Quantization of quadratic Poisson brackets on a polynomial algebra of three variables

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Abstract

Poisson brackets (P.b.) are the natural initial terms for the deformation quantization of commutative algebras. There is an open problem whether any Poisson bracket on the polynomial algebra of \( n \) variables can be quantized. It is known (Poincare–Birkhoff–Witt theorem) that any linear P.b. for all \( n \) can be quantized. On the other hand, it is easy to show that in case \( n = 2 \) any P.b. is quantizable as well.

Quadratic P.b. appear as the initial terms for the quantization of polynomial algebras as quadratic algebras. The problem of the quantization of quadratic P.b. is also open. In the paper we show that in case \( n = 3 \) any quadratic P.b. can be quantized. Moreover, the quantization is given as the quotient algebra of tensor algebra of three variables by relations which are similar to those in the Poincare–Birkhoff–Witt theorem. The proof uses a classification of all quadratic Poisson brackets of three variables, which we also give in the paper. In the appendix we give explicit algebraic constructions of the quantized algebras appeared here and show that they are related to algebras of global dimension three considered by M. Artin, W. Schelter, J. Tate, M. Van Den Bergh and other authors from a different point of view. © 1998 Elsevier Science B.V. All rights reserved.

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1. Deformations of quadratic algebras

Let \( A \) be an associative algebra with unit over a field \( k \) of characteristic zero. We will consider deformations of \( A \) over the algebra of formal power series \( k[[\hbar]] \) in a variable \( \hbar \).

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By a deformation of $A$ we mean an algebra $A_h$ over $k[[h]]$ which is isomorphic to $A[[h]] = A \otimes_k k[[h]]$ as a $k[[h]]$-module and $A_h/hA_h = A$ (the symbol $\otimes$ denotes the tensor product completed in the $h$-adic topology). We will also denote $A$ as $A_0$. Since $k[[h]]$ is a local algebra with the canonical embedding $k \to k[[h]]$, the algebra $A_h$ will be isomorphic to $A[[h]]$ as a $k[[h]]$-module if and only if $A_h$ is a flat $k[[h]]$-module, [5]. According to this, we will say sometimes that $A_h$ is a flat deformation of $A_0$. If $A_0$ is a commutative algebra but $A_h$ is a noncommutative one then $A_h$ is called a (deformation) quantization of $A_0$.

If $A_h'$ is another deformation of $A$, we call the deformations $A_h$ and $A_h'$ equivalent if there exists a $k[[h]]$-algebra isomorphism $A_h \to A_h'$ which induces the identity automorphism of $A_0$.

Let $T = T(V)$ be a tensor algebra over a finite-dimensional vector space $V$ over a field $k$ and let $A$ be a quotient algebra of $T$, i.e., $A = T/\mathcal{I}$ where $\mathcal{I}$ is an ideal in $T$. It is easy to see that if $A_h$ is a deformation of $A$, then $A_h = T[[h]]/\mathcal{I}_h$ where $\mathcal{I}_h$ is an ideal in $T[[h]]$ such that $\mathcal{I} = \mathcal{I}_h/h\mathcal{I}_h$.

Consider $T$ as a graded algebra, $T = \bigoplus_k V^\otimes k$, and let $A$ also be a graded algebra, i.e., $\mathcal{I}$ is a graded ideal. Denote by $\mathcal{I}^k$ the $k$th homogeneous component of $\mathcal{I}$. If $A_h$ is a homogeneous deformation of $A$, then the corresponding ideal $\mathcal{I}_h$ is homogeneous, $\mathcal{I}_h = \bigoplus_k \mathcal{I}^k_h$ where $\mathcal{I}^k_h$ are submodules in $V^\otimes k[[h]]$. Flatness of the deformation $A_h$ means that all the homogeneous components $A^k_h = V^\otimes k[[h]]/\mathcal{I}^k_h$ are free modules of finite rank. It is necessary for this that all the components $\mathcal{I}^k_h$ be splitting $k[[h]]$-submodules in $V^\otimes k[[h]]$ for all $k$. This means that each $\mathcal{I}^k_h$ has a complementary submodule $\mathcal{I}^k_h$, i.e., $V^\otimes k[[h]] = \mathcal{I}^k_h \oplus \mathcal{I}^k_h$.

Now consider the case of quadratic algebras. Recall that an algebra $A$ is called quadratic if $A = T/\mathcal{I}$ and $\mathcal{I}$ is a homogeneous ideal generated by $\mathcal{I}^2$. From now on we use the notations $\mathcal{I}^2 = I$ and $\mathcal{I}^k = I^k$ and perceive each $I^k$ as a subspace in $V^\otimes k$. So, a quadratic algebra is defined by a pair $(V, I)$ where $I$ is a subspace in $V \otimes V$. Define the subspaces $I^k_i$, $i = 1, \ldots, k-1$, in the linear space $V^\otimes k$ as $I^k_i = V \otimes \cdots \otimes I \otimes \cdots \otimes V$ where $I$ is in the $i$th position. It is clear that the component $I^k$ of the ideal $\mathcal{I}$ is equal to the sum $\sum_{i=1}^{k-1} I^k_i$ of the subspaces $I^k_i$ of the linear space $V^\otimes k$.

A deformation of the quadratic algebra $(V, I)$ is defined by a splitting $k[[h]]$-submodule $I_h$ of $V^\otimes 2[[h]]$ such that all the submodules $I^k_h = \sum_{i=1}^{k-1} I^k_i$ are splitting. So, the deformed quadratic algebra is the $k[[h]]$-module $(V[[h]], I_h)$, $I_0 = I$. We shall write $V$ instead of $V[[h]]$, when it does not lead to misunderstanding, and denote the deformed algebra as $(V, I_h)$.

Let $I_h$ be a splitting submodule in $V[[h]]$. Note that in general $I^k_h$ need not be a splitting submodule, even if all $I^m_h$ are splitting ones for $m < k$. But in some cases all $I^k_h$ will be splitting submodules if this is so only for $I^2_h$. We call a quadratic algebra $(V, I)$ weakly Koszul if it satisfies the following distributiveness condition:

$$I^k_i \cap (I^k_2 + I^k_3 + \cdots + I^k_{k-1}) = I^k_i \cap I^k_2 + I^k_1 \cap (I^k_3 + \cdots + I^k_{k-1})$$

for all $k$. 

(1)
This definition is motivated by the fact that a quadratic algebra will be Koszul if and only if for all \( k \) the lattice of subspaces \( I^k_i \), \( i = 1, \ldots, k - 1 \), of \( V^\otimes k \) is distributive under the both operations: taking the sum and the intersections of subspaces (see [3]).

The following result essentially follows from [7] (see also [14]).

**Proposition 1.1.** Let \( I_h \) be a deformation of a subspace \( I \subset V^\otimes 2 \) (i.e. \( I_h \) is a splitting submodule in \( V[[h]] \) such that \( I_0 = I \)). In case \( (V, I) \) is a weakly Koszul algebra, all \( I^k_h \) will be splitting \( k[[h]] \)-submodules if \( I^2_h = V \otimes I_h + I_h \otimes V \) is a splitting \( k[[h]] \)-submodule.

**Proof.** The assertion is valid for \( k = 3 \) by hypothesis of the proposition. Suppose it is valid for all \( m < k \). To prove that the \( k[[h]] \)-submodule \( I^k_h = I_h \otimes V^\otimes (k-2) + V \otimes I_h^{k-1} \) is splitting, it is sufficient to show that the module \( J^a_h = I_h \otimes V^\otimes (k-2) \cap V \otimes I_h^{k-1} \) is splitting. We will prove that the rank of \( J^a_h \) at the general point coincide with the rank of this submodule at the zero point, which implies that \( J^a_h \) is a splitting submodule.

We have the inclusion

\[
I^a_h \otimes V^\otimes (k-2) \cap (V \otimes I_h \otimes V^\otimes (k-3) + V^\otimes 2 \otimes I_h^{k-2}) \supseteq I^{(3)}_h \otimes V^\otimes (k-3) + I^a_h \otimes I_h^{k-2},
\]

where \( I^{(3)}_h \) denotes \( I_h \otimes V \cap V \otimes I_h \).

Now observe that the submodule \( (V \otimes I_h \otimes V^\otimes (k-3) + V^\otimes 2 \otimes I_h^{k-2}) = V \otimes I_h^{k-1} \) is splitting by the induction hypothesis, and the submodules \( I_h \otimes V^\otimes (k-2) \), \( I^{(3)}_h \otimes V^\otimes (k-3) \), and \( I^a_h \otimes I_h^{k-2} \) are splitting as well. At the point \( h = 0 \) the modules in (2) are equal because of the weakly Koszul property. But it is clear that the rank of the intersection of two splitting \( k[[h]] \)-submodules at the general point has to be not more than the rank of their intersection at the zero point, while the rank of the sum of two splitting \( k[[h]] \)-submodules at the general point has to be not less than the rank of their sum at the zero point. Therefore, the ranks of both modules in (2) at the general point must be equal to the rank of these modules at the zero point. Hence, the modules coincide and \( J^a_h \) is a splitting submodule. The proposition is proved. \( \square \)

**Remark.** Let \( I_h \) be an algebraic family of subspaces of \( V \otimes V \), where point \( h \) runs over a connected algebraic variety \( M \), such that the dimension of subspaces \( I_h \otimes V + V \otimes I_h \) of \( V^\otimes 3 \) does not change. The arguments above show that if \( I_\sigma \) satisfies (1) at a point \( \sigma \in M \), then for a fixed \( k > 3 \) the dimension of \( I^k_h \) is constant for \( h \in M \setminus S_k \), where \( S_k \) is an algebraic subset of a smaller dimension. Therefore, for almost all points of \( M \) (the complement to \( \bigcup S_k \)) the dimension of all \( I^k_h \) does not change.

Given a quadratic algebra \( (V, I) \), one can define the dual quadratic algebra \( (V^*, I^*) \) where \( V^* \) is the dual space to \( V \), and \( I^* \) consists of the elements \( \lambda \in V^* \otimes V^* \) such that \( \lambda(I) = 0 \). It follows from duality that in order for \( (V^*, I^*) \) to be weakly Koszul, \( I \) must satisfy the dual to (1) distributiveness condition:

\[
I^k_1 + (I^k_2 \cap \cdots \cap I^k_{k-1}) = (I^k_1 + I^k_2) \cap (I^k_1 + (I^k_3 \cap \cdots \cap I^k_{k-1}))
\]
for all $k$. In particular, if the quadratic algebra $(V, I)$ is Koszul, then its dual quadratic algebra $(V^*, I^*)$ will be Koszul as well.

If $(V, I_h)$ is a deformation of the algebra $(V, I)$, one can build the dual $k[[h]]$-submodule $I_h^*$ in $V^*[ [h]]$.

**Proposition 1.2.** Let $(V, I_h)$ be a deformation of a Koszul quadratic algebra $(V, I)$. Then the pair $(V^*, I_h^*)$ gives a deformation of the dual quadratic algebra $(V^*, I^*)$.

**Proof.** The dual quadratic algebra $(V^*, I_h^*)$ also is a Koszul one, so that it suffices to check that the $k[[h]]$-module $I_h^{*3} = V^* \otimes I_h^* + I_h^* \otimes V^*$ is splitting. But $I_h^{*3}$ is dual to $V \otimes I_h \cap I_h \otimes V$, which is a splitting $k[[h]]$-submodule because $V \otimes I_h + I_h \otimes V$ is splitting. So $I_h^{*3}$ is a splitting submodule as well, which completes the proof. □

Let $x_1, \ldots, x_n$ be a basis in the space $V$, and $k \leq n$. We will perceive the first $k$ variables $x_1, \ldots, x_k$ as even and the other $x_{k+1}, \ldots, x_n$ as odd. Then the tensor algebra $T(V)$ can be regarded as a superalgebra. The super-commutative polynomial superalgebra $P_{k,l}$ ($l = n - k$) is the quotient of $T(V)$ by the ideal generated by the elements

$$x_i \otimes x_j - (-1)^{\xi \eta} x_j \otimes x_i.$$

(Here $(-1)^{\xi} = 1$ if $\xi$ is even and $(-1)^{\xi} = -1$ if $\xi$ is odd.) These elements determine a subspace $I_{l,k}$ in $V \otimes V$. Let $I_{k,l}$ be a subspace determined by the elements

$$x_i \otimes x_j + (-1)^{\xi \eta} x_j \otimes x_i.$$

The subspaces $I_{k,l}$ and $I_{l,k}$ are complemented in $V \otimes V$, so that the natural projection $\pi: T(V) \to P_{k,l}$ gives an isomorphism $\pi_2: I_{k,l} \to P_{k,l}$. Here $P_{k,l}^2$ denotes the component of the degree two in $P_{k,l}$. Further we set $I = I_{l,k}$ and $J = I_{k,l}$. A deformation $I_h$ of the subspace $I$ can be given by a family of linear operators $B_h: I[[h]] \to J[[h]]$, $B_h = h B^1 + h^2 B^2 + \cdots$, such that $I_h$ is a graph of the operator $B_h$. Define a bilinear form $b$ on $P_{k,l}$ in the following way: If $\tilde{x}_i, \tilde{x}_j$ are the images of $x_i, x_j$ under the projection $\pi$, we put

$$b(\tilde{x}_i, \tilde{x}_j) = \pi B^1(x_i \otimes x_j - (-1)^{\xi \eta} x_j \otimes x_i),$$

and on others elements from $P_{k,l}$ the form $b$ is extended by the Leibniz rule

$$b(uv, w) = ub(v, w) + (-1)^{uv} vb(u, w).$$

It is clear that in this way we obtain a one-one correspondence between the operators $B^1 : I \to J$ and the skew-symmetric bilinear forms $b: P_{k,l} \otimes P_{k,l} \to P_{k,l}$ satisfying the Leibniz rule and such that $b(\tilde{x}_i, \tilde{x}_j)$ are homogeneous quadratic forms, $b(\tilde{x}_i, \tilde{x}_j) = \sum_{p} b_{pq}^1 \tilde{x}_p \tilde{x}_q$. We call such a quadratic bracket Poisson if it satisfies the Jacobi identity

$$(-1)^{uv} b(b(u, v), w) + (-1)^{uv} b(b(v, w), u) + (-1)^{uv} b(b(w, u), v).$$

One can show in the standard way that if the family of operators $B_h$ determines a deformation of quadratic algebra $P_{k,l} = (V, I)$, then the form $b$ corresponding to $B^1$ is a quadratic Poisson bracket. It is easy to see that the dual to $P_{k,l}$ quadratic algebra
can be identified with $P_{l,k}$, the supercommutative polynomial superalgebra with $l$ even variables $x_1, \ldots, x^l$ and $k$ odd variables $x^{l+1}, \ldots, x^n$, $n = l + k$. The quadratic Poisson bracket $b$ on $P_{k,l}$ defines a quadratic Poisson bracket $\tilde{b}$ on $P_{l,k}$ in the following way:

$$\tilde{b}(x^i, x^j) = \sum_{pq} b_{pq} \xi^p \xi^q.$$ 

This bracket corresponds obviously to the dual deformation of the dual algebra $P_{l,k}$.

A quadratic Poisson bracket $b$ on $P_{k,l}$ is called quantizable if there exists a deformation of $P_{k,l}$ with $b$ as an initial term. Any deformation of $P_{k,l}$ with initial term $b$ will be called a quantization (or realization) of $b$. Since $P_{k,l}$ is a Koszul algebra, it follows from Proposition 1.2 that:

**Proposition 1.3.** If the bracket $b$ is quantizable on $P_{k,l}$ then the dual bracket $\tilde{b}$ is quantizable on $P_{l,k}$.

In the next sections we describe all quadratic Poisson brackets on the polynomial algebra of three even variables and show that any such bracket can be quantized, i.e. has a realization. Using Proposition 1.3 we can conclude that any quadratic Poisson bracket on the Grassmann algebra of three variables is quantizable as well.

**2. Classification of quadratic Poisson brackets in the three-dimensional case**

Let $b(f, g)$ be a Poisson bracket defined on a polynomial ring $R = k[x_1, \ldots, x_n]$ for which all $y_{i,j} = b(x_i, x_j)$ are homogeneous quadratic forms. Let $P(b)$ be the matrix with elements $p_{i,j} = (\partial / \partial x_j)(\sum_k \partial y_{i,k} / \partial x_k)$. Let $A$ be a matrix from $GL(n, k)$. Matrix $A$ defines an automorphism $a$ of $R$ by $x^a_i = \sum_j a_{i,j} x_j$. It is clear that the same Poisson bracket may be given in terms of $\{x^a_i\}$. Let us denote this Poisson bracket by $b^a$ and let $P^a = P(b^a)$.

**Lemma 2.1.** $P^a = APA^{-1}$.

**Proof.** It is sufficient to check that the lemma is true for elementary automorphisms $x^a_i = x_i$ for $i \neq j$ and $x^a_j = x_j + cx_k$ ($SL(n)$ is generated by these transformations) and for diagonal automorphisms $x^a_i = x_i$ for $i \neq j$ and $x^a_j = cx_j$. This may be done by a straightforward computation. □

(Another way to prove this lemma is to consider $b$ as an element of $V^* \otimes V^* \otimes V \otimes V$. Then $P$ corresponds to the contraction of this tensor.)

**Remark.** Since $y_{i,j} = - y_{j,i}$ one can see that $\text{trace } P = 0$.

Let $b$ be a quadratic Poisson bracket and let $P$ be its corresponding matrix. Let $r = \text{rank } P$. One obtains a natural classification of quadratic Poisson brackets according to the rank and Jordan form of $P$. 
Let us denote by $v_i = \sum_k \delta y_{i,k}/\partial x_k$. For $n = 3$ the Jacobi identity is equivalent to $y_{1,2}v_3 + y_{2,3}v_1 + y_{3,1}v_2 = 0$.

(a) $r = 0$. In this case, all $v_i = 0$ which implies that there exists a cubic form $f$ such that $y_{1,2} = \partial f/\partial x_3$, $y_{2,3} = \partial f/\partial x_1$, and $y_{3,1} = \partial f/\partial x_2$.

(b) $r = 1$. In this case, there exists a linear automorphism after which $v_1 = v_2 = 0$ and $v_3 = x_1$. Therefore, $y_{1,2} = 0$. But then since $v_1 = v_2 = 0$, derivatives $\partial y_{1,3}/\partial x_3 = \partial y_{2,3}/\partial x_3 - 0$ and $v_3 - x_1 = (\partial y_{3,1}/\partial x_1) + (\partial y_{3,2}/\partial x_2)$. This means that there exists a cubic form $f$ such that $y_{1,2} = \partial f/\partial x_3 = 0$, $y_{2,3} = (\partial f/\partial x_1) - x_1x_2$, $y_{3,1} = \partial f/\partial x_2$.

(c) $r = 2$. Here we have two subcases:

- (ca) $v_1 = 0$, $v_2 = -\lambda x_2$, and $v_3 = \lambda x_3$, or
- (cb) $v_1 = 0$, $v_2 = x_1$, and $v_3 = x_2$.

- (ca) There exists a cubic form $f$ such that $y_{1,2} = \partial f/\partial x_3$, $y_{2,3} = (\partial f/\partial x_1) - \lambda x_2 x_3$, $y_{3,1} = \partial f/\partial x_2$. The Jacobi identity then gives $x_3(\partial f/\partial x_3) - x_2(\partial f/\partial x_2) = 0$, and it is easy to see that $f = 2c_1 x_1 x_2 x_3 + c_2 x_1^3$.

- (cb) There exists a cubic form $f$ such that $y_{1,2} = (\partial f/\partial x_3)$, $y_{2,3} = (\partial f/\partial x_1) + x_1 x_3 - \frac{1}{3} x_2^2$, $y_{3,1} = (\partial f/\partial x_2)$. The Jacobi identity then gives $x_2(\partial f/\partial x_3) + x_1(\partial f/\partial x_2) = 0$, and here $f = -2c_1 x_3 x_1^2 + c_1 x_1 x_2^2 + c_2 x_3^3$.

(d) $r = 3$. Here we have three subcases:

- (da) $v_1 = \lambda_1 x_1$, $v_2 = \lambda_2 x_2$, and $v_3 = \lambda_3 x_3$, where $\lambda_1 + \lambda_2 + \lambda_3 = 0$ and $\lambda_1$, $\lambda_2$, and $\lambda_3$ are pairwise different (and different from zero),

- (db) $v_1 = \lambda x_1$, $v_2 = \lambda_2 x_2$, and $v_3 = -2\lambda x_3$ where $\lambda \neq 0$,

- (dc) $v_1 = \lambda x_1$, $v_2 = \lambda(x_2 + x_1)$, and $v_3 = -2\lambda x_3$ where $\lambda \neq 0$.

- (da) There exists a cubic form $f$ such that $y_{1,2} = (\partial f/\partial x_3)$, $y_{2,3} = (\partial f/\partial x_1) + \lambda_2 x_2 x_3$, $y_{3,1} = (\partial f/\partial x_2) - \lambda_1 x_1 x_3$. The Jacobi identity then gives $\lambda_1 x_1(\partial f/\partial x_1) + \lambda_2 x_2(\partial f/\partial x_2) + \lambda_3 x_3(\partial f/\partial x_3) = 0$, and since $\lambda_i \neq 0$ and are pairwise different the only possible $f = 2c_1 x_1 x_2 x_3$.

- (db) There exists a cubic form $f$ such that $y_{1,2} = (\partial f/\partial x_3)$, $y_{2,3} = (\partial f/\partial x_1) + \lambda x_2 x_3$, $y_{3,1} = (\partial f/\partial x_2) - \lambda x_1 x_3$. The Jacobi identity then gives $x_1(\partial f/\partial x_1) + x_2(\partial f/\partial x_2) - 2x_3(\partial f/\partial x_3) = 0$, and here $f = g(x_1, x_2) x_3$.

- (dc) There exists a cubic form $f$ such that $y_{1,2} = (\partial f/\partial x_3) - (\lambda/2) x_1^2$, $y_{2,3} = (\partial f/\partial x_1) + \lambda x_2 x_3$, $y_{3,1} = (\partial f/\partial x_2) - \lambda x_1 x_3$. The Jacobi identity then gives $x_1(\partial f/\partial x_1) + (x_1 + x_2) x_2(\partial f/\partial x_2) - 2x_3(\partial f/\partial x_3) = 0$, and here $f = c x_1^2 x_3$.

(The easiest way to find $f$ in (db) and (dc) is to use the Euler identity $x_1(\partial f/\partial x_1) + x_2(\partial f/\partial x_2) + x_3(\partial f/\partial x_3) = 3f$, and to rewrite Jacobi identities accordingly.)

So in all cases a quadratic Poisson bracket up to a linear transformation is described by a matrix in Jordan form (in case (dc) it is slightly changed for computational convenience) and by a cubic form which is arbitrary in case (a), in variables $x_1$ and $x_2$ in case (b), $f = c_1 x_1 x_2 x_3 + c_2 x_1^3$ in case (ca), $f = -2c_1 x_1 x_2^2 + c_1 x_1 x_2^2 + c_2 x_1^3$ in case (cb), $f = 2c_1 x_1 x_2 x_3$ in case (da), $f = g(x_1, x_2) x_3$ in case (db), and $f = c x_1^2 x_3$ in case (dc).

Now in case (a) $f$ may be reduced by an arbitrary linear transformation and in case (db) by a transformation in variables $x_1, x_2$. 
Remark. It was brought to our attention that essentially the same classification of quadratic Poisson brackets was obtained in [8, 11].

3. Quantization of quadratic Poisson brackets in the three-dimensional case

Let $R = \mathbb{k}[[\hbar]]\langle a_1, a_2, a_3 \rangle$ be a free algebra with three generators over the ring $\mathbb{k}[[\hbar]]$ of formal power series. Let us define on this algebra “Jordan” operation $f \circ g = \frac{1}{2}(fg + gf)$. Let $b$ be a quadratic bracket on $A = \mathbb{k}[x_1, x_2, x_3]$. Then $b$ is defined by $y_{i,j} = b(x_i, x_j)$. Let $y_{i,j} = \sum c_{i,j}^{k,l} x_k x_l$.

Theorem 3.1. The quotient algebra $R_b$ of algebra $R$ by the ideal generated by the relations $[a_i, a_j] = \hbar \sum c_{i,j}^{k,l} a_k \circ a_l$ is a quantization of the bracket $b$. (Here $x_i$ is the image of $a_i$.)

Proof. Let us use the same notations as above, namely $y_{i,j} = b(x_i, x_j)$ and $v_i = \sum_k (\partial y_{i,k}/\partial x_k)$. Here a partial derivative is the following operation: $\circ$ is replaced by commutative multiplication and then ordinary partial derivative is computed. It is easy to check that then in algebra $R$ we have

$$[x_1, y_{2,3}] + [x_2, y_{3,1}] + [x_3, y_{1,2}] = [x_1, x_2] \circ v_3 + [x_2, x_3] \circ v_1 + [x_3, x_1] \circ v_2. \quad (4)$$

So in the case when rank is zero and all $v_i = 0$ the Theorem can be deduced from a result of Drinfeld (see Proposition 1.1) which states that in our setting it is sufficient to check that $W = (I_2 \otimes V) \cap (V \otimes I_2)$ is at least one dimensional. If we denote our relations by $f_{i,j} = y_{i,j} - [x_i, x_j]$, then

$$f_{1,2} x_3 + f_{2,3} x_1 + f_{3,1} x_2 = x_1 f_{1,2} + x_1 f_{2,3} + x_2 f_{3,1},$$

which shows that $\dim(W) > 0$ in this case. Similar proof works for the case $r = 1$ where

$$f_{1,2} x_3 + f_{2,3} x_1 + f_{3,1} x_2 - \hbar f_{1,2} x_1 = x_3 f_{1,2} + x_1 f_{2,3} + x_2 f_{3,1} + \hbar x_1 f_{1,2}$$

and in the second subcase for $r = 2$ where

$$f_{1,2} x_3 + f_{2,3} x_1 + f_{3,1} x_2 - \hbar (c_1 f_{1,2} x_1 + f_{1,2} x_2 + f_{3,1} x_1) = x_3 f_{1,2} + x_1 f_{2,3} + x_2 f_{3,1} - \hbar (c_1 x_1 f_{1,2} - x_2 f_{1,2} - x_1 f_{3,1}).$$

Although it is clear a posteriori that $\dim(W) = 1$ in all the cases, we approach the remaining cases differently, to avoid lengthy computations.

**Rank two case.** In the first subcase

$$[x_1, x_2] = \hbar c_1 x_1 \circ x_2,$$

$$[x_2, x_3] = \hbar \left( c_1 - \frac{\hbar}{2} \right) x_2 \circ x_3 + 3c_2 x_1^2,$$

$$[x_3, x_1] = \hbar c_1 x_3 \circ x_1.$$
Rank three case. In the first subcase
\[ [x_1, x_2] = \hbar c x_1 \circ x_2, \quad [x_2, x_3] = \hbar \left( c + \frac{1}{2} \lambda_2 \right) x_2 \circ x_3, \]
\[ [x_3, x_1] = \hbar \left( c - \frac{1}{2} \lambda_1 \right) x_3 \circ x_1. \]

In the second subcase it is possible to make a linear transformation of variables \( x_1 \) and \( x_2 \) which reduces \( f \) to \((ax_1^2 + bx_1x_2)x_3\). So
\[ [x_1, x_2] = \frac{\hbar}{2}(2ax_1^2 + bx_1 \circ x_2), \quad [x_2, x_3] = \frac{\hbar}{2}(2ax_1 \circ x_3 + (b + \lambda)x_2 \circ x_3), \]
\[ [x_3, x_1] = \frac{\hbar}{2}(b - \lambda)x_3 \circ x_1. \]

In the third subcase relations are
\[ [x_1, x_2] = \frac{\hbar}{2}(2c - \lambda)x_1^2, \quad [x_2, x_3] = \frac{\hbar}{2}(2c x_1 \circ x_3 + \lambda x_2 \circ x_3), \]
\[ [x_3, x_1] = -\frac{\hbar}{2} \lambda x_3 \circ x_1. \]

In all of these cases relations may be rewritten as
\[ x_2x_1 = a_3x_1x_2 + b_3x_1^2, \quad x_3x_2 = a_1x_2x_3 + h_1x_1^2 + c_1x_1x_3, \quad x_3x_1 = a_2x_1x_3. \]

So Bergman's (see [4]) diamond lemma may be applied. It is easy to check that in each of the cases values of parameters are such that the only ambiguity \( x_3x_2x_1 \) is resolvable and the monomials \( x_1^2x_2^2x_3^2 \) form a basis in the corresponding algebras. (As it is shown in [4] resolution of this ambiguity is equivalent to the Jacobi identity so in fact these computations show that \( \dim(W) > 0 \).)

This finishes the proof of the theorem. \( \square \)

Appendix A. Direct constructions of algebras \( R_\hbar \)

In this section we give an alternative proof of the theorem based on direct constructions of algebras isomorphic to \( R_\hbar \).

Let \( \mathcal{F} \) be an algebra of functions on a lattice \( \mathbb{Z}^2 \) with values in \( k(\hbar, w, z) \) where \( w \) and \( z \) are central variables. One of the sources of algebras isomorphic to \( R_\hbar \) will be subalgebras of the algebra \( \mathcal{H} \) of homomorphisms of \( \mathcal{F} \) generated by two "coordinate" shifts and by multiplication by elements of \( \mathcal{F} \). We will be using functional notation \( f(x, y) \) for the elements of \( \mathcal{F} \) and corresponding elements of \( \mathcal{H} \) and denote shifts by \( s_x \) and \( s_y \). When we need only one shift for our construction we will denote it by \( s \).

So we have commuting symbols \( f(x, y) \) and relations are \( s_x f(x, y) = f(x + 1, y)s_x \), \( s_y f(x, y) = f(x, y + 1)s_y \).

Another source will be different subalgebras and extensions of Weyl algebras.
Rank zero case. In this case we can use a classification of cubic forms.

Let \( H(x, y, z) \) be a (homogeneous) cubic form over an algebraically closed field of characteristic zero. Then the following is the list of possible reductions of this form under the action of \( GL(3) \):

\[
\begin{align*}
(1) & \quad 0, \\
(2) & \quad x^3, \\
(3) & \quad x^2y, \\
(4) & \quad xy(x + y), \\
(5) & \quad zx^2 + xy^2, \\
(6) & \quad zx^2 + y^3, \\
(7) & \quad 2zx, \\
(8) & \quad 2zx + x^3, \\
(9) & \quad 2Zxy + x^3 + y^3, \\
(10) & \quad z^2x + y(x + y)(x + cy) \quad \text{where } c \neq 0 \text{ and } c \neq 1.
\end{align*}
\]

A group of order 3 acts on (10).

This list was known of course to Steiner though he did not bother to put it in quite such a form.

Remark. First four orbits correspond to a smaller number of variables. Forms from (5) to (9) are linear in \( z \). Form (10) is the only one with a unique singular point (at the origin). (When \( c=0 \) then (10) is equivalent to (8) and when \( c=1 \) then (10) is equivalent to (9).)

So let us proceed along the list of reduced forms.

(1) corresponds to the ring of polynomials.

(2), (3), and (4) correspond to relations

\[
[x_1, x_2] = 0, \quad [x_2, x_3] = \hbar y_{2,3}, \quad [x_3, x_1] = \hbar y_{3,1},
\]

where \( y_{2,3} \) and \( y_{3,1} \) are some elements of the polynomial ring \( P = \mathbb{k}[[\hbar]][x_1, x_2] \) which depend on the case.

Here we obtain skew polynomial extensions \( P[x_3, \delta] \) of the derivation type of the ring \( P \) (see [6]) and in fact \( y_{2,3} \) and \( y_{3,1} \) can be any elements of \( P \). (We shall use it later.)

These algebras may be realized as subalgebras of the second Weyl algebra \( A_2 \) over \( k(z) \). Say, if generators are \( p_1, q_1 \) and \( p_2, q_2 \) (and relations are \( [p_i, q_i] = 1 \) and all other commutators are zeros), then \( x_1 = p_1, \ x_2 = p_2, \) and \( x_3 = \hbar(q_2 y_{2,3} - q_1 y_{3,1}) + z \) will suffice.

Since the standard monomials in \( p_1, p_2, z \) are linearly independent, it is clear that the standard monomials in \( x_1, x_2, x_3 \) are linearly independent. It implies that the constructed algebras are isomorphic to the corresponding \( R_9 \).

(5) corresponds to relations

\[
[x_1, x_2] = \hbar x_1^2, \quad [x_2, x_3] = \hbar(x_1 \circ x_3 + x_2^2), \quad [x_3, x_1] = \hbar x_1 \circ x_2,
\]

and (6) corresponds to relations

\[
[x_1, x_2] = \hbar x_1^2, \quad [x_2, x_3] = \hbar x_1 \circ x_3, \quad [x_3, x_1] = 3\hbar x_2^2.
\]

These algebras are skew polynomial extensions \( B[x_3; \alpha, \delta] \) of the algebra \( B \) generated by \( x_1 \) and \( x_2 \). (See [6], \( \alpha \) is an automorphism of \( B \) and \( \delta \) is an \( \alpha \) derivation of \( B \).)
Here \( x_1^2 = x_1, \ x_2^2 = x_2 - 2hx_1 \) in both cases and \( x_1^0 = \hbar x_1 \circ x_2, \ x_2^0 = -\hbar(x_2 + \hbar x_1 \circ x_2) \) for (5) and \( x_1^0 = 3\hbar x_2, \ x_2^0 = -3\hbar^2 x_1^2 \) for (6).

Let \( A_1 \) be the first Weyl algebra with generators \( p \) and \( q \) over \( k(\hbar, w, z) \). (Here \( w \) and \( z \) are central variables and \([p, q] = 1\).) Let \( D_1 \) be its field of fractions. The following subalgebras of \( D_1 \) are realizations of corresponding brackets.

For (5) we may consider an algebra generated by \( x_1 = p^{-1}, \ x_2 = -(h(q + w)), \) and \( x_3 = -(fzq + w)^2 p - h(hq + w)^2 + zp^2, \) and for (6) we may consider an algebra generated by \( x_1 = p^{-1}, \ x_2 = -(h(q + w)), \) and \( x_3 = (h(q + w)^3 p^2 + 3h(hq + w^2)p + zp^2. \)

These algebras are homomorphic images of the corresponding \( R_b \).

As above, in both these cases it is clear that the standard monomials of constructed algebras are linearly independent (e.g., it follows from the independence of monomials of \( p^{-1}, w, \) and \( zp^2 \)). So again these algebras are isomorphic to the corresponding \( R_b \).

(7), (8), and (9) correspond to relations

\[
[x_1, x_2] = \hbar x_1 \circ x_2, \quad [x_2, x_3] = \hbar(x_2 \circ x_3 + 3c_1 x_1^3),
\]

\[
[x_3, x_1] = \hbar(x_3 \circ x_1 + 3d_1 x_2^2), \quad \text{where} \ c_7 = d_7 = d_8 = 0 \ \text{and} \ c_8 = c_9 = d_9 = 1.
\]

These algebras are skew polynomial extensions \( Q[x_3; \alpha, \delta] \) of a quantum plane \( Q \) generated by \( x_1 \) and \( x_2 \) (see [6]). Here \( x_2 x_1 = k x_1 x_2 \) where \( k = (1 - \hbar)/(1 + \hbar) \), and these skew extensions are defined by \( x_1^\delta = k x_1, \ x_2^\delta = k x_2, \) and \( x_3^\delta = [c_1/k(1 - \hbar)] x_2^2, \)

\[
x_2^\delta = [c_2 h/(1 + h)] x_1^2,
\]

where \( c_1 \) and \( c_2 \) should be appropriately chosen for each of the cases. It is a straightforward computation to check that \( \delta \) is well defined on \( Q \) by

\[
(x_1 x_2^\delta)^\delta = k^{-i} (1 - k^3)^{-1} [c_3 k(1 - k^3)] x_1^{i-1} x_2^{j+2} + c_4 (1 - k^3) x_1^{i+2} x_2^{j-1},
\]

where \( c_3 = [c_1 h/(1 - \hbar)] \) and \( c_4 = [c_2 h/(1 + \hbar)] \).

We can rewrite our relations as

\[
x_2 x_1 = k x_1 x_2, \quad x_3 x_2 = k x_2 x_3 + c_1 x_1^2, \quad x_3 x_1 = k^{-1} x_1 x_3 + d x_2^2,
\]

where \( k, c, d \in k(\hbar) \), and let \( x_1 = s, \ x_2 = wk^{-x}s, \) and \( x_3 = k(cw^{-1}k^x - dw^2k^{1-2x})(1 - k^3)^{-1}s + zk^xs^{-2} \).

As above, it is easy to check that an algebra generated by \( x_1, x_2, \) and \( x_3 \) is isomorphic to the corresponding \( R_b \).

All of the previously considered cases are also covered by Bergman’s diamond lemma (see [4]). Case (10) (which is not covered by the diamond lemma) requires a more involved construction (see the end of the section).

**Rank one case.** Here

\[
[x_1, x_2] = 0, \quad [x_2, x_3] = h \left( \frac{\partial f}{\partial x_1} - x_1 x_2 \right), \quad [x_3, x_1] = h \frac{\partial f}{\partial x_2},
\]

and this case is analogous to the cases (2), (3), and (4) in the \( r = 0 \) case. The corresponding algebras are skew polynomial extensions of the ring \( P \) of the derivation type and may be realized as subalgebras of \( A_2 \).
Rank two case. The first subcase where

\[ [x_1, x_2] = \hbar c_1 x_1 \circ x_2, \quad [x_2, x_3] = \hbar \left( \left( c_1 - \frac{1}{2} \right) x_2 \circ x_3 + 3c_2 x_1^2 \right), \]

\[ [x_3, x_1] = \hbar c_1 x_3 \circ x_1 \]

is analogous to the case (8) and gives a skew extension of a quantum plane.

Let us rewrite relations:

\[ x_2 x_1 = k x_1 x_2, \quad x_3 x_2 = k_1 x_2 x_3 + c x_1^2, \quad x_3 x_1 = k^{-1} x_1 x_3. \]

The main difference with the previously considered cases is that \( k_1 \neq k \). For this case we may take \( x_1 = s, x_2 = wk^{x-y}k_s y, \) and \( x_3 = cw^{-1}(1 - k_1 k^2)^{-1} k^{1+x-y} k_s x + zk^x s_y. \) (This representation works even when \( c_1 = 0 \) or \( c_1 - \frac{1}{2} = 0. \))

The second subcase where

\[ [x_1, x_2] = \hbar (-2c_1 x_1^2), \quad [x_2, x_3] = \hbar ((-2c_1 + \frac{1}{2}) x_1 \circ x_3 + (c_1 - \frac{1}{2}) x_2^2 + 3c_2 x_1^2), \]

\[ [x_3, x_1] = \hbar c_1 x_1 \circ x_2 \]

gives an extension of the polynomial ring \( P \) when \( c_1 = 0 \). It is analogous to case (5) when \( c_1 \neq 0 \) (and defines an extension of algebra \( B \)).

Let \( c_1 = 0 \). Then \( x_1 \) is a central element and we can take the following two elements of the first Weyl algebra over \( k(h, x_1, w, z) \) for \( x_2 \) and \( x_3 \) (\( t \) denotes \( q p \)):

\[ x_2 = -x_1 (ht + w) \quad \text{and} \quad x_3 = x_1 \left( \frac{1}{2} (ht + w)^2 - 3c_2 + z p \right). \]

Let us assume now that \( c_1 \neq 0 \). Then we may consider algebra generated by \( x_1 = p^{-1}, x_2 = 2c_1 (hq + w), \) and \( x_3 = 2c_1^2 [(hq + w)^2 p + h(q + w)] + (c_1^2 h^2 (4c_1 - 1) - 3c_2)(1 - 6c_1)^{-1} p^{-1} + z p^d, \) where \( d = (2c_1)^{-1} - 2. \) If \( d \) is not an integer we can view this algebra as a subalgebra of \( D_1[p^d] \). When \( 1 - 6c_1 = 0 \) this algebra is not defined and we put \( x_3 = 2c_1^2 [(hq + w)^2 p + h(q + w)] + (c_1^2 h^2 (4c_1 - 1) - 3c_2) (2c_1)^{-1} p^{-1} \ln(p) + z p^{-1}. \) This algebra belongs to \( D_1(\ln(p)) \).

These two extensions of \( D_1 \) are defined as follows. Each element \( a \in D_1 \) may be represented as \( a = \sum_{i=k}^{\infty} a_i p^i \) where \( a_i \in k(h, w, z, q) \). It is possible to extend multiplication from \( D_1 \) on such arbitrary series by

\[ a \star b = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial^j a}{\partial p^j} \frac{\partial^j b}{\partial q^j}, \]

where multiplication in the right-side of this formula is ordinary commutative multiplication of power series.

It is clear that this formula gives well defined expressions when either \( a \) or \( b \) are replaced by \( f \) if \( \partial f / \partial p \) and \( \partial f / \partial q \) are defined and are "sufficiently small". For example, in our cases when \( f = p^d \) or \( f = \ln(p) \) we have \( \partial f / \partial p = p^{-1}(af + b) \) and \( \partial f / \partial q = 0 \) and \( \star \) multiplication is well defined.
Rank three case. The first subcase where

\[
[x_1, x_2] = \hbar c x_1 \circ x_2, \quad [x_2, x_3] = \hbar \left( c + \frac{\lambda_2}{2} \right) x_2 \circ x_3,
\]

\[
[x_3, x_1] = \hbar \left( c - \frac{\lambda_1}{2} \right) x_3 \circ x_1
\]
is analogous to the case (7) and is a "quantum space".

Let us rewrite these relations as \(x_2 x_1 = k_3 x_1 x_2, \; x_3 x_2 = k_1 x_2 x_3, \; x_3 x_1 = k_2 x_1 x_3\) where \(k_i \in k(\hbar)\). We may put \(x_1 = s_x, \; x_2 = k_3^{-x} s_y, \) and \(x_3 = zk_2^{-1} k_1^{-y}\).

In the second subcase it is possible to make a transformation in variables \(x_1\) and \(x_2\) and to reduce \(f\) to either

(a) \(f = 0\) or

(b) \(f = x_1 x_2 x_3\) or

(c) \(f = x_1^2 x_3\).

For (a) relations are

\[
[x_1, x_2] = 0, \quad [x_2, x_3] = \hbar \frac{\lambda}{2} x_2 \circ x_3, \quad [x_3, x_1] = -\hbar \frac{\lambda}{2} x_3 \circ x_1,
\]

and the corresponding algebra may be considered as a quantum space.

For (b) relations are

\[
[x_1, x_2] = \frac{\hbar}{2} x_1 \circ x_2, \quad [x_2, x_3] = \hbar \frac{\lambda + 1}{2} x_2 \circ x_3, \quad [x_3, x_1] = \hbar \frac{1 - \lambda}{2} x_3 \circ x_1,
\]

and this is a quantum space again.

For (c) relations are

\[
[x_1, x_2] = \hbar x_1^2, \quad [x_2, x_3] = \hbar \left( x_1 \circ x_3 + \frac{\lambda}{2} x_2 \circ x_3 \right), \quad [x_3, x_1] = -\hbar \frac{\lambda}{2} x_3 \circ x_1.
\]

In the third subcase relations are

\[
[x_1, x_2] = \frac{\hbar}{2} (2c - \lambda) x_1^2, \quad [x_2, x_3] = \frac{\hbar}{2} (2c x_1 \circ x_3 + \lambda x_2 \circ x_3),
\]

\[
[x_3, x_1] = -\frac{\hbar}{2} \lambda x_3 \circ x_1.
\]

Here is a realization for these two cases. Let us consider the following relations:

\[
[x_1, x_2] = ax_1^2, \quad [x_2, x_3] = (bx_1 - cx_2) \circ x_3, \quad [x_3, x_1] = cx_1 \circ x_3.
\]

Let us rewrite the last relation as \(x_3 x_1 = k x_1 x_3\).

Then \(x_3 x_2 = (k x_2 + k_1 x_1) x_3\), and we have a skew polynomial extension of automorphism type of the subalgebra generated by \(x_1, x_2\).

If \(a = 0\), then it is an extension of polynomial algebra which can be realized as a subalgebra of \(\mathcal{H}\) with \(x_1 = w k^x, \; x_2 = w k_1 x k^{x-1}\), and \(x_3 = s\).

If \(a \neq 0\) then a realization can be found in the tensor product of \(D_1[\rho^d]\) and \(\mathcal{H}\) (which is rather natural since it contains both a quantum plane and a big subalgebra.
of a Weyl algebra). We can take $x_1 = p^{-1}s$, $x_2 = -(aq + w)s$ and $x_3 = z p^d k^{-x}$ where $d = -k_1 k^{-1} a^{-1}$.

Orbit (10) case. To construct algebras which correspond to case (10), one may consider an algebra $B = k(u)[[h, p, q]]$ of formal power series in $h$, $p$, and $q$ over $k(u)$ (where $u$ is a central variable and $[p, q] = h$) and search for a realization of corresponding brackets as subalgebras of $B$ with generators $x_1 = \sum h^i q^j x_{1,i,j}$, $x_2 = \sum h^i q^j x_{2,i,j}$, and $x_3 = z - p$. Here $x_{1,i,j}, x_{2,i,j} \in k(u)[[p]]$.

With the help of rather straightforward computations one may find corresponding solutions and show that the standard monomials are linearly independent in this case as well.

Let us assume that $H(x, y, z) = z^2 x + c_1 x y^2 + c_2 x^2 y + y^3$. Then

$$[x_1, x_2] = h x_1 \circ x_3, \quad [x_2, x_3] = h (x_1^2 + c_1 x_2^2 + c_2 x_1 \circ x_2),$$

$$[x_3, x_1] = h (c_1 x_1 \circ x_2 + c_2 x_1^2 + 3 x_2^2),$$

where we perceive these relations as equations for $x_{1,i,j}$ and $x_{2,i,j}$.

Let us again use notations $[x_i, x_j] = h y_{i,j}$ and rewrite the second and third equations as

$$\sum j h^{i+1} q^{j-1} x_{2,i,j} = - h y_{2,3} \quad \text{and} \quad \sum j h^{i+1} q^{j-1} x_{1,i,j} = h y_{3,1}.$$ 

Now $g(p)q^i = \sum_{k=0}^{i} h^k [p/k] [(i-k)!] q^{-k} g(k)$ where $g(p) \in k[[p]]$ and $g^{(k)}$ is the "ordinary" derivative. So the product of two monomials in $B$ is a linear combination of monomials with the same total $h, q$-degree (which is equal to the sum of degrees of multiples) and $h$-degrees of these monomials are not less than the sum of $h$ degrees of the multiples. Therefore $x_{m,n,0}$ can be expressed through $x_{1,a,b}$ and $x_{2,a,b}$ where $a + b = i + j - 1$ and $a \leq i$. So by induction one can show that $x_{m,n,0}$ can be expressed as polynomials of $x_{1,n,0}^{(k)}$ and $x_{2,n,0}^{(k)}$ where $n \leq i$. Since $x_{m,n,0}$ do not appear in the left-hand-sides of our equations they can be chosen arbitrarily.

Now we have to choose these coefficients in such a way that the first equation will be satisfied. The following observation make this part of the computations more manageable. It is easy to check that $[x_1, y_{2,3}] + [x_2, y_{3,1}] + [x_3, y_{2,1}] = [x_1, x_2] \circ v_3 + [x_2, x_3] \circ v_1 + [x_3, x_1] \circ v_2 = 0$ for any choice of $x_i$. On the other hand, with our choice of $x_i$ the Jacobi identity gives $h [x_1, y_{2,3}] + h [x_2, y_{3,1}] + [x_3, [x_1, x_2]] = 0$. It implies that $[p, [x_1, x_2] - h x_1 \circ p] = 0$ is satisfied with any choice of $x_{m,n,0}$. Therefore, $[x_1, x_2] - h x_1 \circ p$ does not contain terms with $q$, and it is sufficient to check that $[x_1, x_2] - h x_1 \circ x_3$ is an equality only in terms which do not contain $q$. Let us use notation $\equiv$ for such equalities.

It is clear then that $[x_1, x_2] \equiv [\sum h^i x_{1,i,0}, x_2] + [x_1, \sum h^i x_{2,i,0}]$, and that $p \circ x_1 = 2 \sum h^i x_{1,i,0} p + h \sum h^i x_{1,i,1}$.

So the coefficient $A_n$ of $h^n$ in $[x_1, x_2] - h x_1 \circ p$ when $n > 1$ is equal to

$$\frac{1}{x_{1,n-1,0} x_{2,0,1} - x_{2,n-1,0} x_{1,0,1} + x_{1,0,0} x_{2,n-1,1} - x_{2,0,0} x_{1,n-1,1} - 2 x_{1,n-1,0} p + \delta_n}$$
where \( \delta_n \) depends on \( x_{m,k,0} \) with \( k < n - 1 \). The coefficient \( A_1 \) of \( \hbar \) is 
\[
x_{1,k,0} = \frac{\partial^2 H}{\partial x_1 \partial x_2} (x_{1,0,0}, x_{2,0,0}, p) x_{1,k,0} + \frac{\partial^2 H}{\partial x_2 \partial x_2} (x_{1,0,0}, x_{2,0,0}, p) x_{2,k,0} + \delta_{1,k}
\]
and
\[
-x_{2,k,1} = \frac{\partial^2 H}{\partial x_1 \partial x_1} (x_{1,0,0}, x_{2,0,0}, p) x_{1,k,0} + \frac{\partial^2 H}{\partial x_1 \partial x_2} (x_{1,0,0}, x_{2,0,0}, p) x_{2,k,0} + \delta_{2,k}
\]
for \( k > 0 \). (Here \( \delta_{m,k} \) depend on \( x_{m,i,0} \) with \( i < k \).)

For \( k = 0 \) one gets
\[
x_{1,0,1} = \frac{\partial H}{\partial x_2} (x_{1,0,0}, x_{2,0,0}, p) \quad \text{and} \quad -x_{2,0,1} = \frac{\partial H}{\partial x_1} (x_{1,0,0}, x_{2,0,0}, p).
\]

By substituting these expressions in \( A_1 \) one gets
\[
\frac{\partial H}{\partial x_1} (x_{1,0,0}, x_{2,0,0}, p) x_{1,0,0} + \frac{\partial H}{\partial x_2} (x_{1,0,0}, x_{2,0,0}, p) x_{2,0,0} + 2 x_{1,0,0} p = 0.
\]

Since \( 2 x_{1,0,0} p = (\partial H/\partial x_3) (x_{1,0,0}, x_{2,0,0}, p) \) one can conclude that \( \partial H/\partial p = 0 \) and put \( H(x_{1,0,0}, x_{2,0,0}, p) = u \) where \( u \) is a central variable. Substitution in \( A_n \) for \( n > 1 \) gives
\[
\frac{\partial}{\partial p} \left( \frac{\partial H}{\partial x_1} (x_{1,0,0}, x_{2,0,0}, p) x_{1,n-1,0} + \frac{\partial H}{\partial x_2} (x_{1,0,0}, x_{2,0,0}, p) x_{2,n-1,0} \right) + e_n = 0
\]
accordingly. (Here \( e_n \) depend on \( x_{m,i,0} \) with \( i < n - 1 \).)

Let us put finally \( x_{1,i,0} = 0 \) for all \( i \) and \( x_{2,0,0} = v \) where \( v \) is a central variable. Then \( u = v^3 \) and \( x_1 = 3v^2 q + \cdots, x_2 = v + \cdots, x_3 = p \) where omitted terms are of higher \( \hbar \), \( q \)-degree. Since the standard monomials in the lowest terms are linearly independent, we can conclude that the standard monomials in \( x_1, x_2, x_3 \) are also linearly independent.

**Remark.** (1) Algebras which appeared here are considered from a different point of view in \([1,2,9,10,12]\).

(2) We thank the referee who brought to our attention that the quantization in the cases of rank 0 and 1 can be deduced from Proposition 4.7 of the paper \([13]\).

**References**


