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On the Drazin inverse of the sum of two operators and its application to operator matrices ☆

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ABSTRACT

Given two bounded linear operators F, G on a Banach space $\mathcal{X}$ such that $G^2F = GF^2 = 0$, we derive an explicit expression for the Drazin inverse of $F+G$. For this purpose, firstly, we obtain a formula for the resolvent of an auxiliary operator matrix in the form $M = \begin{pmatrix} F & I \\ GF & G \end{pmatrix}$. From the provided representation of $(F+G)^D$ several special cases are considered. In particular, we recover the case $GF = 0$ studied by Hartwig et al. [R.E. Hartwig, G. Wang, Y. Wei, Some additive results on Drazin inverse, Linear Algebra Appl. 322 (2001) 207–217] for matrices and by Djordjević and Wei [D.S. Djordjević, Y. Wei, Additive results for the generalized Drazin inverse, J. Aust. Math. Soc. 73 (1) (2002) 115–126] for operators. Finally, we apply our results to obtain representations for the Drazin inverse of operator matrices in the form $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ which are extensions of some cases given in the literature.

1. Introduction and preliminaries

The Drazin inverse for bounded linear operators on complex Banach spaces was investigated by Caradus [3]. Therein it was established that the Drazin inverse of a bounded operator A on a Banach space $\mathcal{X}$ exits if and only if 0 is at most a pole of the resolvent $R(\lambda, A)$. A generalization of the Drazin inverse which is defined whenever 0 is not an accumulation point of the spectrum of A was studied by Koliha in [16]. The continuity of the conventional and the generalized Drazin inverse for bounded linear operator was studied by Rakočević and by Koliha et al. in [20] and [17], respectively.

The Drazin inverse finds its applications in a number of areas such that differential and difference equations, Markov chains and control theory [1,2].

If A and B are two Drazin invertible operators such that $AB = BA = 0$ then $(A+B)^D = A^D + B^D$. This result was originally proved by Drazin [11] in the contexts of associative rings and semigroups. Hartwig, Wang and Wei [14] gave an expression for $(A+B)^D$, for complex square matrices, when only the one side condition $AB = 0$ is required and this result was extended for operators by Djordjević and Wei in [10]. Expressions of the Drazin inverse of the sum of two matrices under the weaker conditions $A^DB = AB^D = 0$ and $(I - BB^D)AB(I - AA^D) = 0$ were given in [4], and it was extended for elements in a Banach algebra in [6]. Further, an expression for $(a+b)^D$, where a and b are elements in a Banach algebra, was given in [8] under conditions $a = ab^\pi$, $b^\pi ba^\pi = b^\pi$, $b^\pi a^\pi ba = b^\pi a^\pi ab$, where we denote $a^\pi = 1 - aa^D$ for any element a in the Banach algebra.

In this paper we concentrate on the Drazin inverse for bounded operators and continue the investigation of additive perturbations for the Drazin inverse mentioned in the preceding paragraph.

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A related topic is to obtain representations of the Drazin inverse of $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Meyer and Rose [19] gave a representation for the Drazin inverse of a 2×2 block triangular matrix in terms of the individual blocks. Djordjević and Stanimirović [9] considered the extension of Meyer and Rose result to the setting of triangular operator matrices. Several authors have considered the problem for 2×2 block matrices, with square diagonal blocks, under certain conditions on the individual terms [5,9,13,18]. We apply our main results to obtain representations for the Drazin inverse of block operator matrices which are extensions of some results given in [7,9,13].

Let $\mathcal{B}(\mathcal{X})$ denote the Banach algebra of all bounded operators on the complex Banach space $\mathcal{X}$. An operator $A \in \mathcal{B}(\mathcal{X})$ is said to be *Drazin invertible* if there exists an operator $A^D \in \mathcal{B}(\mathcal{X})$ such that

$$AA^D = A^D A, \qquad \left(A^D\right)^2 A = A^D, \qquad A^{k+1}A^D = A^k \quad \text{for some integer } k \geqslant 0,$$

in which case it is unique and it is called the *Drazin inverse* of A. The smallest integer $k \geqslant 0$ in the latter identity is called the *index* $\operatorname{ind}(A)$ of A. If we define $A^0 = I$, then the previous conditions hold with $k = 0$ if and only if A is invertible. We note that if A is nilpotent, then it is Drazin invertible, $A^D = 0$, and $\operatorname{ind}(A) = r$, where r is the power of nilpotency of A.

We write $\sigma(A)$, $\rho(A)$ and $r(A)$ for the spectrum, the resolvent set and the spectral radius of A, respectively. For $\lambda \in \rho(A)$ we denote the resolvent $(\lambda I - A)^{-1}$ by $R(\lambda, A)$. If 0 is an isolated point of $\sigma(A)$, then the *spectral projection* of A associated with $\{0\}$ is defined by

$$A^\pi = \frac{1}{2\pi i}\int_\gamma R(\lambda, A)\, d\lambda,$$

where γ is a small circle surrounding 0 and separating 0 from $\sigma(A) \setminus \{0\}$. If A is Drazin invertible then $A^\pi = I - AA^D$.

We recall that A is Drazin invertible and $\operatorname{ind}(A) = r \geqslant 1$ if and only if $\alpha(A) = \delta(A) = r$ where $\alpha(A)$ and $\delta(A)$ denote the ascent and the descent of A, respectively. In this case $\mathcal{X} = \mathcal{R}(A^r) \oplus \mathcal{N}(A^r)$, $R(\lambda, A)$ has a pole of order r at $\lambda = 0$ and it can be expressed by [3]

$$R(\lambda, A) = \sum_{n=1}^{r} \frac{A^{n-1}A^\pi}{\lambda^n} - \sum_{n=0}^{\infty} \lambda^n \left(A^D\right)^{n+1} \tag{1.1}$$

in the region $0 < |\lambda| < (r(A^D))^{-1}$.

The proof of the following lemma can be found in [12, Theorem 11.1.2].

Lemma 1.1. *Let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. If there exists the Drazin inverse of BA, then there exists the Drazin inverse of AB, $\operatorname{ind}(AB) \leqslant \operatorname{ind}(BA) + 1$ and*

$$(AB)^D = A\left((BA)^D\right)^2 B = A\left((BA)^2\right)^D B.$$

The outline of this paper is as follows. Let $F, G \in \mathcal{B}(\mathcal{X})$ two Drazin invertible operators on a Banach space. We first give a representation for the resolvent of an operator matrix in the form $M = \left(\begin{smallmatrix} F & I \\ GF & G \end{smallmatrix}\right)$, under conditions $G^2F = GF^2 = 0$, and then we use this result to derive a formulae for $(F+G)^D$ in Section 2. Several special cases are analyzed in Section 3. Finally, applications of our results will be presented for operator matrices under some conditions in Section 4.

2. Drazin inverse of the sum of two operators

We first prove a result which gives a representation for the resolvent of a type of operator matrices.

Lemma 2.1. *Let $F, G \in \mathcal{B}(\mathcal{X})$ be Drazin invertibles such that $G^2F = GF^2 = 0$, and GF is Drazin invertible. Let M be the operator defined on the Banach space $\mathcal{X} \times \mathcal{X}$ by the operator matrix $M = \left(\begin{smallmatrix} F & I \\ GF & G \end{smallmatrix}\right)$. Then the resolvent of M has the representation*

$$R(\lambda, M) = \begin{pmatrix} \lambda^2 R(\lambda, F)R(\lambda^2, GF) & \lambda^2 R(\lambda, F)R(\lambda^2, GF)R(\lambda, G) \\ R(\lambda^2, GF)GF & \lambda^2 R(\lambda^2, GF)R(\lambda, G) \end{pmatrix} \tag{2.1}$$

in the region $0 < |\lambda| < \min\{(r(G^D))^{-1}, (r(F^D))^{-1}, (r((GF)^D))^{-2}\}$.

Proof. Let $S(\lambda) = \lambda - G - GFR(\lambda, F)$ and let $\rho(S)$ denote the set of all $\lambda \in \mathbb{C}$ such that $S(\lambda)^{-1}$ is a bounded linear operator in $\mathcal{X}$. By [15, Proposition H], we have $\rho(M) \cap \rho(F) = \rho(F) \cap \rho(S)$ and for any λ in this set

$$R(\lambda, M) = \begin{pmatrix} R(\lambda, F)(I + S(\lambda)^{-1}GFR(\lambda, F)) & R(\lambda, F)S(\lambda)^{-1} \\ S(\lambda)^{-1}GFR(\lambda, F) & S(\lambda)^{-1} \end{pmatrix}. \tag{2.2}$$

Let us introduce the punctured neighborhood of 0, $\Gamma = \{\lambda \in \mathbb{C}\colon\ 0 < |\lambda| < \delta\}$ where $\delta = \min\{(r(G^D))^{-1}, (r(F^D))^{-1}, (r((GF)^D))^{-2}\}$. Now, let $Z(\lambda) = \lambda^2 R(\lambda^2, GF)R(\lambda, G)$. We claim that $S(\lambda)^{-1} = Z(\lambda)$ for any $\lambda \in \Gamma$. Indeed, first since G is Drazin invertible and $G^2F = 0$, thus $G^D GF = 0$ also holds, we get

$$R(\lambda, G)GF = \left(\sum_{n=1}^{t} \frac{G^{n-1}G^{\pi}}{\lambda^n} - \sum_{n=0}^{\infty} \lambda^n \left(G^D\right)^{n+1}\right) GF = \frac{1}{\lambda} GF, \quad t = \operatorname{ind}(G),$$

in the region $0 < |\lambda| < (r(G^D))^{-1}$. Analogously, since $GF^2 = 0$, we obtain $GFR(\lambda, F) = \frac{1}{\lambda} GF$ in the region $0 < |\lambda| < (r(F^D))^{-1}$. Hence, for any $\lambda \in \Gamma$,

$$Z(\lambda)S(\lambda) = \lambda^2 R\left(\lambda^2, GF\right) - R\left(\lambda^2, GF\right)GF = I.$$

On the other hand, since GF is Drazin invertible, by noting that $G(GF)^D = 0$, we can see that $GR(\lambda^2, GF) = \frac{1}{\lambda^2} G$ in the region $0 < |\lambda| < (r((GF)^D))^{-2}$. Further,

$$S(\lambda)Z(\lambda) = \left(\lambda - \frac{1}{\lambda} GF - G\right) \lambda^2 R\left(\lambda^2, GF\right) R(\lambda, G) = \lambda R(\lambda, G) - GR(\lambda, G) = I,$$

for any $\lambda \in \Gamma$.

By using that $S(\lambda)^{-1} = \lambda^2 R(\lambda^2, GF)R(\lambda, G)$ and $R(\lambda, G)GF = \lambda^{-1}GF = GFR(\lambda, F)$ we obtain

$$R(\lambda, F)\left(I + S(\lambda)^{-1} GFR(\lambda, F)\right) = R(\lambda, F)\left(I + R\left(\lambda^2, GF\right)GF\right) = \lambda^2 R(\lambda, F) R\left(\lambda^2, GF\right).$$

Therefore, we get (2.1) for any $\lambda \in \Gamma$, which give us the desired result. □

Previous to the main result we give the following lemma. For any integer k, we denote by $\lfloor k/2 \rfloor$ the integer part of $k/2$.

Lemma 2.2. *Let $F, G \in \mathcal{B}(\mathcal{X})$ as in Lemma 2.1. Let $s = \operatorname{ind}(F)$, $t = \operatorname{ind}(G)$ and $r = \operatorname{ind}(GF)$. For any $0 \leqslant k$, by denoting $k' = \lfloor (k-1)/2 \rfloor$ and $\alpha = 0$ if k is even and $\alpha = 1$ otherwise, we have*

(i) *If $\mathcal{B}_{k+2}$ is the coefficient at λ^{-k-2} of $R(\lambda, F)R(\lambda^2, GF)$, then*

$$\mathcal{B}_{k+2} = -F^k X - \left(F^D\right)^{\alpha} U (GF)^{k'+1} + Z_k, \tag{2.3}$$

where

$$X = \sum_{j=1}^{\lfloor s/2 \rfloor} F^{2j-1} F^{\pi} \left((GF)^D\right)^j, \qquad U = \sum_{j=0}^{r-1} \left(F^D\right)^{2j+1} (GF)^j (GF)^{\pi}, \tag{2.4}$$

$$Z_0 = 0, \qquad Z_k = \sum_{j=0}^{k'} F^{\pi} F^{k-1-2j} (GF)^j (GF)^{\pi}, \quad k \geqslant 1. \tag{2.5}$$

(ii) *If Γ_{k+2} is the coefficient at λ^{-k-2} of $R(\lambda^2, GF)R(\lambda, G)$, then*

$$\Gamma_{k+2} = -YG^k - (GF)^{k'+1} V \left(G^D\right)^{\alpha} + T_k, \tag{2.6}$$

where

$$Y = \sum_{j=1}^{\lfloor t/2 \rfloor} \left((GF)^D\right)^j G^{2j-1} G^{\pi}, \qquad V = \sum_{j=0}^{r-1} (GF)^{\pi} (GF)^j \left(G^D\right)^{2j+1}, \tag{2.7}$$

$$T_0 = 0, \qquad T_k = \sum_{j=0}^{k'} (GF)^{\pi} (GF)^j G^{k-1-2j} G^{\pi}, \quad k \geqslant 1. \tag{2.8}$$

Proof. We consider the Laurent series

$$R(\lambda, F) = \sum_{n=1}^{s} \frac{F^{n-1}F^{\pi}}{\lambda^n} - \sum_{n=0}^{\infty} \lambda^n \left(F^D\right)^{n+1}$$

and

$$R\left(\lambda^2, GF\right) = \sum_{n=1}^{r} \frac{(GF)^{n-1}(GF)^{\pi}}{\lambda^{2n}} - \sum_{n=0}^{\infty} \lambda^{2n} \left((GF)^D\right)^{n+1}$$

in a punctured neighborhood of 0. Hence, the coefficient at λ^{-k-2} of $R(\lambda,F)R(\lambda^2,GF)$ is given by

$$\mathcal{B}_{k+2} = -X_k - U_k + Z_k,$$

where

$$X_k = \sum_{j=1}^{\lfloor (s-k)/2\rfloor} F^\pi F^{2j+k-1}\big((GF)^D\big)^j = F^k X,$$

$$U_k = \sum_{j=k'+1}^{r-1} \big(F^D\big)^{2j-k+1}(GF)^j(GF)^\pi = \big(F^D\big)^\alpha U(GF)^{k'+1},$$

$$Z_k = \sum_{j=0}^{k'} F^\pi F^{k-1-2j}(GF)^j(GF)^\pi, \quad k \geqslant 1, \qquad Z_0 = 0,$$

with X, U are defined as in (2.4) and $\alpha = 0$ if k is even, otherwise $\alpha = 1$. This completes the proof of (i). Analogously, it is proved (ii). □

Now, we are in position to state the main result.

Theorem 2.3. *Let $F, G \in \mathcal{B}(\mathcal{X})$ be Drazin invertibles such that $G^2F = GF^2 = 0$, and let GF be Drazin invertible. Then $F+G$ is Drazin invertible and*

$$(F+G)^D = UG^\pi + F^\pi V + X(I+YG)G^\pi + F^\pi(I+FX)Y + FUV + UVG + \sum_{k=0}^{2r+t-2}\big(F^D\big)^{k+1}\Gamma_{k+2}G + \sum_{k=0}^{2r+s-2} F\mathcal{B}_{k+2}\big(G^D\big)^{k+1}, \tag{2.9}$$

where X, Y, U, V, $\mathcal{B}_{k+2}$ and Γ_{k+2} are defined as in Lemma 2.2, Eqs. (2.3)–(2.8)*. Moreover,* $\operatorname{ind}(F+G) \leqslant 2r+s+t-1$.

Proof. Define the operators $A = (\,F\ \ I\,): \mathcal{X}\oplus\mathcal{X} \to \mathcal{X}$ and $B = \binom{I}{G}: \mathcal{X} \to \mathcal{X}\oplus\mathcal{X}$. We note that $F+G = AB$ and $BA = \begin{pmatrix} F & I \\ GF & G\end{pmatrix}$. If BA is Drazin invertible, then we can apply Lemma 1.1 to obtain

$$(F+G)^D = (AB)^D = A\big((BA)^D\big)^2 B. \tag{2.10}$$

Next we will obtain an operator matrix representation of $(BA)^D$. First, we can apply Lemma 2.1 to get

$$R(\lambda, BA) = \begin{pmatrix} \lambda^2 R(\lambda,F)R(\lambda^2,GF) & \lambda^2 R(\lambda,F)R(\lambda^2,GF)R(\lambda,G) \\ R(\lambda^2,GF)GF & \lambda^2 R(\lambda^2,GF)R(\lambda,G)\end{pmatrix}, \tag{2.11}$$

in a punctured neighborhood of 0. Hence, it follows that BA has a pole at $\lambda = 0$ of order at most $\nu = 2r+s+t-2$ (where r, s and t are defined as in Lemma 2.2) and, consequently, BA is Drazin invertible and $R(\lambda, BA)$ has the Laurent series

$$R(\lambda, BA) = \sum_{n=1}^{\nu} \lambda^{-n}(BA)^{n-1}(BA)^\pi - \sum_{n=0}^{\infty}\lambda^n\big((BA)^D\big)^{n+1} \tag{2.12}$$

in a punctured neighborhood of 0. We also note that $\operatorname{ind}(F+G) \leqslant \operatorname{ind}(BA)+1 \leqslant 2r+s+t-1$.

Now, in view of (2.11) and (2.12), comparing the coefficients at λ^0 and using Lemma 2.2,

$$(BA)^D = \begin{pmatrix} X+U & -W \\ (GF)^D GF & Y+V\end{pmatrix}, \tag{2.13}$$

where W is the coefficient at λ^{-2} of $R(\lambda,F)R(\lambda^2,GF)R(\lambda,G)$. Further, from (2.10) and (2.13), by noting that $X^2 = XU = UX = 0$, $U^2 = F^D U$, $Y^2 = YV = VY = 0$, $V^2 = VG^D$, and $GFU = GFX = YGF = VGF = 0$, it follows that

$$\begin{aligned}(F+G)^D &= (\,F\quad I\,)\begin{pmatrix} F^D U - W(GF)^D GF & -(X+U)W - W(Y+V) \\ 0 & VG^D - GF(GF)^D W\end{pmatrix}\begin{pmatrix} I \\ G\end{pmatrix} \\ &= U + V - FW(GF)^D GF(I+YG) - (I+FX)GF(GF)^D WG - F(UW+WV)G.\end{aligned} \tag{2.14}$$

Now, since $GFR(\lambda,F) = \lambda^{-1}GF$ and $R(\lambda,G)G = \lambda R(\lambda,G) - I$, it follows that

$$GFR(\lambda,F)R\big(\lambda^2,GF\big)R(\lambda,G)G = GF\big(R\big(\lambda^2,GF\big)R(\lambda,G) - \lambda^{-1}R\big(\lambda^2,GF\big)\big).$$

Hence, using that $-Y-V$ is the coefficient at λ^{-2} of $R(\lambda^2,GF)R(\lambda,G)$, we get

$$GF(GF)^\pi WG = -GFV, \qquad (GF)^D GFWG = -Y. \tag{2.15}$$

Analogously, we can see that

$$FW(GF)^D GF = -X. \tag{2.16}$$

On the other hand,

$$FF^D R(\lambda, F)R(\lambda^2, GF)R(\lambda, G)G = -\left(\sum_{n=0}^{\infty} \lambda^n (F^D)^{n+1}\right) R(\lambda^2, GF)R(\lambda, G)G.$$

Therefore,

$$FF^D WG = -\sum_{k=0}^{2r+t-2} (F^D)^{k+1} \Gamma_{k+2} G,$$

where Γ_{k+2} is defined as in (2.6). Using the above expression and (2.15)

$$\begin{aligned} FUWG &= FF^D(I - GF(GF)^D)WG + F\sum_{j=1}^{r-1}(F^D)^{2j+1}(GF)^j(GF)^\pi WG \\ &= -\sum_{k=0}^{2r+t-2} (F^D)^{k+1} \Gamma_{k+2} G + FF^D Y + F(F^D - U)V. \end{aligned} \tag{2.17}$$

Analogously,

$$FWVG = -\sum_{k=0}^{2r+s-2} F\mathcal{B}_{k+2}(G^D)^{k+1} + XGG^D + U(G^D - V)G. \tag{2.18}$$

By substituting the expressions (2.15)–(2.18) in (2.14) we conclude the result of this theorem. □

3. Special cases

In this section we assume that F and G verify the conditions of Theorem 2.3 and we analyze some special cases of the above mentioned result.

The second part of the following corollary is an extension to the infinite dimensional case of [7, Lemma 2.1].

Corollary 3.1. *If G is nilpotent, then*

$$(F+G)^D = U + X(I+YG) + F^\pi(I+FX)Y + \sum_{k=0}^{2r+t-2} (F^D)^{k+1}(T_k - YG^k)G, \tag{3.1}$$

where X, U are as in (2.4), Y as in (2.7) and T_k as in (2.8).

In the case $G^2 = 0$, then

$$(F+G)^D = U + X + F^\pi(I+FX)(GF)^D G + F^D UG. \tag{3.2}$$

Proof. If G is nilpotent then $V = 0$ and if $G^2 = 0$, we also have $Y = (GF)^D G$ and $YG^k = 0$ for all $k \geqslant 1$. So, the results follow from (2.9). □

Corollary 3.2. *If F and G are nilpotent, then*

$$(F+G)^D = X(I+YG) + (I+FX)Y,$$

where X is defined as in (2.4) and Y is defined as in (2.7).

In the case $G^2 = 0$, then

$$(F+G)^D = X + (I+FX)(GF)^D G. \tag{3.3}$$

Moreover, if $F^2 = G^2 = 0$, then

$$(F+G)^D = F(GF)^D + (GF)^D G.$$

Proof. Since F and G are nilpotent then $U = V = 0$. In the case $G^2 = 0$, we also have $Y = (GF)^D G$ and $YG = 0$. Further if $F^2 = 0$ then we get $X = F(GF)^D$ and $FX = 0$. Therefore, the results follow from (2.9). □

Corollary 3.3. *If GF is nilpotent, then*

$$(F+G)^D = UG^\pi + F^\pi V + FUV + UVG + \sum_{k=0}^{2r+t-2} \left(F^D\right)^{k+1}\left(T_k - (GF)^{k'+1} V \left(G^D\right)^\alpha\right)G$$
$$+ \sum_{k=0}^{2r+s-2} F\left(Z_k - \left(F^D\right)^\alpha U (GF)^{k'+1}\right)\left(G^D\right)^{k+1},$$

where U is defined as in (2.4), *V as in* (2.7), *Z_k as in* (2.5), *T_k as in* (2.8), *α and k' as in the premises of Lemma* 2.2.

Proof. It follows from Theorem 2.3 by noting that $X = Y = 0$. □

Corollary 3.4. *If* $\operatorname{ind}(GF) = 1$, *then*

$$(F+G)^D = X(I+YG)G^\pi + F^\pi (I+FX)Y + \sum_{k=0}^{t-1}\left(F^D\right)^{k+1}\left((GF)^\pi G^\pi - YG\right)G^k$$
$$+ \sum_{k=0}^{s-1} F^k\left(F^\pi (GF)^\pi - FX\right)\left(G^D\right)^{k+1},$$

where X is defined as in (2.4) *and Y is defined as in* (2.7).

Proof. Since $\operatorname{ind}(GF) = 1$ then $(GF)^D = (GF)^\sharp$ and $(GF)(GF)^\pi = 0$. In this case, U defined as in (2.4) and V as in (2.7) simplify to $U = F^D(GF)^\pi$ and $V = (GF)^\pi G^D$. On the other hand, by computing $\mathcal{B}_{k+2}$ and Γ_{k+2} as in (2.3) and (2.6), respectively, we get

$$\mathcal{B}_{k+2} = \begin{cases} -X - F^D(GF)^\pi, & k=0, \\ F^{k-1}(F^\pi (GF)^\pi - FX), & k \geqslant 1, \end{cases} \qquad \Gamma_{k+2} = \begin{cases} -Y - (GF)^\pi G^D, & k=0, \\ ((GF)^\pi G^\pi - YG)G^{k-1}, & k \geqslant 1. \end{cases}$$

Finally, substituting the above relations in (2.9) we obtain the result. □

Applying Corollary 3.3 or Corollary 3.4 to the case $GF = 0$, we obtain the representation given in [14] for matrices and in [10] for bounded operators.

Corollary 3.5. *If $GF = 0$, then*

$$(F+G)^D = \sum_{k=0}^{t-1}\left(F^D\right)^{k+1} G^k G^\pi + \sum_{k=0}^{s-1} F^\pi F^k \left(G^D\right)^{k+1}.$$

Proof. Since $GF = 0$ we have $X = Y = 0$, $U = F^D$, $V = G^D$, and $T_k = G^{k-1}G^\pi$, $Z_k = F^\pi F^{k-1}$, for $k \geqslant 1$. □

4. Application to bounded operator matrices

Let $\mathcal{Y}$, $\mathcal{Z}$ be two Banach spaces, $\mathcal{X} = \mathcal{Y} \times \mathcal{Z}$ and let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a bounded linear operator matrix on $\mathcal{X}$. We illustrate an application of our results to derive representations for M^D under some conditions.

The following result due to Djordjević and Stanimirović [9] is an extension to the setting of triangular operator matrices of a well-known result of Meyer and Rose [19] concerning the Drazin inverse of a 2×2 block upper triangular matrix.

Lemma 4.1. *Let $M = \left(\begin{smallmatrix} A & 0 \\ C & D \end{smallmatrix}\right)$ an operator matrix. If* $\operatorname{ind}(A) = r$ *and* $\operatorname{ind}(D) = s$, *then M is Drazin invertible,* $\max\{r, s\} \leqslant \operatorname{ind}(M) \leqslant r + s$ *and*

$$M^D = \begin{pmatrix} A^D & 0 \\ \mathcal{N} & D^D \end{pmatrix},$$

where

$$\mathcal{N} = \left(D^D\right)^2 \left(\sum_{i=0}^{r-1}\left(D^D\right)^i C A^i\right) A^\pi + D^\pi \left(\sum_{i=0}^{s-1} D^i C \left(A^D\right)^i\right)\left(A^D\right)^2 - D^D C A^D. \tag{4.1}$$

If M is Drazin invertible and $0 \notin \operatorname{acc}\sigma(A) \cup \operatorname{acc}\sigma(D)$, *then A and D also are Drazin invertibles.*

Some results have been provided for the general case under certain conditions. The case $BC = 0$, $DC = 0$, and $BD = 0$ has been considered in [9] and the case $BC = 0$, $DC = 0$ (or $BD = 0$), and D nilpotent in [13]. We focus our attention in the generalization of the mentioned results.

Theorem 4.2. *Let $A \in \mathcal{B}(\mathcal{Y})$, $D \in \mathcal{B}(\mathcal{Z})$ be Drazin invertibles and BC be Drazin invertible, and let $\nu_1 = \operatorname{ind}(A)$, $\nu_2 = \operatorname{ind}(BC)$, $\nu_3 = \operatorname{ind}(D)$. If $BCA = 0$, $BD = 0$, and BC nilpotent then*

$$M^D = \begin{pmatrix} \Gamma & A^D \Gamma B \\ \Sigma & D^D + (\mathcal{N}\Gamma + D^D\Sigma)B \end{pmatrix},$$

where $\mathcal{N}$ as in formula (4.1), and for $k \geqslant 1$, $\mathcal{N}_k = \sum_{i=0}^{k-1} (D^D)^{k-1-i}\mathcal{N}(A^D)^i$ and

$$\Gamma = \sum_{j=0}^{\nu_2-1} \left(A^D\right)^{2j+1}(BC)^j, \qquad \Sigma = \sum_{j=0}^{\nu_2-1} \mathcal{N}_{2j+1}(BC)^j. \tag{4.2}$$

Proof. We consider the splitting $M = F + G$ where $F = \left(\begin{smallmatrix} A & 0 \\ C & D \end{smallmatrix}\right)$ and $G = \left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$. We note that $G^2 = 0$ and, using that $BD = 0$ and $BCA = 0$, we obtain $GF^2 = 0$. Moreover, since BC nilpotent it follows that GF is nilpotent, and thus, $(GF)^D = 0$. So, applying the second part of Corollary 3.1, we have that

$$(F+G)^D = U + F^D UG, \qquad U = \sum_{j=0}^{\nu_2-1} \left(F^D\right)^{2j+1}(GF)^j.$$

By using Lemma 4.1, we get

$$\left(F^D\right)^k = \begin{pmatrix} (A^D)^k & 0 \\ \mathcal{N}_k & (D^D)^k \end{pmatrix}, \quad k \geqslant 1,$$

where $\mathcal{N}_k$ as in the statement of this theorem. Further, with Γ and Σ as in (4.2), the final result is obtained using the following expressions

$$U = \begin{pmatrix} \Gamma & 0 \\ \Sigma & D^D \end{pmatrix}, \qquad F^D UG = \begin{pmatrix} 0 & A^D\Gamma B \\ 0 & \mathcal{N}_1 \Gamma B + D^D \Sigma B \end{pmatrix}. \qquad \square$$

The following result is a straightforward application of the above theorem.

Corollary 4.3. *Let $A \in \mathcal{B}(\mathcal{Y})$, $D \in \mathcal{B}(\mathcal{Z})$ be Drazin invertibles. If $BC = 0$ and $BD = 0$ then*

$$M^D = \begin{pmatrix} A^D & (A^D)^2 B \\ \mathcal{N}_1 & D^D + \mathcal{N}_2 B \end{pmatrix},$$

where $\mathcal{N}_i$, $i = 1, 2$, as in Theorem 4.2.

Theorem 4.4. *Let $A \in \mathcal{B}(\mathcal{Y})$, $D \in \mathcal{B}(\mathcal{Z})$ be Drazin invertibles and BC be Drazin invertible, and let $\nu_1 = \operatorname{ind}(A)$, $\nu_2 = \operatorname{ind}(BC)$, and $\nu_3 = \operatorname{ind}(D)$. If $BCA = 0$, $DC = 0$, and $BD = 0$, then*

$$M^D = \begin{pmatrix} A\Psi & \Psi B \\ C\Psi & D^D + C(A^D\Psi + (A\Psi - A^D)(BC)^D)B \end{pmatrix},$$

where

$$\Psi = \sum_{j=0}^{\nu_2-1} \left(A^D\right)^{2j+2}(BC)^j(BC)^\pi + \sum_{j=0}^{\lfloor \nu_1/2 \rfloor} A^\pi A^{2j}\left((BC)^D\right)^{j+1}. \tag{4.3}$$

Proof. We consider the splitting $M = F + G$ where $F = \left(\begin{smallmatrix} A & 0 \\ C & D \end{smallmatrix}\right)$ and $G = \left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$. We note that $G^2 = 0$ and, using that $BD = 0$ and $BCA = 0$, we obtain $GF^2 = 0$. Applying the second part of Corollary 3.1, we have that $(F+G)^D$ is given by expression (3.2). Now, we will derive matrix representations for the terms in the mentioned formulae. From Lemma 4.1 it follows that F and GF are Drazin invertibles, and

$$F^D = \begin{pmatrix} A^D & 0 \\ C(A^D)^2 & D^D \end{pmatrix}, \qquad F^\pi = \begin{pmatrix} A^\pi & 0 \\ -CA^D & D^\pi \end{pmatrix}, \qquad \max\{\nu_1, \nu_3\} \leqslant \operatorname{ind}(F) \leqslant \nu_1 + \nu_3,$$

$$(GF)^D = \begin{pmatrix} (BC)^D & 0 \\ 0 & 0 \end{pmatrix}, \qquad (GF)^\pi = \begin{pmatrix} (BC)^\pi & 0 \\ 0 & I \end{pmatrix}, \qquad \operatorname{ind}(GF) = \nu_2.$$

On the other hand, we have

$$F^k = \begin{pmatrix} A^k & 0 \\ CA^{k-1} & D^k \end{pmatrix}, \qquad \left(F^D\right)^k = \begin{pmatrix} (A^D)^k & 0 \\ C(A^D)^{k+1} & (D^D)^k \end{pmatrix}, \quad k \geqslant 1.$$

By denoting $\Psi_1 = \sum_{j=0}^{\nu_2-1} (A^D)^{2j+2}(BC)^j(BC)^\pi$ and $\Psi_2 = \sum_{j=0}^{\lfloor \nu_1/2 \rfloor} A^\pi A^{2j}((BC)^D)^{j+1}$, we obtain

$$U = \sum_{j=0}^{\nu_2-1} \left(F^D\right)^{2j+1}(GF)^j(GF)^\pi = \begin{pmatrix} A\Psi_1 & 0 \\ C\Psi_1 & D^D \end{pmatrix}$$

and

$$X = \sum_{j=1}^{\lfloor (\nu_1+\nu_3)/2 \rfloor} F^\pi F^{2j-1}\left((GF)^D\right)^j = \begin{pmatrix} A\Psi_2 & 0 \\ C\Psi_2 & 0 \end{pmatrix}.$$

Further, we get

$$F^D U G = \begin{pmatrix} 0 & \Psi_1 B \\ 0 & CA^D\Psi_1 B \end{pmatrix}$$

and

$$F^\pi (GF)^D G + FX(GF)^D G = \begin{pmatrix} 0 & A^\pi (BC)^D B \\ 0 & -CA^D(BC)^D B \end{pmatrix} + \begin{pmatrix} 0 & A^2\Psi_2(BC)^D B \\ 0 & CA\Psi_2(BC)^D B \end{pmatrix}.$$

Finally, substituting the preceding representations in (3.2) and by denoting $\Psi = \Psi_1 + \Psi_2$, we get the result. □

We remark that from Theorem 4.4 we derive a representation for a 2×2 operator matrix M under conditions $BCA = 0$ and $D = 0$, which is an extension of the result given in [7] for block matrices.

Theorem 4.5. *Let A, BC be Drazin invertibles and let $\nu_1 = \operatorname{ind}(A)$ and $\nu_2 = \operatorname{ind}(BC)$. If $BCA = 0$, $DC = 0$, and D nilpotent with index of nilpotency ν_3. Then*

$$M^D = \begin{pmatrix} A\Psi & (A\Psi - A^D)\Phi D + \Psi BC\Phi + \Omega \\ C\Psi & C((\Psi - (A^D)^2)\Phi D + (A\Psi - A^D)\Phi + A^D\Omega) \end{pmatrix},$$

where $\nu = 2\nu_2 + \nu_3 - 1$, Ψ is defined as in (4.3) and

$$\Phi = \sum_{j=0}^{\lfloor \nu_3/2 \rfloor} \left((BC)^D\right)^{j+1} BD^{2j}, \qquad \Omega = \sum_{k=1}^{\nu} \left(A^D\right)^{k+1} \left(\sum_{j=0}^{k'} (BC)^\pi (BC)^j BD^{k-2j-1} - \Phi D^{k+1} \right), \tag{4.4}$$

where $k' = \lfloor k - 1/2 \rfloor$.

Proof. We consider the splitting $M = F + G$ with $F = \left(\begin{smallmatrix} A & 0 \\ C & 0 \end{smallmatrix}\right)$ and $G = \left(\begin{smallmatrix} 0 & B \\ 0 & D \end{smallmatrix}\right)$. Since D is nilpotent it follows that G is nilpotent and, thus, $G^D = 0$. We observe that $\nu_3 \leqslant \operatorname{ind}(G) \leqslant \nu_3 + 1$. From Lemma 4.1 it follows that F, GF are Drazin invertibles and, using $DC = 0$, we get

$$F^D = \begin{pmatrix} A^D & 0 \\ C(A^D)^2 & 0 \end{pmatrix}, \qquad F^\pi = \begin{pmatrix} A^\pi & 0 \\ -CA^D & I \end{pmatrix}, \qquad (GF)^D = \begin{pmatrix} (BC)^D & 0 \\ 0 & 0 \end{pmatrix}, \qquad (GF)^\pi = \begin{pmatrix} (BC)^\pi & 0 \\ 0 & I \end{pmatrix}.$$

On the other hand,

$$F^k = \begin{pmatrix} A^k & 0 \\ CA^{k-1} & 0 \end{pmatrix}, \qquad G^k = \begin{pmatrix} 0 & BD^{k-1} \\ 0 & D^k \end{pmatrix}, \quad k \geqslant 1.$$

Now, using $BCA = 0$ and $DC = 0$ we obtain that $G^2F = 0$ and $GF^2 = 0$. So we can apply Corollary 3.1 to conclude that $(F+G)^D$ is given by expression (3.1). In the sequel we derive the matrix representations of the terms in the mentioned formulae. We consider Ψ defined as in (4.3) and Φ as in the statement of this theorem. We compute

$$U + X = \begin{pmatrix} A\Psi & 0 \\ C\Psi & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}, \qquad XYG = \begin{pmatrix} 0 & A\Psi\Phi D \\ 0 & C\Psi\Phi D \end{pmatrix}.$$

Further, we have

$$F^{\pi}(I+FX)Y=\begin{pmatrix}0 & \Psi BC\Phi\\ 0 & C(A\Psi-A^D)\Phi\end{pmatrix},$$

$$\left(F^D\right)^{k+1}YG^{k+1}=\begin{pmatrix}0 & (A^D)^{k+1}\Phi D^{k+1}\\ 0 & C(A^D)^{k+2}\Phi D^{k+1}\end{pmatrix},\quad k\geqslant 0,$$

$$\left(F^D\right)^{k+1}T_kG=\begin{pmatrix}0 & (A^D)^{k+1}\Omega_k\\ 0 & C(A^D)^{k+2}\Omega_k\end{pmatrix},\quad k\geqslant 1,$$

where $\Omega_k=\sum_{j=0}^{k'}(BC)^{\pi}(BC)^jBD^{k-2j-1}$. Consequently, by substituting the above expressions in (3.1), the explicit formula for M^D is obtained. □

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