

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