The extremal function for $K_8^-$ minors

Zi-Xia Song

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA

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Abstract

A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges. Let $K_8^-$ be the graph obtained from $K_8$ by deleting one edge. We prove a conjecture of Jakobsen that every simple graph on $n \geq 8$ vertices and at least $(11n - 35)/2$ edges either has a $K_8^-$ minor, or is isomorphic to a graph obtained from disjoint copies of $K_1, 2, 2, 2, 2$ and/or $K_7$ by identifying cliques of size five.

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1. Introduction

All graphs considered in this paper are finite and simple. Let $G$ be a graph and let $x$ and $y$ be adjacent vertices in $G$. We denote by $G/xy$ the graph obtained from $G$ by contracting the edge $xy$, i.e., by replacing $x$ and $y$ by one new vertex adjacent to every vertex that is adjacent to $x$ or $y$ in $G$. A graph is a minor of another if the first can be obtained from a subgraph of the second by contracting edges. We say that a graph $G$ has an $H$ minor (denoted by $G \succ H$) if $G$ has a minor isomorphic to $H$.

One of the central problems of Graph Theory is the following conjecture due to Hadwiger [3].
Conjecture 1.1. For every integer \( t \geq 1 \), every graph with no \( K_{t+1} \) minor is \( t \)-colorable.

Hadwiger’s conjecture is trivially true for \( t \leq 2 \), and reasonably easy for \( t = 3 \), as shown by Dirac [2]. However, for \( t \geq 4 \), Hadwiger’s conjecture implies the Four Color Theorem. (To see that, let \( H \) be a planar graph, and let \( G \) be obtained from \( H \) by adding \( t - 4 \) vertices, each joined to every other vertex of the graph. Then \( G \) has no \( K_{t+1} \) minor, and hence is \( t \)-colorable by Hadwiger’s conjecture, and hence \( H \) is 4-colorable.) Wagner [14] proved that the case \( t = 4 \) of Hadwiger’s conjecture is, in fact, equivalent to the Four Color Theorem, and the same was shown for \( t = 5 \) by Robertson et al. [10]. Hadwiger’s conjecture remains open for \( t \geq 6 \). For \( t = 6 \), Kawarabayashi and Toft [8] proved that any 7-chromatic graph has either \( K_7 \) or \( K_{4,4} \) as a minor. Jacobsen [4] proved that every \( 7 \)-chromatic graph has a \( K_7^\ominus \) minor, where for integer \( p > 0 \), \( K_p^\ominus \) (resp. \( K_p^= \)) denotes the graph obtained from \( K_p \) by removing one edge (resp. two edges).

Mader [9] showed that for \( p \leq 7 \) every graph with \( e(G) \geq (p - 2)|G| - \left( \binom{p-1}{2} \right) + 1 \) has a \( K_p \) minor. For \( p = 6 \), this result was instrumental in the proof of Hadwiger’s conjecture for \( t = 5 \) mentioned above, and so it is reasonable to expect that further progress will be tied to a suitable generalization of Mader’s result. Unfortunately, Mader’s theorem does not extend for \( p \geq 8 \): \( K_{2,2,2,2,2,2} \) is a counterexample for \( p = 8 \), and further counterexamples may be constructed by adding new vertices joined to all existing ones. On the other hand, Jørgensen [7] proved that every graph \( G \) with \( e(G) \geq 6|G| - 20 \) either has a \( K_8 \) minor or is a \( (K_2,2,2,2,2,5) \)-cockade, where cockades are defined recursively as follows. Let \( H_1, H_2 \) be graphs and let \( k \) be an integer. Any graph isomorphic to \( H_1 \) or \( H_2 \) is an \( (H_1, H_2, k) \)-cockade. Now let \( G_1, G_2 \) be \( (H_1, H_2, k) \)-cockades and let \( G \) be obtained from the disjoint union of \( G_1 \) and \( G_2 \) by identifying a clique of size \( k \) in \( G_1 \) with a clique of the same size in \( G_2 \). Then the graph \( G \) is also an \( (H_1, H_2, k) \)-cockade, and every \( (H_1, H_2, k) \)-cockade can be constructed this way. In the case when \( H_1 = H_2 = H \), it will be called an \( (H, k) \)-cockade. Thomas and the author [12] proved that every graph \( G \) with \( e(G) \geq 7|G| - 27 \) either has a \( K_9 \) minor or is a \( (K_2,2,2,2,2,6) \)-cockade, or is isomorphic to \( K_{2,2,2,3,3} \). More generally, Seymour and Thomas (see [12]) conjectured the following:

Conjecture 1.2. For every \( p \geq 1 \) there exists a constant \( N = N(p) \) such that every \( (p - 2) \)-connected graph on \( n \geq N \) vertices and at least \( (p - 2)n - \binom{p-1}{2} + 1 \) edges has a \( K_p \) minor.

In [1], Chen, Gould, Kawarabayashi, Pfender and Wei proved that every simple graph on \( n \) vertices and at least \( 9n - 46 \) edges has a \( K_9^- \) minor, and used that to deduce that if, in addition, \( G \) is 6-connected, then it is 3-linked. The work of Chen, Gould, Kawarabayashi, Pfender and Wei suggested that there may be interest in the extremal problem for \( K_p^- \) minors.

Jakobsen [4,5] proved the following:

Theorem 1.3. For \( p = 5, 6, 7 \), if \( G \) is a graph with \( n \geq p \) vertices and at least \( (p - \frac{5}{2})n - \frac{1}{2}(p - 3)(p - 1) \) edges, then \( G \geq K_p^- \), or \( G \) is a \( (K_{p-1}, p - 3) \)-cockade when \( p \neq 7 \), or \( p = 7 \) and \( G \) is a \( (K_{2,2,2,2}, K_6, 4) \)-cockade.
In [5], Jakobsen also conjectured that Theorem 1.3 extends to $p = 8$ as follows:

**Conjecture 1.4.** If $G$ is a graph with $n \geq 8$ vertices and at least $\frac{11n - 35}{2}$ edges, then $G > K_8^-$ or $G$ is a $(K_1, 2, 2, 2, K_7, 5)$-cockade.

The purpose of this paper is to prove Conjecture 1.4, as follows.

**Theorem 1.5.** If $G$ is a graph with $n \geq 8$ vertices and at least $\frac{11n - 35}{2}$ edges, then $G > K_8^-$ or $G$ is a $(K_1, 2, 2, 2, K_7, 5)$-cockade.

Jakobsen [5] pointed out that the graph $K_2, 2, 2, 2, 3$ contains no $K_9^-$ minor. In fact, there are many more small counterexamples to an analogue of Conjecture 1.4 for $p = 9$: $K_1, 1, 2, 2, 2, 2$, $K_1, 2, 2, 3, 3$, $K_3, 3, 3, 3$ and $K_2, 3, 3, 4$. Thus an analogue of Conjecture 1.4 for $p = 9$ will have to include the conclusion that $G$ is isomorphic to one of these graphs.

**2. Preliminaries**

We need to introduce more notation. For a graph $G$, we use $|G|$ and $e(G)$ to denote the order and size of $G$, respectively. The *complement* $\overline{G}$ of a graph $G$ has the same vertex set as $G$, and distinct vertices $u, v$ are adjacent in $\overline{G}$ just when they are not adjacent in $G$. The complement of a complete graph $K_v$ will be denoted by $\overline{K}_v$. For any vertex $v$ of a graph $G$, we use $N(v)$ or $N_G(v)$ to denote the subgraph of $G$ spanned by the neighbors of $v$. The subgraph spanned by $x$ and the neighbors of $x$ is denoted by $N[x]$ or $N_G[x]$. For any subgraph $H$ of $G$ we denote by $N(H)$ the subgraph of $G$ spanned by the vertices in $V(G) \setminus V(H)$ that are adjacent to a vertex in $H$.

For a graph $G$, $A, B \subseteq V(G)$ and two nonadjacent vertices $x, y \in V(G)$, we will use $e_G(A, B)$ to denote the number of edges between $A$ and $B$ in $G$ and $G + xy$ to denote the graph obtained from $G$ by adding an edge joining $x$ to $y$. The join $G + H$ (resp. union $G \cup H$) of two vertex disjoint graphs $G$ and $H$ is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$ (resp. $E(G) \cup E(H)$). The following results will be needed later. Theorem 2.1 is a result of Jørgensen [7], Theorem 2.2 was first proved by Jung [6]. For a complete characterization of the graphs with no pair of such paths, see [11, 13].

**Theorem 2.1.** Let $G$ be a graph with $n \leq 11$ vertices and $\delta(G) \geq 6$. Then $G \supseteq K_6 \cup K_1$ or $G$ is one of the graphs $K_2, 2, 2, 2$, $K_3, 3, 3$ or the complement of the Petersen graph. In particular, $G \supseteq K_6^- \cup K_1$.

**Theorem 2.2.** Let $G$ be a 4-connected graph and let $x_1, x_2, y_1, y_2$ be vertices in $G$. If $G$ does not contain an $x_1 - y_1$ path and an $x_2 - y_2$ path that are disjoint, then $G$ is planar and $e(G) \leq 3|G| - 7$.

In the proof of Theorem 1.5, we shall consider graphs with $n$ vertices and exactly $\lceil \frac{11n - 35}{2} \rceil$ edges. Such graphs have vertices of degree at most 10. Since we want to consider contraction
in the graph spanned by the neighbors of a vertex of minimum degree, we need some results about contractions in graphs with at most 10 vertices.

**Lemma 2.3.** Let \( G \) be a graph with 8 vertices and \( \delta(G) \geq 5 \). Then \( G \not\cong K^−_6 \cup K_1 \) or \( G \) is isomorphic to \( C_8, C_4 + C_4, K_3 + C_5, K_2 + C_6, \) or \( K_{2,3,3} \). In particular, all these graphs are edge maximal subject to not having a \( K^−_6 \cup K_1 \) minor. Moreover, \( C_8 > K_6 \) and \( C_4 + C_4 > K_6 \).

**Proof.** It is not hard to verify that the graphs listed are edge maximal subject to not having a \( K^−_6 \cup K_1 \) minor. Thus we may assume that every edge of \( G \) is incident with a vertex of degree five. Let \( x \in V(G) \) be such that \( d(x) = 5 \). If \( e(G - x) \geq \frac{1}{2}(7|G - x| - 15) = 17 \), by Theorem 1.3, \( G - x \not\cong K^−_6 \) or \( G - x = K_3 + (K_2 \cup K_2) \). In the second case, \( x \) is adjacent to the four vertices of degree 4 in \( K_3 + (K_2 \cup K_2) \). It is easy to check that \( G > K^−_6 \cup K_1 \). Hence we may assume \( e(G - x) \leq 16 \), and so \( 20 \leq e(G) \leq 21 \). If \( e(G) = 20 \), then \( G \) is 5-regular on 8 vertices. Thus \( \overline{G} \) is 2-regular. It follows that \( \overline{G} \) is isomorphic to \( C_8, C_4 \cup C_4, \) or \( C_3 \cup C_5 \), and so the lemma holds. If \( e(G) = 21 \), then \( G \) has either one vertex of degree 7 and seven vertices of degree 5 or two vertices of degree 6 and six vertices of degree 5. In the first case, let \( y \) be the vertex of degree 7. Then \( G - y \) is 4-regular on 7 vertices. Thus \( \overline{G - y} = C_7 \) or \( C_3 \cup C_4 \). It is easy to check that \( G - y \not\cong K^−_5 \cup K_1 \) and thus \( G > K^−_6 \cup K_1 \). For the latter, let \( z, w \) be the two vertices of degree 6. Since \( G \) is edge minimal, we have \( zw \notin E(G) \). It follows that \( G - \{z, w\} \) is 3-regular on 6 vertices. Thus \( G \) is \( K_2 + \overline{C_6} \) or \( K_{2,3,3} \). The last assertion is easy to verify. □

**Lemma 2.4.** Let \( G \) be a graph with \( 9 \leq n \leq 10 \) vertices and \( \delta(G) \geq 5 \). Then \( G \not\cong K^−_6 \cup K_1 \) or \( G \) is isomorphic to \( J \) (given in Fig. 1).

**Proof.** Lemma 2.4 can be checked by computers. However, a computer-free proof is given in the appendix. □

By Lemmas 2.3 and 2.4, it follows that
Corollary 2.5. Let \( G \) be a graph with \( 8 \leq |G| \leq 10 \) and \( \delta(G) \geq 5 \). Then \( G \supseteq K_6^- \cup K_1 \) or \( G \) is isomorphic to \( C_8, C_4 + C_4, K_3 + C_5, K_2 + C_6, K_2, 3, 3, \) or \( J \). In particular, all these graphs are edge maximal (subject to not having a \( K_6^- \cup K_1 \) minor) with maximum degree \( \leq |G| - 2 \). Moreover, \( C_8 \supseteq K_6, C_4 + C_4 \supseteq K_6, \) and \( J \supseteq K_6 \).

Finally, we need some results about contractions in \((K_1, 2, 2, 2, K_7, 5)\)-cockades. Our proof of Conjecture 1.4 uses induction by deleting and contracting edges of \( G \). We need to investigate graphs \( G \) such that the new graph \( G - xy \) or \( G/xy \) is a \((K_1, 2, 2, 2, K_7, 5)\)-cockade, where \( xy \in E(G) \). It turns out that contracting an edge of \( G \) in the proof of Conjecture 1.4 will not produce a \((K_1, 2, 2, 2, K_7, 5)\)-cockade. So we only consider the case when \( G - xy \) is a \((K_1, 2, 2, 2, K_7, 5)\)-cockade. We do that next.

Lemma 2.6. Let \( G \) be a \((K_1, 2, 2, 2, K_7, 5)\)-cockade and let \( x \) and \( y \) be nonadjacent vertices in \( G \). Then \( G + xy \) is contractible to \( K_8^- \).

**Proof.** This is obviously true if \( G \) is \( K_1, 2, 2, 2, 2 \). So we may assume that \( G \) is obtained from \( H_1 \) and \( H_2 \) by identifying on \( K_5 \), where both \( H_1 \) and \( H_2 \) are \((K_1, 2, 2, 2, K_7, 5)\)-cockades. If both \( x \), \( y \in V(H_1) \), then \( H_1 \supseteq K_8^- \) by induction. So we may assume that \( x \in V(H_1) - V(H_2) \) and \( y \in V(H_2) - V(H_1) \). If there exists \( z \in V(H_1) \cap V(H_2) \) such that \( yz \notin E(G) \), then by contracting \( V(H_1) - V(H_1) \cap V(H_2) \) to \( z \), the resulting graph will have a \( K_8^- \) minor by induction. So we may assume \( y \) is adjacent to all vertices in \( V(H_1) \cap V(H_2) \). Similarly, we may assume that \( x \) is adjacent to all vertices in \( V(H_1) \cap V(H_2) \). Hence there exists \( w \in V(H_1) \) such that \( H_1[\{w, x, V(H_1) \cap V(H_2)\}] \) is a \( K_7 \) subgraph in \( H_1 \). Clearly, \( G[\{w, x, y, V(H_1) \cap V(H_2)\}] + xy \supseteq K_8^- \). \( \square \)

It is easy to observe that

Lemma 2.7. Let \( G \) be a \((K_1, 2, 2, 2, K_7, 5)\)-cockade. Then \( e(G) = \frac{11|G|-35}{2} \).

3. Proof of Theorem 1.5

In this section we prove Theorem 1.5 by induction on \( n \). The only graphs \( G \) with 8 vertices and \( e(G) \geq \frac{11 \times 8 - 35}{2} \) are \( K_8^- \) and \( K_8 \). So we may assume that \( n \geq 9 \) and the assertion holds for smaller values of \( n \).

Suppose \( G \) is a graph with \( n \) vertices and \( e(G) \geq \frac{11n - 35}{2} \) but \( G \) is not contractible to \( K_8^- \) and \( G \) is not a \((K_1, 2, 2, 2, K_7, 5)\)-cockade. By Lemma 2.6, we may assume that \( e(G) = \lceil \frac{11n - 35}{2} \rceil \).

If \( G \) has a vertex \( x \) with \( d(x) \leq 5 \), then \( e(G - x) \geq \frac{11n - 35}{2} - 5 > \frac{11|G-x|-35}{2} \). By the induction hypothesis and Lemma 2.7, \( G - x \supseteq K_8^- \), a contradiction. Thus

(1) \( \delta(G) \geq 6 \).

(2) \( \delta(N(x)) \geq 5 \) for any \( x \in V(G) \).
Proof. Suppose that there exists \( y \in N(x) \) such that \( d_{N(x)}(y) \leq 4 \). Then \( e(G/xy) \geq \frac{11(n-1)-34}{2} \geq \frac{11(G/xy)-35}{2} \). By the induction hypothesis and Lemma 2.7, \( G - x > K_8^- \), a contradiction. □

Let \( S \) be a minimal separating set of vertices in \( G \), and let \( G_1 \) and \( G_2 \) be proper subgraphs of \( G \) so that \( G = G_1 \cup G_2 \) and \( G_1 \cap G_2 = G[S] \). For \( i = 1, 2 \), let \( d_i \) be the largest integer so that \( G_i \) contains disjoint set of vertices \( V_1 \), \( V_2 \), \ldots, \( V_p \) so that \( G_i[V_j] \) is connected and \( |S \cap V_i| = 1 \), \( 1 \leq j \leq p = |S| \), and so that the graph obtained from \( G_i \) by contracting \( V_1 \), \( V_2 \), \ldots, \( V_p \) and deleting \( V(G) - (\cup_j V_j) \) has \( e(G[S]) + d_i \) edges. Let \( G_i' \) (resp. \( G_2' \)) be obtained from \( G_i \) (resp. \( G_2 \)) by adding \( d_i \) (resp. \( d_1 \)) edges to \( G[S] \). By (1), \( |G_i| \geq 7 \), \( i = 1, 2 \). Hence we may assume that \( e(G_i) \leq \frac{11|G_i|-35}{2} - d_2 \) (otherwise \( e(G_i') > \frac{11|G_i|-35}{2} \), in which case, \( G_i' > K_8^- \) by induction). Similarly, we may assume that \( e(G_2) \leq \frac{11|G_2|-35}{2} - d_1 \). Consequently,

\[
(3) \quad \frac{11n-35}{2} \leq e(G) = e(G_1) + e(G_2) - e(G[S]) \leq \frac{11n+11|S|-70}{2} - d_1 - d_2 - e(G[S]),
\]

and so

\[
(4) \quad 11|S| \geq 35 + 2d_1 + 2d_2 + 2e(G[S]).
\]

(5) \( G \) is 5-connected.

Proof. It follows from (4) that \( |S| \geq 4 \). Note that \( d_i \geq |S| - 1 - \delta(G[S]), i = 1, 2 \), and \( 2e(G[S]) \geq |S|\delta(G[S]) \). By (4), we have \( 7|S| \geq 31 + (|S| - 4)\delta(G[S]) \), which implies that \( |S| \geq 5 \). □

(6) There is no minimal separating set \( S \) so that \( G[S] \) is complete.

Proof. Suppose that \( G[S] \) is complete. By (5), \( |S| \geq 5 \). If \( |S| \geq 6 \), by contracting \( V(G_1) - S \) and \( V(G_2) - S \) into two new vertices, we get \( G > K_8^- \). So we may assume \( |S| = 5 \). Note that when \( G[S] = K_5 \), we get equality in (3). Thus \( e(G_i) = \frac{11|G_i|-35}{2} \) for \( i = 1, 2 \) and \( e(G) = \frac{11n-35}{2} \). It follows by induction that \( G \) is a \((K_1,2,2,2,2, K_7, 5)\)-cockade, a contradiction. □

(7) There is no minimal separating set \( S \) with a vertex \( x \) so that \( G[S - x] \) is complete.

Proof. Suppose that \( G[S - x] \) is complete. By (5), \( |S| \geq 5 \). By (6), we may assume \( \delta(G[S]) \leq |S| - 2 \). Then \( d_1 = d_2 = |S| - 1 - \delta(G[S]) \) and \( 2e(G[S]) = (|S| - 1)(|S| - 2) + 2\delta(G[S]) \). By (4), \( 11|S| \geq 35 + 4(|S| - 1 - \delta(G[S])) + (|S| - 1)(|S| - 2) + 2\delta(G[S]) = |S|^2 + |S| + 33 - 2\delta(G[S]) \geq |S|^2 + |S| + 33 - 2(|S| - 2). \) It follows that \( |S|^2 - 12|S| + 37 \leq 0 \), which is impossible. □

(8) \( 7 \leq \delta(G) \leq 10 \).

Proof. Let \( x \in V(G) \) be a vertex such that \( d(x) = \delta(G) \). By (1), \( d(x) \geq 6 \). If \( d(x) = 6 \), by (2), \( N(x) = K_6 \). Now \( K_6 \) will be a minimal separating set, which contradicts (6). Thus \( \delta(G) = d(x) \geq 7 \). On the other hand, since \( e(G) = \frac{11n-35}{2} \), we have \( d(x) \leq 10 \). □

(9) \( \delta(G) \geq 8 \).
Thus by (7), we may assume $K$ and let $v$ (9b), $w$ (9c) for any $v$ (9b), $w$ (9c), respectively. Hence

\begin{align*}
\text{Proof.} & \quad \text{Suppose that there exists } N(x) = [y, z_1, z_2, z_3, w_1, w_2, w_3] \text{ so that } y \text{ is adjacent to all vertices in } N(x) - y \text{ and } z_i w_i \notin E(G). \text{ Suppose that } G - N[x] \text{ is disconnected. Let } K \text{ and } K' \text{ be two components of } G - N[x]. \text{ Since } N(x) = K_{1,2,2,2}, \text{ by (7), } N(K) \text{ and } N(K') \text{ contain two pairs of nonadjacent vertices of } N(x), \text{ respectively. We may assume that } z_1, w_1 \in N(K) \text{ and } z_2, w_2 \in N(K'). \text{ Let } P \text{ be a } z_1-w_1 \text{ path in } K \text{ and } P' \text{ be a } z_2-w_2 \text{ path in } K'. \text{ Then by contracting all but one of the edges of } P \text{ and } P', \text{ respectively, we get a } K_8^- \text{ minor of } G, \text{ a contradiction.}

(9a) G - N[x] \text{ is connected.}

(9b) There is no vertex in } G - N[x] \text{ that is adjacent to a pair of nonadjacent vertices in } N(x).

\begin{align*}
\text{Proof.} & \quad \text{Suppose that there exists } v \in V(G) - N[x] \text{ adjacent to, say } z_1 \text{ and } w_1. \text{ Let } K \text{ be a component of } G - N[x] - v. \text{ If } N(K) \text{ contains a pair of nonadjacent vertices of } \{z_2, z_3, w_2, w_3\}, \text{ say } z_2 \text{ and } w_2, \text{ then there is a } z_2-w_2 \text{ path } P \text{ in } K. \text{ Now by contracting } v \text{ to } z_1 \text{ and all but one of the edges of the path } P, \text{ we get a } K_8^- \text{ minor of } G, \text{ a contradiction.}

\text{Thus by (7), we may assume } z_1, w_1 \in N(K). \text{ Let } K' = G - N[x] - K. \text{ Clearly, } K' \text{ is connected. If } N(K') \text{ contains a pair of nonadjacent vertices, other that } z_1 \text{ and } w_1 \text{ of } N(x), \text{ then } G \text{ would have a } K_8^- \text{ minor, a contradiction. Therefore, we may assume that } w_2, w_3 \in N(K) - N(K') \text{ and } z_2, z_3 \in N(K') - N(K). \text{ Since } w_2 z_3 \in E(G), w_2 \text{ and } z_3 \text{ have at least one common neighbor in } G - N[x]. \text{ It follows that } w w_2, w z_3 \in E(G) \text{ and thus } w_2 \in N(K'), \text{ a contradiction.}

(9c) \text{ for any } v \in N(x) - y, v \text{ has at least three neighbors in } G - N[x]. \text{ Hence,}

\text{Suppose that } w \text{ is a cut-vertex of } G - N[x]. \text{ Let } K \text{ be a component of } G - N[x] - w \text{ and let } K' = G - N[x] - K. \text{ Then } K' \text{ is connected. Since } N(x) = K_{1,2,2,2}, \text{ by (7), } N(K) \text{ and } N(K') \text{ contain at least one pair of nonadjacent vertices of } N(x), \text{ respectively. If } N(K) \text{ and } N(K') \text{ contain distinct pairs of nonadjacent vertices of } N(x), \text{ then } G \text{ would have a } K_8^- \text{ minor by the existence of such two disjoint paths in } K \text{ and } K', \text{ respectively. So we may assume that } z_1, w_1 \in N(K) \cap N(K') \text{ and } N(K) \text{ and } N(K') \text{ contain no pair of nonadjacent vertices of } N(x) \text{ other than } z_1, w_1. \text{ Thus we may assume that } z_2, z_3 \in N(K') - N(K) \text{ and } w_2, w_3 \in N(K) - N(K'). \text{ Since } w_2 z_3 \in E(G), w_2 \text{ and } z_3 \text{ have at least one common neighbor in } G - N[x]. \text{ It follows that } w w_2, w z_3 \in E(G), \text{ and thus } w_2 \in N(K'), \text{ a contradiction. Therefore}

(9d) G - N[x] \text{ is 2-connected.}
Consider the graph \( H = G - \{x, y, z_3, w_3\} \). We next show that \( H \) is 4-connected.

Let \( S \) be a minimal separating set of at most three vertices in \( H \). By (9c) and (9d), \(|S| \geq 2\) and \(|S \cap N(x)| \leq 1\). If \(|S \cap N(x)| = 1\), we may assume that \( w_1 \in S \). Since \( z_1z_2, z_1w_2 \in E(G) \), \( z_1, z_2, w_2 \) are in the same component of \( H - S \). Denote this component by \( K \). If \( w_1 \notin S \), then also \( w_1 \in K \), and in this case we assume that \( S \) and \( w_1 \) are chosen so that \(|S \cap N(w_1)|\) is maximal. We next show that there exist \( z'_2 \) and \( w'_2 \) in \( G - N[x] - S \) adjacent to \( z_2 \) and \( w_2 \), respectively. By (9b) and (9c), we may assume that \( w_2 \) has exactly three neighbors in \( G - N[x] \), say \( a, b, c \), and \( S = \{a, b, c\} \). Clearly, \( w_1 \notin S \). By the assumption that \(|S \cap N(w_1)|\) is maximal, it follows that \( w_1 \) is adjacent to all vertices in \( S \). Since \( w_2z_1 \in E(G) \), by (2), \( z_1 \) and \( w_2 \) have at least one common neighbor in \( G - N[x] \). Since \( w_2 \) has only three neighbors \( a, b, c \) in \( G - N[x] \), we may assume \( z_1a \in E(G) \). Now \( a \) is adjacent to both \( z_1 \) and \( w_1 \), which contradicts (9b). This proves that there exist \( z'_2, w'_2 \in (V(G) - N[x] - S) \) such that \( z_2z'_2, w_2w'_2 \in E(G) \).

Clearly, \( z'_2, w'_2 \in K \). By (9d), \( G - N[x] \) contains two independent \( z'_2w'_2 \) paths. One of these paths is contained in \( G[K \cup S] \).

Since \( G \) is not contractible to \( N[x] + z_2w_2 + z_3w_3 \), there is no \( z_3w_3 \) path in \( G[K' \cup \{z_3, w_3\}] \), where \( K' \neq K \) is another component of \( H - S \). But this implies \( K' \) is separated from \( x \) by \( S \) and two adjacent vertices in \( N(x) \). We may assume that such two vertices are \( \{y, w_3\} \). Since \( G \) is 5-connected, \(|S| = 3\). Let \( S = \{s_1, s_2, s_3\} \), where \( s_1 = w_1 \) if \( w_1 \notin S \), and \( S' = S \cup \{y, w_3\} \). Then \( S' \) is a minimal separating set of \( G \). Let \( H_1 = G[K' \cup S'] \) and \( H_2 = G - K' \). Let \( d_1 \) and \( d_2 \) be defined as in the paragraph following (2). Clearly, \( K \cup \{x, z_3\} \) is contained in \( H_2 \). By Menger’s theorem, there exist three disjoint paths between \( \{x, w_1, z_2\} \) and \( S \) in \( G - \{y, w_3\} \). By contracting those paths, we get \( d_2 + e_G(S') = e(K_5) = 10 \). By (2), \( d_1 \geq 1 \). By (4), \( 55 = 11 \times 5 \geq 35 + 2(d_2 + e(S')) + 2d_1 = 35 + 20 + 2 = 57 \), a contradiction. Thus \( H \) is 4-connected.

Since \( G \) is not contractible to \( K_5^- \), it follows from Theorem 2.2 applied to the vertices \( z_1, z_2, w_1, w_2 \) that \( e(H) \leq 3|H| - 7 = 3(n - 4) - 7 \). Since the vertices \( z_3 \) and \( w_4 \) have no common neighbor in \( G - N[x] \), they together have at most \(|G| - |N[x]| = n - 8 \) neighbors in \( G - N[x] \). The vertices \( \{z_3, w_3\} \) are incident with 8 edges of \( N[x] \). Thus

\[
\frac{11n - 35}{2} \leq e(G) \leq d(x) + d(y) - 1 + e(H) + (n - 8) + 8
\]

\[
\leq 7 + n - 2 + 3(n - 4) - 7 + (n - 8) + 8 = 5n - 14.
\]

It follows that \( n \leq 7 \), which contradicts the fact that \( n \geq \delta(G) + 1 \geq 8 \) by (8).

\( \square \)

(10) Let \( x \) be a vertex such that \( 8 \leq d(x) \leq 10 \). Then there is no component \( K \) of \( G - N[x] \) such that \( N(K) = N(x) \).

**Proof.** Suppose such a component \( K \) exists. By (2), \( \delta(N(x)) \geq 5 \). By Corollary 2.5, \( N(x) > K_6^- \cup K_1 \) or \( N(x) > K_6 \) or \( N(x) \in \{K_3 + C_5, K_2, 3, 3, K_2 + C_6\} \). In the first case, there is a vertex \( y \in N(x) \) such that \( N(x) - y > K_6^- \). By contracting \( V(K) \cup \{y\} \) to a single vertex we see that \( G > K_5^- \), a contradiction. We will use this argument repeatedly later, and we shall refer to it as “contracting \( K \) onto a free vertex of \( N(x) \)”.

If \( N(x) > K_6 \), then we obtain the same conclusion by contracting \( K \) to a vertex. So we may assume that \( N(x) \in \{K_3 + C_5, K_2, 3, 3, K_2 + C_6\} \). We claim that \( G - N[x] \) is connected. Suppose
$G - N[x]$ is disconnected. Let $K' \neq K$ be another component of $G - N[x]$. By (6), $N(K')$ is not complete. Let $a, b \in N(K')$ be such that $ab \notin E(G)$. Let $P$ be an $a$-$b$ path in $K'$. By Corollary 2.5, $N(x)$ is edge maximal, and so $N[x] \cup P > K_7^-$, a contradiction. Thus $G - N[x]$ is connected, as claimed. We consider the following two cases.

**Case 1**: $G - N[x]$ is 2-connected.

Suppose $N(x) = \{K_2 + C_6, K_2, 3, 3\}$. By (2), there exist $x_1, x_2, y_1, y_2 \in N(x)$ such that $x_1, x_2, y_1, y_2 \in E(G)$, $x_1$ and $x_2$ (resp. $y_1$ and $y_2$) have at least two common neighbors in $G - N[x]$, and $x_1 y_1, x_2 y_2 \notin E(G)$ but $N[x] + x_1 y_1 + x_2 y_2 > K_8^-$. Let $u_1, u_2 \in V(K)$ be two distinct common neighbors of $x_1$ and $x_2$, and $w_1, w_2 \in V(K')$ be two distinct common neighbors of $y_1$ and $y_2$, respectively. By Menger’s Theorem, $K'$ contains two disjoint paths from $\{u_1, u_2\}$ to $\{w_1, w_2\}$. Thus $G$ has two disjoint paths with interiors in $K$, one with ends $x_1, y_1$, and the other with ends $x_2, y_2$. Then $G > K_8^-$ by the existence of those two paths, a contradiction.

Suppose $N(x) = \overline{K_3} + C_5$. Let $V(\overline{K_3}) = \{a_1, a_2, a_3\}$ and let $\overline{C_5}$ have vertices $y_1, y_2, y_3, y_4, y_5$ in order. Let $w \in V(G - N[x])$. Then $G - N[x] - w$ is connected and each vertex of $N(x)$ is adjacent to at least one vertex of $G - N[x] - w$. If $w$ is adjacent to two vertices of $a_1, a_2, a_3$, say $a_1, a_2$, then $G > N[x] + a_1 a_2 + y_1 y_2 + y_3 y_5 > K_8^-$ by contracting $w a_1$ and $V(G - N[x] - w)$ onto $y_2$, respectively. Similarly, if $w$ is adjacent to two nonadjacent vertices of $y_1, y_2, \ldots, y_5$, say $y_1, y_2$, then $G > N[x] + y_1 y_2 + y_2 y_3 + y_3 y_4 > K_8^-$ by contracting $w y_1$ and $V(G - N[x] - w)$ onto $y_3$, respectively. So we may assume that any pair of nonadjacent vertices of $N(x)$ have no common neighbor in $G - N[x]$. By (2), there exist $w_1, w_2, w_3, w_4 \in V(G - N[x])$ such that $w_i$ is a common neighbor of $y_i$ and $a_i, i = 1, 2, 3$, and $w_4$ a common neighbor of $y_2$ and $y_5$. Since any pair of nonadjacent vertices of $N(x)$ have no common neighbor in $G - N[x]$, we have $w_i \neq w_j$ for $i \neq j$. As $G - N[x]$ is 2-connected, there exist two disjoint paths, say $P_1, P_2$, between $\{w_1, w_4\}$ and $\{w_2, w_3\}$ in $G - N[x]$. We may assume that $P_1$ is a $w_1$-$w_3$ path. Now $G > N[x] + a_1 a_3 + y_1 y_2 + y_1 y_5 > K_8^-$ by contracting $a_1 w_1, y_1 w_2$ and all but one of the edges of each of $P_1, P_2$, a contradiction.

**Case 2**: $G - N[x]$ is not 2-connected.

In this case, $G - N[x]$ is connected. Let $w$ be a cut-vertex of $G - N[x]$ and let $H_1$ be a connected component of $G - N[x] - w$ with $N(H_1)$ minimal, and let $H_2 = G - N[x] - H_1$. Clearly, $H_2$ is also connected. If $N(H_1) \subseteq N(H_2)$ or $N(H_2) \subseteq N(H_1)$, say the latter. Then $N(H_1) = N(K) = N(x)$. By (6), there exists $e = ab \in E(N(H_2))$. By Corollary 2.5, there exists $u \in N(x)$ such that $N[x] + e - u > K_6^-$. Then $G > K_8^-$ by contracting the $a$-$b$ path in $H_2$ and contracting $V(H_1)$ to $u$. So we may assume that there exist $a \in N(H_1) - N(H_2)$ and $b \in N(H_2) - N(H_1)$. By (2), any two adjacent vertices in $N(x)$ have at least one common neighbor in $G - N[x]$. Thus $ab \notin E(G)$, $N_{N(x)}(a) \subseteq N(H_1)$ and $N_{N(x)}(b) \subseteq N(H_2)$. Suppose $N(x) = \{K_2 + \overline{C_6}, K_2, 3, 3\}$. Since $ab \notin E(G)$, there exist $x_1, y_1 \in N_{N(x)}(a)$ and $x_2, y_2 \in N_{N(x)}(b)$ such that $x_1 y_1, x_2 y_2 \notin E(G)$ but $N[x] + x_1 y_1 + x_2 y_2 > K_8^-$. Then $G > K_8^-$ by the existence of $x_i$-$y_i$ path in $H_i, i = 1, 2$, a contradiction. Suppose $N(x) = \overline{K_3} + C_5$. Let $V(\overline{K_3}) = \{a_1, a_2, a_3\}$ and let $\overline{C_5}$ have vertices $y_1, y_2, y_3, y_4, y_5$ in order. If $a, b \in \{a_1, a_2, a_3\}$, then $y_i \in (N_{N(x)}(a) \cap N_{N(x)}(b))$ for all $i = 1, 2, \ldots, 5$. Thus $G > K_8^-$ by contracting $V(H_1)$ to $y_1$ and $V(H_2)$ to $y_2$, a contradiction.
respectively. So we may assume that $a, b \in \{y_1, \ldots, y_5\}$, say $a = y_1$ and $b = y_2$. Clearly, $a_1, a_2, a_3, y_3, y_4 \in N(H_1)$ and $a_1, a_2, a_3, y_4, y_5 \in N(H_2)$. By (2), $y_3$ and $y_5$ have at least one common neighbor, say $y$, in $G - N[x]$. We may assume that $y \in V(H_1)$. Then $y_5 \in N(H_1)$ and so $G > K_{8^-}$ by contracting $V(H_1)$ to $y_4$ and $V(H_2)$ to $a_1$, respectively, a contradiction. \[\Box\]

(11) Let $x$ be a vertex such that $8 \leqslant d(x) \leqslant 10$. Then there is no component $K$ of $G - N[x]$ such that $N(K') \subseteq N(K)$ for every component $K'$ of $G - N[x]$.

**Proof.** Suppose such a component $K$ exists. Among all vertices $x$ with $8 \leqslant d(x) \leqslant 10$ for which such a component exists, choose $x$ to be of minimal degree. By (10), $N(K) \neq N(x)$. Let $y \in N(x) - N(K)$ be of smallest degree. Then $N(y) \subseteq N[x]$. Note that $d(y) \leqslant d(x) \leqslant d(y) + 2$. Suppose $d(y) = d(x)$. Then each vertex of $N(x)$ is either adjacent to all vertices in $N[y]$ or contained in $N(K)$, and $d_N(y) = |N(x)| - 1$. By Corollary 2.5, $N(x) > K_{6^-} \cup K_1$. By contracting $N(K)$ to a free vertex of $N(x)$, we obtain $G > K_{8^-}$, a contradiction. Next, suppose $d(y) = d(x) + 1$. Let $\{z\} = N(x) - N[y]$. Then $z \notin N(K)$, for otherwise we would have chosen $y$ for $x$. By the choice of $y$, $d(z) = d(x) - 1$. Thus $\{z\}$ is a component of $G - N(y)$ such that $N(\{z\}) = N(y)$, which contradicts (10). Finally, suppose $d(y) = d(x) + 2$. Then $d(x) = 10$. Let $\{z, w\} = N(x) - N[y]$. Clearly, $z$ and $w$ are not both in $N(K)$, otherwise we would have chosen $y$ for $x$. So we may assume that $z \notin N(K)$. If $zw \notin E(G)$, then $\{z\}$ is a component of $G - N[y]$ such that $z$ is adjacent to all the vertices in $N(y)$, which contradicts (10). So we may assume $zw \in E(G)$, and thus $w \notin N(K)$ (otherwise we would have chosen $y$ for $x$, because $K \cup \{z, w\}$ is a component in $G - N[y]$ satisfying (11)). By the choice of $y$, $d(z), d(w) \geqslant d(y)$. Now $e(N(x)) \geqslant (d(y) - 1) + (d(z) - 2) + (d(w) - 2) + 1 + \frac{4|N(x)| |N(y)|}{2} \geqslant 3d(y) - 4 + 2(d(y) - 1) = 5d(y) - 6 = 5(d(x) - 2) - 6 = 5d(x) - 16 > \frac{9|N(x)|}{2} - 12$. By Theorem 1.3, $N(x) > K_7^-$ and so $G > N[x] > K_{8^-}$, a contradiction. \[\Box\]

It follows from (11) that

(12) $G - N[x]$ is disconnected.

(13) Let $x$ be a vertex such that $8 \leqslant d(x) \leqslant 10$. Then there is no component $K$ of $G - N[x]$ with one vertex $w$ so that $d_G(y) \geqslant 11$ for every vertex $y \neq w$ in $K$ and $d_G(w) \geqslant d_G(x)$.

**Proof.** Assume that such a component $K$ exists. Let $G_1 = G - K$ and $G_2 = G[K \cup N(K)]$. Let $d_1$ be defined as in the paragraph following (2). Let $G'_2$ be a graph with $V(G'_2) = V(G_2)$ and $e(G'_2) = e(G_2) + d_1$ edges obtained by contracting edges in $G_1$. By (9), $|G'_2| \geqslant 9$. If $e(G'_2) > \frac{11|G'_2|^2 - 35}{2}$, then $G > G'_2 > K_{8^-}$ by induction, a contradiction. Thus $e(G_2) = e(G'_2) - d_1 \geqslant \frac{11|G'_2|^2 - 35}{2} - d_1 = \frac{11|N(K)| + 11|K| - 35}{2} - d_1$. On the other hand, for any $u \in N(K)$, there exists $w \in K$ such that $uw \in E(G)$. By (2), $d_{G_2}(u) \geqslant 6$. Thus $e(G_2) \geqslant \frac{1}{2} (6 \times |N(K)| + 11(|K| - 1) + d_G(w)) \geqslant \frac{6|N(K)| + 11|K| - 11 + d_G(w)}{2}$. It follows that

(13a) $5|N(K)| \geqslant 24 + d(x) + 2d_1$ and so $|N(K)| \geqslant 7$ by (9).

Let $t = e_G(N(K), K)$ and $d = \delta(N(K))$. Then $e(G_2) = e(G[K]) + t + e(N(K)) \geqslant \frac{11(|K| - 1) + d_G(w) - t}{2} + t + \frac{|N(K)| \times d}{2} \geqslant \frac{11|K| - 11 + d(x) + t + |N(K)| \times d}{2}$. It follows that
By (13), there exists another vertex adjacent to every vertex in $N(K)$. By (9), $G$ has a vertex of degree 8, 9 or 10. Among the vertices of degree 8, 9 or 10 for which the order of the largest component of $G - N[x]$ is maximum, choose $x$ so that its degree is minimum. Let $K$ be a largest component of $G - N[x]$.

By (12), there is another component $K'$ of $G - N[x]$. By (13), there is a vertex $x'$ in $K'$ of degree $d_G(x') \leq 10$. By the maximality of the order of $K$, $N(K) \subseteq N(x') \cap N(x)$. Thus $N(K) \subseteq N(K')$ and $K$ is also a component of $G - N[x']$. By the choice of $x$, $d(x') \geq d(x)$. By (13), there exists another vertex $y' \neq x'$ in $K'$ of degree $d(x') \leq d(y') \leq 10$. Clearly, $y'$ is adjacent to every vertex in $N(K)$. By (11), there is a third component $K''$ of $G - N[x]$. By symmetry, $K''$ has two vertices $x''$, $y''$ of degree at most 10 in $G$ and $N(K) \subseteq N(x'') \cap N(y'')$. Let $G_1 = G - K$, $G_2 = G[N(K) \cup K]$ and let $d_1$ and $d_2$ be as in the paragraph following (2).

Since $\delta(N(x)) \geq 5$, $\delta(N(K)) \geq 5 - (10 - |N(K)|) = |N(K)| - 5$. Therefore there is a subgraph $T$ of $N(K)$ with $|N(K)| - 5$ vertices and at least $|N(K)| - 6$ edges. Contract the vertices in $N(K) - T$ with different vertices in $\{x, x', y, x'', y''\}$, which are adjacent to every vertex in $N(K)$. It is easy to see that

$$d_1 + e(N(K)) \geq e(K_5) + 5(|N(K)| - 5) + (|N(K)| - 6) = 6|N(K)| - 21.$$

(13c) $|K|^2 - 12|K| + 11 + s \geq 0$.

By (10), $N(K) \neq N(x)$. This, together with (13a), implies that $7 \leq |N(K)| \leq 9$. Thus $|K| \geq (\Delta(G_2) + 1) - |N(K)| \geq (11 + 1) - 9 = 3$. We next show that $t \leq d(x) + s$, where $s = 14$.

By (2), $d \geq 5 - (|N(x)| - |N(K)|)$. If $|N(K)| = 7$, by (6) and (13a), we have $d_1 \geq 1$ and $d(x) + 2d_1 \leq 11$. Thus $d(x) \leq 9$ and $d \geq 5 - (9 - 7) = 3$. By (13b), $\frac{-t + d(x)}{2} \geq 1 + d(x) + 12 + \frac{3(|N(K)| - 11|N(K)|)}{2} \geq -7$. If $|N(K)| = 8$, then $d(x) \leq 10$ and $d \geq 5 - (10 - 8) = 3$. By (13b), $\frac{-t + d(x)}{2} \geq 11 + d(x) + \frac{d(|N(K)| - 2)}{2} - \frac{9|N(K)|}{2} \geq 11 + d(x) + 3\times\frac{(8 - 2)}{2} - \frac{9\times8}{2} \geq -7$. If $|N(K)| = 9$, then $d(x) \leq 10$ and $d \geq 5 - (10 - 9) = 4$. By (13b), $\frac{-t + d(x)}{2} \geq 11 + d(x) + \frac{d(|N(K)| - 2)}{2} - \frac{9|N(K)|}{2} \geq 11 + d(x) + \frac{4\times8 - 9}{2} - \frac{9\times9}{2} > -7$. In all cases, we have $t \leq d(x) + 14$ and $s = 14$.

Since $s = 14$ and $|K| \geq 3$, by (13c), $|K| > 8$. Note that $e(G[K]) \geq \frac{11(|K|-1)+d(w)-t}{2} \geq \frac{11(|K|-1)\times25}{2}$. It follows that $G[K] \geq K_8$ by induction, a contradiction. □
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Appendix. Proof of Lemma 2.4

Here we give a computer-free proof of Lemma 2.4. We first prove two lemmas.

**Lemma A.1.** Let $G$ be a graph on 8 vertices. Let $u, w \in V(G)$ be such that $d(u) \geq 4$, $d(w) = 7$, and $d(v) \geq 5$ for every $v \neq u, w$. Then $G \nrightarrow K_6 - K_1$.

**Proof.** Suppose $d(u) \geq 5$. Then $\delta(G) \geq 5$ and $\Delta(G) = 7$. By Lemma 2.3, $G \nrightarrow K_6 - K_1$. So we may assume that $d(u) = 4$. Then $e(G) \geq \lceil \frac{4+7+5+6}{2} \rceil = 21$. Note that $e(G - u) = e(G) - 4 \geq 17$ and $G - u$ has at most three vertices of degree 4. By Theorem 1.3, we have $G - u \nrightarrow K_6^c$. □

**Lemma A.2.** Let $G$ be a graph on 9 vertices. Let $uw \in E(G)$ be such that $d(u) = 4$, $d(w) \geq 7$ and $d(v) \geq 5$ for every $v \neq u, w$. Then $G \nrightarrow K_6 - K_1$.

**Proof.** Suppose $G$ is not contractible to $K_6^c - K_1$. We may assume that $G$ is edge minimal. We claim that $d(w) = 7$. Suppose $d(w) = 8$. Since the number of odd vertices of any graph is even, there exists another vertex, say $v \in V(G)$, such that $d(v) \geq 6$. Clearly, $vw \in E(G)$ and $d_{G-vw}(w) \geq 7, d_{G-vw}(u) = 4, d_{G-vw}(v) \geq 5$ for any $v \neq u, w$, which contradicts the fact that $G$ is edge minimal. Hence $d(w) = 7$, as claimed.

We first show that $G$ is 4-connected. Let $S$ be a minimal separating set of $G$ with $|S| \leq 3$. Since $|G| = 9$ and $d(v) \geq 5$ for any $v \neq u, w$, we have $|S| = 3$. Let $H_1$ and $H_2$ be the two connected components of $G - S$. Then $|H_1| = |H_2| = 3$. We may assume that $H_1 = K_3$ and each vertex of $H_1$ is adjacent to all vertices of $S$. Note that there exists a vertex, say $a \in V(H_2)$, adjacent to all vertices in $S$. Let $b \in S$. Now $G/ab - V(H_2 - a) \nrightarrow K_6^c$. This proves that $G$ is 4-connected.

Since $uw \in E(G)$, let $V(N(u)) = \{w, a, b, c\}$ and $A = V(G) - V(N[u]) = \{d, e, f, g\}$. We next prove the following claim.

**Claim.** For any $v \in \{a, b, c\}$, if $vw \in E(G)$, then $d_{N(u)}(v) \geq 2$.

**Proof.** Suppose otherwise. We may assume that $aw \in E(G)$ and $ab, ac \notin E(G)$. Let $w'$ be the new vertex in $G/ua$. Then $d_{G/ua}(w) = 6, d_{G/ua}(w') \geq 6$ and $ww' \in E(G)$. Note that $\delta(G/ua - ww') \geq 5$. By Lemma A.2, $G/ua \nrightarrow K_6^c - K_1$. □

Suppose that $w$ is adjacent to all vertices of $A$. Since $d_G(w) = 7$, we may assume that $cw \notin E(G)$. If $ca \notin E(G)$ or $d_G(a) \geq 6$, then $\Delta(G/uc) = 7, d_{G/uc}(b) \geq 4$ and $d_{G/uc}(v) \geq 5$ for any $v \in V(G/uc - b)$. By Lemma A.1, $G/uc \nrightarrow K_6^c - K_1$. Hence $ca \in E(G)$ and $d_G(a) = 5$. Similarly, $cb \in E(G)$ and $d_G(b) = 5$. Note that $e_G(v, \{a, b, c\}) \geq 1$ for any $v \in A$. If
Then, we have $G/da/db - u > K_6^-$. So we may assume that $e(G[A]) \leq 4$. Thus $e_G(A, N(u)) \geq 20 - 2e(G[A]) \geq 12$ and $e_G([a, b, c], A) = e_G(A, N(u)) - e_G(w, A) \geq 12 - 4 = 8$. Note that $d_G(a) = d_G(b) = 5$. It follows that $ab \notin E(G)$ and $c$ is adjacent to all vertices of $A$. Hence $d_G(c) = 7, d_G(v) = 5$ for any $v \in A$, and $e(G[A]) = 4$. So $G[A] = C_4$ or $K_1 + (K_2 \cup K_1)$. In the first case, we may assume that $G[A]$ has vertices $d, e, f, g$ in order and $ad, de \in E(G)$. Then by symmetry, either $af, bg \in E(G)$ or $ae, eg \in E(G)$. If $af \in E(G)$, then $be, bg \in E(G)$ and so $G/ad/be - u = K_6^-$. If $ae \in E(G)$, then $bf, bg \in E(G)$ and so $G/uv/de - a = K_6^-$. In the second case, we may assume that $ed, ef, eg, fg \in E(G)$. Then $d$ is adjacent to all vertices of $N(u)$. Note that either $af, bg \in E(G)$ or $ag, bf \in E(G)$. In either case, $G/da/db - u > K_6^-$. This proves the case when $w$ is adjacent to all vertices of $A$.

Suppose $w$ is adjacent to all vertices of $N(u)$. Then $d_G(w, A) = 3$. By Claim, $\delta(G([a, b, c])) \geq 1$. We may assume that $ab, bc \in E(G)$. Note that $e_G(v, [a, b, c]) \geq 1$ for any $v \in A$. If $G[A] = K_4$, then $G/ab/bc - u > K_6^-$. So we may assume that $e(G[A]) \leq 5$. It follows that $e_G(A, N(u)) \geq 20 - 2e(G[A]) \geq 10$ and so $e_G([a, b, c], A) = e_G(A, N(u)) - e_G(w, A) \geq 10 - 3 = 7$. Thus $ca \notin E(G)$ (otherwise, since $G$ is edge minimal, at most one of $a, b, c$ could be of degree $\geq 5$, and so $e([a, b, c], A) \leq 4 + 1 + 1 = 6$, a contradiction).

If $a$ is adjacent to all vertices of $A$, then $\Delta(G/uc) = 7, d_G/uc(b) = 4$ and $d_G/uc(v) \geq 5$ for any $v \in V(G/uc - b)$. By Lemma A.1, $G/uc > K_6^- \cup K_1$. Hence $a, \text{ similarly } c$, is adjacent to at most three vertices of $A$. Thus $e_G(N(u), A) \leq 3 + 3 + 1 + 3 = 10 \leq e_G(A, N(u))$. It follows that $G[A] = K_4^-, a \text{ (resp. } c) \text{ is adjacent to exactly three vertices of } A, \text{ all vertices of } A \text{ are of degree five. Since } G[A] = K_4^-, \text{ we may assume that } de \notin E(G)$. Note that $e_G(b, A) = 1$, we may assume that $be \notin E(G)$. Then $ew, ea, ec \in E(G)$. Observe that $e_G(d, N(u)) = 3$ and if $v \in N(u)$ is not adjacent to $d$, then $vf \in E(G)$ or $vg \in E(G)$, say the later. Clearly, $G/ae/dg - f > K_6^-$. □

**Proof of Lemma 2.4.** We may assume that $G$ is minor minimal subject to $\delta(G) \geq 5$ and $|G| \geq 9$. If $\delta(G) \geq 6$, by Theorem 2.1, $G > K_6^- \cup K_1$. So we may assume that $\delta(G) = 5$. We first prove two claims.

**Claim 1.** Every edge of $G$ is in at least two triangles.

**Proof.** Suppose $e = uv \in E(G)$ is in at most one triangle in $G$. Let $w$ be the new vertex in $G/e$. Then $d_G(w) \geq 7$, and $d_G(y) \geq 4$, where $y$ is the common neighbor of $u$ and $v$ in $G$. Clearly, $wy \in E(G/e)$ and $d_G/e(v) \geq 5$ for any $v \neq w, y$. Since $G$ is minor minimal, by Lemmas A.1 and A.2, $G > G/e > K_6^- \cup K_1$. □

**Claim 2.** There is no edge of $G$ with both ends of degree at least six in $G$.

**Proof.** Suppose $e = uv \in E(G)$ is such that $d(u), d(v) \geq 6$. Then $\delta(G - e) \geq 5$ and $|G| \geq 9$, which contradicts the fact that $G$ is minor minimal. □

We next show that $G$ is 4-connected. Let $S$ be a minimal separating set of $G$ with $|S| \leq 3$. Let $H_1$ be a component of $G - S$ with minimal order and $H_2 = G - S - H_1$. If $|S| \leq 2$, then, since $\delta(G) \geq 5$, $|H_1|, |H_2| \geq 4$, and hence $|S| = 2, H_1$ and $H_2$ are isomorphic to $K_4$,
because $|G| \leq 10$. But then, clearly, $G \not\simeq K_6^- \cup K_1$. Suppose $|S| = 3$. Then $H_1 = K_3$ and $3 \leq |H_2| \leq 4$. Note that every vertex of $H_1$ is adjacent to every vertex of $S$. If there is a vertex \( b \in V(H_2) \) such that \( b \) is adjacent to all vertices in $S$, then $G/ab - V(H_2 - b) > K_6^-$, where $a \in S$. Otherwise $H_2 = K_4$. By the minimality of $|S|$, $G$ has a matching from $S$ into $H_2$. By contracting this matching, it follows that $G > K_6^- \cup K_4$. This shows that $G$ is 4-connected.

Since \( \delta(G) = 5 \), let \( x \in V(G) \) be such that $d(x) = 5$. We may assume that $V(N(x)) = \{a, b, c, d, e\}$ and $A = V(G) - V(N[x]) = \{y_1, y_2, \ldots, y_{|G| - 6}\}$.

**Claim 3.** $N(x)$ contains no subgraph isomorphic to $K_{2,3}$.

**Proof.** Suppose that $N(x)$ has a subgraph $H$ isomorphic to $K_{2,3}$. We may assume that $d_H(a) = d_H(e) = 3$ and $d_H(b) = d_H(c) = d_H(d) = 2$. Suppose that there exists a vertex of $A$, say $y_1$, such that $y_1b, y_1c, y_1d \in E(G)$. If $G[\{b, c, d\}] \not\simeq K_3$, say $bc, ae \in E(G)$, then $G/y_1d - y_2 > K_6^-$. So we may assume that $G[\{b, c, d\}] = K_3$. If two of $b, c, d$, say $b, c$, have a common neighbor, say $y_2$, of $A - y_1$ in $G$, then $G/b_2y_2/dy_1 - y_3 > K_6^-$. It follows that any two vertices of $b, c, d$ have no common neighbors in $A$, thus there is a matching $M$ from $\{b, c, d\}$ into $A - y_1 = \{y_2, y_3, y_4\}$, and $V(M) \cap A$ is not a stable set in $G$. We may assume that $y_2y_3 \in E(G)$ and $by_2, cy_3 \in M$. Now $G/b_2y_2y_3/dy_1 - y_4 > K_6^-$. This proves that there is no vertex of $A$ adjacent to all $b, c, d$ in $G$. Next, suppose that $G[\{b, c, d\}]$ induces at least two edges, say $bc, cd \in E(G)$. We may assume that $bd, ae \notin E(G)$, otherwise $N[x] \not\simeq K_6^-$. Among $a, b, c, d, e$, by Claim 2, we may assume that $d_G(e) = 5$. Let $c_1 \in E(G)$. If $c_1 \notin E(G)$, by Claim 1, $\delta(N(e)) \geq 2$. Thus $by_1, dy_1 \in E(G)$ and so $G/b_1y_1 - y_2 > K_6^-$. It follows that $c_1y_1 \in E(G)$. Then $d_G(c) \geq 6$. By Claim 2, $d_G(a) = d_G(b) = d_G(d) = 5$. By Claim 1 and the symmetry of $b$ and $d$, we may assume that $c_1y_1 \in E(G)$. Then $dy_1 \notin E(G)$, otherwise $y_1$ is adjacent to all $b, c, d$ in $G$. Similarly, let $d_2y_2 \in E(G)$. Then $c_2y_2 \notin E(G)$ and $by_2, ey_2 \notin E(G)$. Thus $a_2y_2 \in E(G)$. Now $y_3$ is only adjacent to $c, y_1, y_2, y_4$, which contradicts the fact that $d_G(y_3) \geq 5$. This proves that $G[\{b, c, d\}]$ contains at most one edge. We may assume that $bc, bd \notin E(G)$.

Suppose that $d_G(a), d_G(e) \geq 6$. Then $\delta(G/xb) \geq 5$. Since $G$ is minor minimal, we have $|G| = 9$. Let $w$ be the new vertex in $G/xb$. Then $d_G/w(b) \geq 6$. If $d_G/w(a) > 6$ or $d_G/w(e) > 6$, say the latter, then $\delta(G/xb - ew) \geq 5$. By Lemma 2.3, $G/xb > K_6^- \cup K_1$. It follows that $d_G(a) = d_G(e) = 6$. Since $|G| = 9$ and the number of odd vertices of a graph is even, there exists a vertex of $A$, say $y_1$, such that $d_G(y_1) \geq 6$. Then $d_G/w(y_1) \geq 6$ and $w_1y_1 \in E(G/xb)$. Now $\delta(G/xb - w_1y_1) \geq 5$. By Lemma 2.3, $G/xb > K_6^- \cup K_1$. Consequently, $d_G(a) = 5$ or $d_G(e) = 5$. We may assume that $d_G(a) = 5$. If $ae \in E(G)$, then, since $G$ is 4-connected, $e$ has at least one neighbor in $A$. It follows that $d_G(e) > 6$ and so $d_G(b) = d_G(c) = d_G(d) = 5$.

Now $x$ and $b$ have exactly two common neighbors $a$ and $e$ in $G$. If $d_G(e) \geq 8$, then in $G/xb$, $\Delta(G/xb) = 7, d_G/xb(a) = 4$ and $d_G/xb(v) \geq 5$ for any $v \in V(G/xb - a)$. By Lemmas A.1 and A.2, $G/xb > K_6^- \cup K_1$. So we may assume that $e$ is adjacent to at most two vertices of $A$ in $G$. Then $e_G(N(x), A) \leq 8$. It follows that $e_G(N(x), A) = 8, |A| = 4, G[A] = K_4$, and $G[\{b, c, d\}] = K_3$. We may assume that $by_1, cy_4 \in E(G)$. Then $G/by_1y_1/y_2y_3y_4 - y_4 > K_6^-$. Hence $ae \notin E(G)$. Let $a_1y_1 \in E(G)$. Then $cd \in E(G)$, otherwise, by Claim 1, $\delta(N(a)) \geq 2$, but then $y_1$ is adjacent to all $b, c, d$ in $G$. Again, by Claim 1, $by_1 \in E(G)$. By symmetry of $c$ and $d$, we may assume that $c_1y_1 \in E(G)$ and so $dy_1 \notin E(G)$ (otherwise $y_1$ is adjacent to all $b, c, d$). Let $d_2y_2 \in E(G)$. Then $a_2y_2 \notin E(G)$ and $y_2$ is adjacent to at most
one of $b$ and $c$ in $G$. It follows that either $y_2y_1 \in E(G)$ (in this case $G/by_1/y_1y_2 - y_3>K_6^−$) or $y_2y_3, y_2y_4 \in E(G)$ and $y_1$ is adjacent to at least one of $y_3, y_4$, say $y_3$ (in this case $G/by_1/y_1y_3/y_3y_2 - y_4>K_6^−$). □

Claim 4. $N(x)$ contains no subgraph isomorphic to $K_1 + (K_2 \cup K_2)$.

Proof. Suppose that $N(x)$ has a subgraph $H$ isomorphic to $K_1 + (K_2 \cup K_2)$. We may assume that $d_H(c) = 4$, and $ab, de \in E(H)$. By Claim 3, there exists at most one edge between $\{a, b\}$ and $\{d, e\}$ in $G$. Suppose such an edge exists. By symmetry, we may assume that $ad \in E(G)$. By Claim 1, $\delta(N(a)) \geq 2$. By Claim 3, we may assume that $cy_1 \in E(G)$. It follows that $d_G(c) \geq 6$ and by Claim 2, $d_G(b) = d_G(d) = d_G(e) = 5$. If $e_G(c, A) \geq 3$, then $d_{G/xe}(c) \geq 7, d_{G/xe}(d) = 4$ and $d_{G/xe}(v) \geq 5$ for any $v \neq e$. By Lemmas A.1 and A.2, $G/xe > K_6^− \cup K_1$. Hence $e_G(c, A) \leq 2$. By counting the number of edges between $N(x)$ and $A$ in $G$, it follows that $e_G(A, N(x)) = 8$ and $G[A] = K_4$. Let $by_1, ey_j \in E(G)$, where $y_1, y_j, y_1$ could be the same. Clearly, $G/ejy_j /y_1y_i - (A - \{y_1, y_i, y_j\}) = K_6^−$. This shows that there exists no edge between $\{a, b\}$ and $\{d, e\}$ in $G$. By Claim 2, we may assume that $d_G(b) = d_G(e) = 5$. Let $by_1, ey_2 \in E(G)$.

Suppose that $d_G(c) = 5$. Then by Claim 1, $y_1y_2, ay_1, ay_2 \in E(G)$. Let $y_i, y_j$ be the two neighbors of $e$ in $A$. By Claim 1, $y_1y_2, dy_1, dy_j \in E(G)$. If $y_1 = y_1$ and $y_2 = y_2$, then $G/ejy_j /y_1y_i - (A - \{y_1, y_i\}) = K_6^−$. Hence, by symmetry, we may assume that $y_i, y_j \neq y_1, y_2$ and so $ey_3, ey_4 \in E(G)$. Clearly, $G/xe > K_6^− \cup K_1$ or $G$ is isomorphic to $J$. This proves that $d_G(c) \geq 6$. By Claim 2, $d_G(a) = d_G(b) = d_G(d) = d_G(e) = 5$. If $d_G(c) \geq 8$, then $d_{G/xa}(c) \geq 7$, $d_{G/xa}(b) = 4$ and $d_{G/xa}(v) \geq 5$ for any $v \neq c, b$. By Lemmas A.1 and A.2, $G/xe > K_6^− \cup K_1$. It follows that $6 \leq d_G(c) \leq 7$. Since $by_1, ey_2 \in E(G)$, by the symmetry of $a, b, d, e$, we may assume that $cy_1 \notin E(G)$. By Claim 1, $y_1y_2, ay_1 \in E(G)$.

Suppose $ey_1 \in E(G)$. By Claim 1, $dy_1 \in E(G)$. If $dy_2 \in E(G)$ or $ey_2 \in E(G)$, say the latter, then $G/ey_1 /y_2y_3 - y_3>K_6^−$. So we may assume that $dy_2, ey_2 \notin E(G)$. Let $ey_3 \in E(G)$. By Claim 1, $y_1y_3 \in E(G)$. By symmetry of $a, b, d, e, ay_3, by_3 \notin E(G)$. If $|A| = 3$, then by Claim 1, $cy_2, cy_3, ay_2, dy_3, y_2y_3 \in E(G)$ and so $G/xd/dy_3 - e = K_6^−$. If $|A| = 4$, since $y_4$ is adjacent to at least two vertices other than $b, e$ of $H$, we may assume that $ay_4 \in E(G)$. Then $G/ey_4 /by_1 - \{y_2, y_3\} = K_6^−$ if $dy_4 \in E(G)$, otherwise $y_3y_4 \in E(G)$ and $G/ey_1 /y_3y_4 - y_2 = K_6^−$. This proves that $ey_1 \notin E(G)$ and similarly, $dy_1 \notin E(G)$. Thus $y_1y_2, y_2y_3 \in E(G), i = 2, 3, 4$, and $d_{G}(y_1) = 5$. We claim that $G[A] = K_4$. If $dy_2 \in E(G)$, by Claim 1, $\delta(N(y_1)) \geq 2$ and so $G[A] = K_4$. If $dy_2 \notin E(G)$, we may assume that $ay_3 \in E(G)$. By Claim 1, $\delta(N(y_1)) \geq 2$ and so $cy_2, cy_3 \in E(G)$. Since $d_G(c) \leq 7$, we have $cy_4 \notin E(G)$ and so $y_4$ is adjacent to $d, e, y_1, y_2, y_3$. Then either $G[A] = K_4$ or $y_2y_3 \notin E(G)$ (in this case, we may assume that $y_3 \in E(G)$). Then $G/ey_1 /y_1y_4 \in E(G)$ and $G/ey_3 /y_3y_2 - y_2 = K_6^−$. Hence $G[A] = K_4$, as claimed. Since $e_G(N(x), A) \geq 9$, there exists a vertex $y_i \in A$ such that $d_G(y_i) \geq 6$. Note that $d_G(y_1) = 5$, we have $y_i \neq y_1$. By Claim 2, $cy_i \notin E(G)$ and so $e_G(y_i, \{a, b, d, e\}) \geq 3$. Since $y_1e, y_1d \notin E(G)$, let $ey_j, dy_k \in E(G)$, where $y_j, y_k \neq y_i$. If $y_i \neq y_2$, then $G/ey_j /by_1/y_1y_j > K_6^− \cup K_1$. So we may assume that $y_i = y_2$. If $ay_2 \notin E(G)$,
then $G/xy ≤ y ≤ K_6^− \cup K_1$. If $ay \in E(G)$, we may assume that $ey \in E(G)$. Then $G/xy \in y \le K_6^− \cup K_1$. □

By Claim 1, $δ(N(x)) ≥ 2$. Hence, by Claims 3 and 4, $N(x)$ is isomorphic to either $C_5$ or $C_5$ with exactly one chord.

Suppose that $N(x)$ is isomorphic to $C_5$ and $N(x)$ has vertices $a, b, c, d$ and $e$ in order. By Claim 2, $N(x)$ contains at most two vertices of degree $≥ 6$. Suppose that $N(x)$ contains exactly two vertices of degree $≥ 6$, say $b$ and $d$. Then $δ(G/xc) ≥ 5$. Since $G$ is minor minimal, we have $|G| ≥ 9$, $d_G(b) = d_G(d) = 6$, and by Claim 2, $d_G(v) = 5$ for any $v ∈ V(G – \{b, d\})$, which contradicts the fact the number of odd vertices of $G$ is even. This implies that $N(x)$ contains at most one vertex of degree greater than five (we may assume $d_G(e) ≥ 6$ if such a vertex exists). Thus $d_G(a) = d_G(b) = d_G(c) = d_G(d) = 5$. Let $c, y ∈ E(G)$. By Claims 3 and 4, $N(c)$ contains no subgraph isomorphic to $K_{2,3}$ and $K_1 + (K_2 \cup K_2)$. Thus by Claim 1, $y \in E(G)$. We may assume that $by \in E(G)$.

Then $by, dy$ cannot be both in $E(G)$, otherwise $N(c) > K_{2,3}$.

Suppose $by, dy \notin E(G)$. Since $d_G(b) = 5$, let $by \in E(G)$. By Claim 1, $ay, y \in E(G)$. We claim that $dy \notin E(G)$. Suppose $dy \in E(G)$. By Claim 1, $y, ey \in E(G)$. Thus $d_G(y) ≥ 6$ and so $d_G(e) = d_G(y) = 5$. If $|A| = 3$, by Claim 1, $ay, ey \in E(G)$. Clearly, $G/xy \not\in y \le K_6^−$. If $|A| = 4$, then $y_4$ is adjacent to $a, e, y_1, y_2, y_3$, and so $G/xy \in E(G)$. This proves that $dy \notin E(G)$. Since $d_G(d) = 5$, let $dy \in E(G)$. Then by Claim 1, $dy \in E(G)$. If $ay \notin E(G)$, then $d_G(y) = 5$ and $y_4y_1, y_4y_3 \in E(G)$. By Claim 1, $\delta(N(y_4)) ≥ 2$ and so $ey_3 \in E(G)$. Note that $a$ is adjacent to exactly one vertex of $\{y_1, y_2\}$. Now $G/xy \not\in y \le K_6^−$ if $ay_1 \in E(G)$ or $G/xy \not\in y \le K_6^−$ if $ey_3 \in E(G)$. This proves that $ay \in E(G)$. By Claim 1, $\delta(N(a)) ≥ 2$ and so $y_3y_4 \in E(G)$. Clearly, $y_1y_4 \not\in E(G)$ (otherwise $d_G(y_4) ≥ 6$ and so by Claim 2, $e$ is adjacent to exactly one of $y_2$ and $y_3$, say $y_2$. Then $d_G(y_3) = 4$, which is a contradiction). It follows that $ey_1 \in E(G)$ and $d_G(y_1) = 5$. By Claim 1, $\delta(N(y_1)) ≥ 2$, we have $ey_2, ey_3 \in E(G)$. Now $G/xa/xc/xy \not\in y \le K_6^−$.

Suppose $by \notin E(G)$ but $dy \in E(G)$. Since $d_G(b) = 5$, let $by \in E(G)$. By Claim 1, $ay, y \in E(G)$. Suppose $|A| = 4$. Then $y_4$ is adjacent to $a, e, y_1, y_2, y_3$. Then $d_G(y) ≥ 5$. By Claim 2, $d_G(y) = d_G(y_3) = 5$. By Claim 1, $\delta(N(y_3)) ≥ 2$, we have $ey_3 \in E(G)$. Now $G/ab/cy_2/xy_1 \not\in y \le K_6^−$. So we may assume that $|A| = 3$. Since $cy_3, dy_3 \notin E(G)$, it follows that $d_G(y_3) = 5$ and $y_3e, y_3y_2 \in E(G)$. By Claim 1, $ey_2 \in E(G)$. Note that $a$ is adjacent to exactly one vertex of $y_1, y_2$. Now $G/xa/xy \not\in y \le K_6^−$ if $ay \in E(G)$ or $G/xa/y \not\in y \le K_6^−$ if $ay \in E(G)$.

Finally, assume that $dy \notin E(G)$ but $by \in E(G)$. Since $d_G(d) = 5$, let $dy \in E(G)$. By Claim 1, $ey_3, y_3y_2 \in E(G)$. Suppose $|A| = 4$. Then $y_4$ is adjacent to $a, e, y_1, y_2, y_3$. Thus $d_G(y_2) ≥ 6$. By Claim 1, $\delta(N(y_4)) ≥ 2$ and so $ay_1 \in E(G)$. Since $d_G(y_2) ≥ 6$, by Claim 2, $y_3$ is only adjacent to $d, e, y_2, y_4$, which contradicts the fact that $d_G(y_3) ≥ 5$. So we may assume that $|A| = 3$. Since $cy_3, dy_3 \notin E(G)$, it follows that $d_G(y_3) = 5$ and $y_3a, y_3y_1 \in E(G)$. Suppose $ay_1 \in E(G)$. By Claim 2, $e$ is adjacent to exactly one vertex of $y_1, y_2$. Thus $G/xy \not\in y \le K_6^−$ if $ey_1 \in E(G)$ or $G/xy \not\in y \le K_6^−$ if $ey_2 \in E(G)$. Suppose $ay_1 \notin E(G)$. Then $ey_1, ay_2 \in E(G)$. Now $G/xy \not\in y \le K_6^−$. This completes the proof that $N(x)$ is isomorphic to $C_5$. 
It remains to consider the case when \( N(x) \) is isomorphic to \( C_5 \) with exactly one chord. We may assume that \( E(N(x)) = \{ab, bc, cd, de, ea, be\} \). By Claim 2, one of \( b \) and \( e \), say \( e \), is of degree five in \( G \). Let \( ey_1 \in E(G) \). By Claim 1, \( \delta(N(e)) \geq 2 \) and so \( dy_1 \in E(G) \). Suppose \( ay_1, by_1 \in E(G) \). We claim that \( d_G(a) \geq 6 \). Suppose \( d_G(a) = 5 \). Let \( ay_2 \in E(G) \). By Claim 1, \( \delta(N(a)) \geq 2 \) and so \( y_2y_1 \in E(G) \). It follows that \( N(a) > K_1 + (K_2 \cup K_2) \), which contradicts Claim 4. Hence \( d_G(a) \geq 6 \), as claimed. By Claims 1 and 2, \( d_G(b) = 5 \) and \( cy_1 \in E(G) \). But now \( N(b) > K_{2,3} \), which contradicts Claim 3. This proves that at most one of \( by_1, ay_1 \) are in \( E(G) \).

Suppose \( by_1 \in E(G) \) but \( ay_1 \notin E(G) \). If \( d_G(b) = 5 \), then \( \delta(N(b)) \geq 2 \), \( cy_1 \in E(G) \). By Claim 1, \( \delta(N(a)) \geq 2 \). Hence \( ay_1 \in E(G) \), \( i = 1, 2, 3, 4 \), and \( G[\{y_2, y_3, y_4\}] = K_3 \).

Since there is no edge between \( \{b, e\} \) and \( \{y_2, y_3, y_4\} \) in \( G, e_G(\{y_2, y_3, y_4\}, \{c, d, y_1\}) \geq 6 \).

However, by Claim 2, \( e_G(\{c, d, y_1\}, \{y_2, y_3, y_4\}) \leq 5 \), which is a contradiction. So we may assume that \( d_G(b) \geq 6 \). By Claim 2, \( d_G(a) = 5 \). Let \( ay_2, ay_3 \in E(G) \). By Claim 1, \( \delta(N(a)) \geq 2 \) and so \( y_2y_3, by_2, by_3 \in E(G) \). Then \( N(a) > K_1 + (K_2 \cup K_2) \), which contradicts Claim 4.

Finally, suppose \( ay_1 \in E(G) \) but \( by_1 \notin E(G) \). We claim that \( d_G(d) \geq 6 \). Suppose \( d_G(d) = 5 \). Let \( dy_2 \in E(G) \). We may assume that \( N(d) \neq C_5 \). By Claim 1, \( \delta(N(d)) \geq 2 \) and so \( y_1y_2, cy_1, cy_2 \in E(G) \). It follows that \( G[ay_1/ey_1 - y_1 > K_6^-] \) if \( ay_2 \notin E(G) \). So we may assume that \( ay_2, by_2 \notin E(G) \). Then \( y_2y_3, y_2y_4 \in E(G) \). Since \( by_1, by_2 \notin E(G) \), we may assume that \( by_3 \in E(G) \). Now \( G[ay_1/ey_1 - y_3 > K_6^-] \). This proves that \( d_G(d) \geq 6 \). By Claim 2, \( d_G(c) = 5 \) and so \( d_G(b) \geq 6 \) (otherwise, by symmetry of \( b \) and \( e \), \( d_G(c) \geq 6 \)). Now \( \delta(G/xc) \geq 5 \).

Since \( G \) is minor minimal, we have \( |G| = 9 \). Let \( w \) be the new vertex in \( G/xc \). Then \( d_G/xc(w) \geq 6 \).

If \( d_G/xc(b) \geq 6 \) or \( d_G/xc(d) \geq 6 \), say the latter, then \( \delta(G/xb - dw) \geq 5 \). By Lemma 2.3, \( G/xc > K_6^- \cup K_1 \). It follows that \( d_G(b) = d_G(d) = 6 \). Since \( |G| = 9 \) and the number of odd vertices of \( G \) is even, there exists a vertex, say \( y_1 \), of \( A \) such that \( d_G(y_1) \geq 6 \). Note that \( d_G/xc(y_1) \geq 6 \) and \( y_1c \in E(G) \). Now \( y_1w \in E(G/xc) \) and \( \delta(G/xc - y_1w) \geq 5 \).

By Lemma 2.3, \( G/xc > K_6^- \cup K_1 \).

\[ \square \]

References


[8] K. Kawarabayashi, B. Toft, Any 7 chromatic graph has a \( K_7 \) or \( K_{4,4} \) as a minor, Combinatorica 25 (2005) 327–353.