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Matroidizing Set Systems: A New Approach to Matroid Theory

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Abstract. For a finite nonempty set E we associate in a canonical way to every antichain $\mathcal{B} \subseteq \mathcal{P}(E)$ a matroid $M(\mathcal{B})$ such that $M(\mathcal{B}) = M_0$ if $\mathcal{B}$ is the set of bases of a matroid M_0. We do this by first associating to $\mathcal{B}$ a closure operator $<\ldots>=<\ldots>_{\mathcal{B}}: \mathcal{P}(E) \to \mathcal{P}(E)$ and to a closure operator $<\ldots>$ the antichain $\mathcal{B}_{<\ldots>}$, consisting of all minimal generating sets. For $n \geq 0$ we define new antichains $P^n(\mathcal{B})$, where $P^0(\mathcal{B}) := \mathcal{B}$ and $P^{n+1}(\mathcal{B}) := P^n(\mathcal{B}_{(<\ldots>_{\mathcal{B}})})$ for all such n. Then $P^1(\mathcal{B}) = \mathcal{B}$ if and only if $\mathcal{B}$ is the set of bases of some matroid. We show that there exists some $m \geq 0$, depending only on the cardinality $\#E$, such that $P^{m+1}(\mathcal{B}) = P^m(\mathcal{B})$ for every antichain $\mathcal{B} \subseteq \mathcal{P}(E)$ and, hence, may define $M(\mathcal{B})$ to be the matroid with $P^m(\mathcal{B})$ as its set of bases. This simple construction has many intriguing properties, which we believe deserve further study.

In the sequel we assume that E is a finite nonempty set. For an antichain $\mathcal{B} \subseteq \mathcal{P}(E)$ we define an operator $<\ldots>_{\mathcal{B}}: \mathcal{P}(E) \to \mathcal{P}(E)$ by

$$< F >_{\mathcal{B}} := \{e \in E \mid \text{ for every } B \in \mathcal{B} \text{ with } e \in B \text{ there exists some } f \in F \text{ with } (B \setminus \{e\}) \cup \{f\} \in \mathcal{B}\}. \tag{1}$$

Lemma. *$<\ldots>_{\mathcal{B}}: \mathcal{P}(E) \to \mathcal{P}(E)$ is a closure operator for any antichain $\mathcal{B} \subseteq \mathcal{P}(E)$; that means:*

$$\text{For } F \subseteq E \text{ we have } F \subseteq < F >_{\mathcal{B}} = << F >_{\mathcal{B}} >_{\mathcal{B}}; \tag{C1}$$

$$\text{For } F_1 \subseteq F_2 \subseteq E \text{ we have } < F_1 >_{\mathcal{B}} \subseteq < F_2 >_{\mathcal{B}} . \tag{C2}$$

Proof: The only nontrivial assertion is $< F >_{\mathcal{B}} = << F >_{\mathcal{B}} >_{\mathcal{B}}$ for $F \subseteq E$. Assume $e \in << F >_{\mathcal{B}} >_{\mathcal{B}}$. For $e \in B \in \mathcal{B}$ there exists $f' \in < F >_{\mathcal{B}}$ with $B' := (B \setminus \{e\}) \cup \{f'\} \in \mathcal{B}$. Thus there exists also some $f \in F$ with

$$(B \setminus \{e\}) \cup \{f\} = (B' \setminus \{f'\}) \cup \{f\} \in \mathcal{B};$$

this means $e \in < F >_{\mathcal{B}}$.

For $\mathcal{A} \subseteq \mathcal{P}(E)$ we put

$$\mathcal{A}_{\min} := \{A \in \mathcal{A} \mid A' \in \mathcal{A} \text{ and } A' \subseteq A \text{ imply } A' = A\}.$$

Now assume that $<\ldots>: \mathcal{P}(E) \to \mathcal{P}(E)$ denotes some closure operator. Put

$$\mathcal{B}_{<\ldots>} := \{B \subseteq E \mid < B >= E\}_{\min}. \tag{2}$$

If M denotes some matroid defined on E with $\mathcal{B}$ as its set of bases and $<\ldots>$ as its closure operator, then we have $<\ldots>=<\ldots>_{\mathcal{B}}$ and $\mathcal{B} = \mathcal{B}_{<\ldots>}$. More generally, we have

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PROPOSITION. *Assume $\mathcal{B} \subseteq \mathcal{P}(E)$ is some antichain and $<\ldots>: \mathcal{P}(E) \to \mathcal{P}(E)$ denotes some closure operator. Then the following statements are equivalent:*

(i) $<\ldots>=<\ldots>_{\mathcal{B}}$ *and* $\mathcal{B} = \mathcal{B}_{<\ldots>}$.

(ii) $<\ldots>=<\ldots>_{\mathcal{B}}$ *and $\mathcal{B}$ is the set of bases of some matroid; that means: for all $B_1, B_2 \in \mathcal{B}$ and $b \in B_1 \setminus B_2$ there exists some $b' \in B_2 \setminus B_1$ with $(B_1 \setminus \{b\}) \cup \{b'\} \in \mathcal{B}$.*

(iii) $\mathcal{B} = \mathcal{B}_{<\ldots>}$ *and $<\ldots>$ is the closure operator of some matroid; that means: for all $F \subseteq E$ and $e, f \in E$ we have*

$$f \in < F \cup \{e\} > \setminus < F > \text{ if and only if } e \in < F \cup \{f\} > \setminus < F > .$$

PROOF: (i) $\Rightarrow$ (ii): Assume $B_1, B_2 \in \mathcal{B}$ and $b \in B_1 \setminus B_2$. $B_2 \in \mathcal{B} = \mathcal{B}_{<\ldots>}$ implies $< B_2 >_{\mathcal{B}} = < B_2 > = E$. Thus by the definition of $<\ldots>_{\mathcal{B}}$ there exists some $b' \in B_2$ with $B_1' := (B_1 \setminus \{b\}) \cup \{b'\} \in \mathcal{B}$. Since $\mathcal{B}$ is an antichain and $b' \neq b$, we have clearly $b' \in B_2 \setminus B_1$. (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) follow immediately from the fact that an antichain $\mathcal{B}$ which satisfies (ii) and a closure operator $<\ldots>$ which satisfies (iii) define a matroid, respectively.

The operations

$$P : \mathcal{B} \mapsto <\ldots>_{\mathcal{B}} \mapsto \mathcal{B}_{(<\ldots>_{\mathcal{B}})},$$
$$Q : <\ldots> \mapsto \mathcal{B}_{<\ldots>} \mapsto <\ldots>_{(\mathcal{B}_{<\ldots>})}$$

define maps from the set of antichains or the set of closure operators, respectively, into themselves. By our proposition we have $P(\mathcal{B}) = \mathcal{B}$ and $Q(<\ldots>) = <\ldots>$ if and only if $\mathcal{B}$ is the set of bases of some matroid and $<\ldots>$ is the closure operator of some matroid. By induction we define

$$P^0(\mathcal{B}) := \mathcal{B}, \; P^{m+1}(\mathcal{B}) := P(P^m(\mathcal{B})) \tag{3}$$

for any antichain $\mathcal{B} \subseteq \mathcal{P}(E)$ and

$$Q^0(<\ldots>) := <\ldots>, \; Q^{m+1}(<\ldots>) = Q(Q^m(<\ldots>)) \tag{4}$$

for any closure operator $<\ldots>: \mathcal{P}(E) \to \mathcal{P}(E)$ and $m \geq 1$.

It is natural to ask: Does the iterated application of P or Q always lead to a fixed point and thus to a matroid? Otherwise, there were purely periodic antichains and closure operators with respect to P and Q, respectively. That this is impossible is the main assertion of the following.

THEOREM. *Assume $\mathcal{B} \subseteq \mathcal{P}(E)$ is some antichain and $<\ldots>: \mathcal{P}(E) \to \mathcal{P}(E)$ denotes some closure operator. Then we have:*

(i) *For every $B' \in P(\mathcal{B})$ with $B' \neq \emptyset$ there exists some $B \in \mathcal{B}$ with $B \subseteq B'$.*

(ii) *There exists some $m = m(\#E)$ such that*

$$P^{m+1}(\mathcal{B}) = P^m(\mathcal{B}) \text{ and } Q^{m+1}(<\ldots>) = Q^m(<\ldots>)$$

for all antichains $\mathcal{B} \subseteq \mathcal{P}(E)$ and for all closure operators $<\ldots>: \mathcal{P}(E) \to \mathcal{P}(E)$. In particular, $P^m(\mathcal{B})$ and $Q^m(<\ldots>)$ define matroids $M(\mathcal{B})$ and $M(<\ldots>)$, respectively, on E.

(iii) *If r denotes the rank of the matroid $M(\mathcal{B})$, then we have*

$$r \geq \min \{\#B \mid B \in \mathcal{B}\}. \tag{5}$$

PROOF:

(i) Assume $B' \in P(\mathcal{B})$ and $B' \neq \emptyset$. Then $\mathcal{B} \neq \emptyset$. Choose some $B \in \mathcal{B}$ such that $\#(B \setminus B')$ is as small as possible. If suffices to show that $B \subseteq B'$. Assume $e \in B$. $B' \in P(\mathcal{B})$ means that $< B' >_{\mathcal{B}} = E$; thus there exists some $f \in B'$ with $B_0 := (B \setminus \{e\}) \cup \{f\} \in \mathcal{B}$. Clearly, we have $B_0 \setminus B' \subseteq B \setminus B'$ and thus $B_0 \setminus B' = B \setminus B'$ by our choice of B. This implies $e \in B'$ in case $e \neq f$, while in case $e = f$ there is nothing to show.

(ii) follows directly from i) and the above Proposition, since E is finite.

(iii) is an immediate consequence of i), because the rank of a matroid equals the cardinality of every minimal spanning subset.

Example 1. Assume $< \ldots >: \mathcal{P}(E) \to \mathcal{P}(E)$ denotes a closure operator satisfying the anti-exchange property; that is, for $F \subseteq E$ and $e_1, e_2 \in E \setminus < F >$ with $e_1 \neq e_2$ and $e_1 \in < F \cup \{e_2\} >$ one has $e_2 \notin < F \cup \{e_1\} >$. By [**2**, Theorem 2.3] there exists a unique $B \subseteq E$ with $\mathcal{B}_{<\ldots>} = \{B\}$. Therefore $M(< \ldots >)$ is the matroid with $E \setminus B$ as its set of loops and B as its set of coloops.

Example 2. Assume $2 \leq n \leq \#E$ and $\mathcal{B} \subseteq \mathcal{P}_n(E) := \{A \subseteq E \mid \#A = n\}$. Put

$$\mathcal{H} := \mathcal{H}(\mathcal{B}) := \{H \subseteq E \mid \mathcal{B} \cap \mathcal{P}_n(H) = \emptyset \neq \mathcal{B} \cap \mathcal{P}_n(H \cup \{e\}) \text{ for all } e \in E \setminus H\}. \tag{6}$$

Furthermore, put $\widetilde{E} := E \,\dot{\cup}\, \mathcal{H}$, and define $\widetilde{\mathcal{B}} \subseteq \mathcal{P}_n(\widetilde{E})$ by

$$\widetilde{\mathcal{B}} := \mathcal{P}_n(E) \,\dot{\cup}\, \{\{e_1, \ldots, e_{n-1}, H\} \mid H \in \mathcal{H}, \{e_1, \ldots, e_{n-1}\} \in \mathcal{P}_{n-1}(E) \setminus \mathcal{P}_{n-1}(H)\}. \tag{7a}$$

We claim

$$\mathcal{B}_{(<\ldots>_{\widetilde{\mathcal{B}}})} = \mathcal{B}. \tag{7b}$$

To prove (7b) it suffices to show

$$< B >_{\widetilde{\mathcal{B}}} = \widetilde{E} \text{ for every } B \in \mathcal{B}, \tag{7c}$$

$$< H \,\dot{\cup}\, \mathcal{H} >_{\widetilde{\mathcal{B}}} \neq \widetilde{E} \text{ for every } H \in \mathcal{H}. \tag{7d}$$

Assume $B \in \mathcal{B}$. (7c) follows from the following observations:

- for every $A \in \mathcal{P}_{n-1}(E)$ there exists $b \in B$ with $A \cup \{b\} \in \mathcal{P}_n(E) \subseteq \widetilde{\mathcal{B}}$,
- for all $H \in \mathcal{H}, \{e_1, \ldots, e_{n-1}\} \in \mathcal{P}_{n-1}(E) \setminus \mathcal{P}_{n-1}(H)$, and $1 \leq i \leq n-1$ there exists some $b \in B$ with $\{b\} \cup (\{e_1, \ldots, e_{n-1}, H\} \setminus \{e_i\}) \in \widetilde{\mathcal{B}}$, just choose $b \in B \setminus H$ if $\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{n-1}\} \subseteq H$ and $b \in B \setminus \{e_1, \ldots, e_{n-1}\}$ otherwise.

To verify (7d) assume $H \in \mathcal{H}$ and $e \in E \setminus H$. Then by (6) there exists some $B = \{e_1, \ldots, e_n\} \in \mathcal{B}$ with $e \in B \subseteq H \cup \{e\}$, say $e = e_1$. We have $\{e_1, \ldots, e_{n-1}, H\} \in \widetilde{\mathcal{B}}$, but there does not exist any $s \in H \dot{\cup} \mathcal{H}$ with $\{s, e_2, \ldots, e_{n-1}, H\} \in \widetilde{\mathcal{B}}$. This means $e \notin < H \dot{\cup} \mathcal{H} >_{\widetilde{\mathcal{B}}}$ and therefore $< H \,\dot{\cup}\, \mathcal{H} >_{\widetilde{\mathcal{B}}} \neq \widetilde{E}$.

By a repeated application of (7b) it follows that for every $n \geq 2$, every finite set E with $n \leq \#E$, every $\mathcal{B} \subseteq \mathcal{P}_n(E)$ and every $m \in \mathbb{N}$ there exists some finite set $\widetilde{E} \supseteq E$ and some $\widetilde{\mathcal{B}} \subseteq \mathcal{P}_n(\widetilde{E})$ with $P^m(\widetilde{\mathcal{B}}) = \mathcal{B}$ and $P^k(\mathcal{B}) \subsetneq P^{k-1}(\mathcal{B})$ for $1 \leq k \leq m$. In particular, this holds if $\mathcal{B}$ is the set of bases of some matroid M defined on E in which case $\mathcal{H}(\mathcal{B})$ is the set of hyperplanes of M.

Assume — still more specifically — $E = E_0 = \{e_1, e_2\}, \mathcal{B} = \mathcal{B}_0 = \{E_0\}$ and therefore $n = \#E = 2$. Then we have $\mathcal{H}_0 := \mathcal{H}(\mathcal{B}_0) = \{\{e_1\}, \{e_2\}\}$. For $\nu \geq 0$ put

$$\begin{aligned} E_{\nu+1} :=& E_\nu \dot\cup \mathcal{H}_\nu, \\ \mathcal{B}_{\nu+1} :=& \mathcal{P}_2(E_\nu) \dot\cup \{\{e, H\} \mid H \in \mathcal{H}_\nu, e \in E_\nu \setminus H\}, \\ \mathcal{H}_{\nu+1} :=& \mathcal{H}(\mathcal{B}_{\nu+1}) = \{X \subseteq E_{\nu+1} \mid X = \mathcal{H}_\nu \text{ or} \\ & X = \{e\} \cup \{H \in \mathcal{H}_\nu \mid e \in H\} \text{ for some } e \in E_\nu\}, \\ a_\nu :=& \#E_\nu, b_\nu := \#\mathcal{H}_\nu. \end{aligned}$$

For $\nu \geq 0$ we have $\dot{P}(\mathcal{B}_{\nu+1}) = \mathcal{B}_\nu$ by (7b). Furthermore, $a_0 = b_0 = 2, a_{\nu+1} = a_\nu + b_\nu$ and $b_{\nu+1} = 1 + a_\nu$ for $\nu \geq 0$. This implies $b_1 = 3$ and $b_{\nu+1} = b_\nu + b_{\nu-1}$ for $\nu \geq 1$. Hence, $b_\nu = 1 + a_{\nu-1}$ is nothing but the $(\nu+2)nd$ Fibonacci number!

It follows from the above theorem that to every antichain $\mathcal{B} \subseteq \mathcal{P}(E)$ we can associate the matroid $M(\mathcal{B})$ whose set of bases is $P^{m(\#E)}(\mathcal{B})$ and that indeed $M(\mathcal{B}) = M_0$ if $\mathcal{B}$ is the set of bases of the matroid M_0, defined on E. It might be of interest to study this new invariant for all the various set systems which are considered in combinatorics and in particular in combinatorial optimization. It may also be of interest to find good upper bounds for $m(\#E)$ and to study for each matroid M_0, defined on E, the variety of antichains $\mathcal{B} \subseteq \mathcal{P}(E)$ with $M(\mathcal{B}) = M_0$ as well as $m(M_0) := \min\{m \in \mathbb{N} \mid P^{m+1}(\mathcal{B}) = P^m(\mathcal{B})$ for all antichains $\mathcal{B} \subseteq \mathcal{P}(E)$ with $M(\mathcal{B}) = M_0\}$.

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