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Solutions of Equivariance for Iterative Differential Equations

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Abstract—By equivariance under the action of a group of invertible linear transformations on a Euclidean space, we describe symmetries of mappings. Based on known results on existence of solutions for iterative differential equations, in this paper, we discuss the special class of solutions which possess equivariance on **R**. Existence, uniqueness, and smooth dependence are given by using fixed-point theorems and by virtue of properties of finitely generated Lie groups. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Iterative differential equation, Equivariance, Finitely generated, Lie group, Fixed-point theorem.

1. INTRODUCTION

Among functional differential equations [1] there is an important class involving iteration. In particular, Cooke [2] points out that it is highly desirable to establish the existence and stability of periodic solutions for the equation

$$x'(t) + ax(t - h(t, x(t))) = F(t), \qquad \forall\, t \in \mathbf{R}.$$

In [3], Stephan obtains existence of periodic solutions for the equation

$$x'(t) + ax(t - r + \mu h(t, x(t))) = F(t), \qquad \forall\, t \in \mathbf{R}.$$

Eder [4] discusses a special iterative differential equation

$$x'(t) = x(x(t)),$$

for all $t \in \mathbf{R}$ and proves that every solution either vanishes identically or is strictly monotonic. The existence of solutions for the equation

$$x'(t) = f(x(x(t)))$$

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is given in [5] when $f \in C^1(\mathbf{R})$ and generalized without smoothness of f in [6]. Recently, a more general form

$$x'(t) = \sum_{j=1}^{m} a_j(t) x^j(t) + F(t)$$

is considered and existence, uniqueness, and C^1 dependence of its smooth solutions are obtained in [7,8]. Besides, a result of analytic solutions for the equation $x'(t) = x^m(t)$ is given in [9]. The general form of iterative differential equations can be written as

$$x'(t) = G\left(x^{n_1}(t), x^{n_2}(t), \dots, x^{n_k}(t)\right), \tag{1}$$

where $x^0(t) = t$, $x^1(t) = x(t)$, $x^k(t) = x(x^{k-1}(t))$, $k = 2, 3, \dots$. As mentioned in [2,3,10], it is related to infection models and also to motions of charged particles with retarded interaction. Further investigations are of practical interest (see, e.g., [2,4–7,9,11]).

Equivariance is used to describe symmetries of mappings (seen in [12,13]). Let Ω be a Banach space and Γ be a Lie group of linear transformations on Ω. A mapping $x : \Omega \to \Omega$ is said to be Γ-*equivariant* if

$$x(\gamma t) = \gamma x(t), \qquad \forall\, t \in \Omega, \quad \forall\, \gamma \in \Gamma.$$

Sometimes the restriction of such a mapping x on a subset of Ω is considered to be of Γ-*equivariance*. It is an interesting problem to find solutions of equivariance for (1).

In this paper, we study solutions of equivariance for equation (1), where $t \in I := [-\eta, \eta]$, $\eta > 0$, G maps I^k, the product of k intervals of the same I, to $\mathbf{R}$ continuously, x is the unknown mapping, and all n_js ($j = 1, 2, \dots, k$) are positive integers. We consider the action of groups of invertible linear transformations on $\mathbf{R}$ and prove existence and uniqueness of its solutions of equivariance by using fixed-point theorems, where difficulties from compactness are overcome by virtue of properties of finitely generated Lie groups. C^1 dependence on given functions is discussed further. Two concrete equations with nonlinear iterates, one of which is considered in [5] and the other is not, are given under the action of the group $\mathbf{Z}_2$. However, we also remark that it is not easy to obtain a solution of equivariance even if an equation only with linear iterates as discussed in [4] is considered.

2. MAIN RESULTS

Obviously, invertible linear transformations of $\mathbf{R}$ take the form $t \to \gamma t$, where $0 \neq \gamma \in \mathbf{R}$. Without loss of generality, we assume that any Lie group acting linearly on $\mathbf{R}$ can be identified with a subgroup of $GL(\mathbf{R})$, the multiplicative topological group of nonzero reals, which we can identify with $\mathbf{R}_0 = \mathbf{R} \setminus \{0\}$. Let $V \subset \mathbf{R}_0$ be a set of generators for a subgroup $\Gamma < GL(\mathbf{R})$ and write $\Gamma = \langle V \rangle$. Γ is referred as to be *finitely generated* if V has finite elements. Moreover, we say Γ is *topologically finitely generated* if it is the closure in $GL(\mathbf{R})$ of a finitely generated group. V is *minimal* of Γ if no proper subset generates Γ. Clearly, $\mathbf{Z}_2$ is the finitely generated subgroups and all odd functions, being symmetric to the origin, are $\mathbf{Z}_2$-equivariant.

Consider the action of a topologically finitely generated Lie group Γ on $\mathbf{R}$. Suppose that

(A_1) $G(0, 0, \dots, 0) = 0$ and

$$|G(t_1, t_2, \dots, t_k) - G(s_1, s_2, \dots, s_k)| \leq \sum_{i=1}^{k} C_i |t_i - s_i|, \qquad \forall\, t_i, s_i \in I, \quad i = 1, 2, \dots, k,$$

for some constants $C_i \geq 0$, $i = 1, 2, \dots, k$, which do not all vanish but satisfy

$$\eta \sum_{i=1}^{k} C_i \leq 1; \tag{2}$$

(A_2) $G(\gamma t_1, \gamma t_2, \dots, \gamma t_k) = G(t_1, t_2, \dots, t_k)$ for all $\gamma \in \Gamma$ and for all $t_i \in I$ with $\gamma t_i \in I$, $i = 1, 2, \dots, k$.

The following result gives existence for solutions of equivariance.

THEOREM 1. *Suppose that Γ is a topologically finitely generated Lie group acting on $\mathbf{R}$ and that Hypotheses (A_1) and (A_2) hold. Then equation (1) has a solution x of Γ-equivariance on I which satisfies $x(0) = x'(0) = 0$, $|x(t) - x(s)| \leq |t - s|$, and $|x'(t) - x'(s)| \leq \sum_{i=1}^{k} C_i |t - s|$, for all $t, s \in I$.*

In Theorem 1, Condition (A_2) requires an invariant property of the multivariable function G with respect to the action of the group Γ. Many functions have such a property. A function $G(t_1, t_2, \ldots, t_k)$ is invariant with respect to $\mathbf{Z}_2$ when G is an even function in each variable. For example, $G(t_1, t_2, t_3) = t_1^2 + t_2^4 + t_3^6$ and $G(t_1, t_2) = \cos t_1 + \sin^2 t_2$.

The next one concerns uniqueness and C^1 dependence.

THEOREM 2. *If all conditions in Theorem 1 hold and*

$$(\eta + 1) \sum_{i=1}^{k} n_i C_i < 1 \tag{3}$$

additionally, then the solution obtained in Theorem 1 for equation (1) is unique. Moreover, this solution is C^1 smoothly dependent on the given G.

3. COMPACTNESS OF SPACES FOR SOLUTIONS

Let $C^1(I)$ denote the set of all continuously differentiable functions on the interval I. Define $\|x\| = \max_{x \in I}\{|x(t)|\}$ and $\|x\|_1 = \|x\| + \|x'\|$, where $x \in C^1(I)$. Then $C^1(I)$ is a Banach space with the norm $\|\cdot\|_1$. Take notations

$$\begin{aligned} \mathcal{X}^1(I; K_0, K_1) &:= \{x \in C^1(I) : x(0) = x'(0) = 0,\ |x(t) - x(s)| \leq K_0 |t - s|, \\ &\qquad |x'(t) - x'(s)| \leq K_1 |t - s|,\ \forall\, t, s \in I\}, \\ \mathcal{X}^1_\Gamma(I) &:= \{x \in C^1(I) : x(\gamma t) = \gamma x(t),\ \forall\, \gamma \in \Gamma,\ \forall\, t \text{ with } \gamma t \in I\}, \end{aligned}$$

and $\mathcal{X}^1_\Gamma(I; K_0, K_1) := \mathcal{X}^1(I; K_0, K_1) \cap \mathcal{X}^1_\Gamma(I)$.

LEMMA 1. *Suppose that Γ is topologically finitely generated. Then $\mathcal{X}^1_\Gamma(I)$ is a convex closed subset of $C^1(I)$ in the topology of norm $\|\cdot\|_1$.*

PROOF. For each $\gamma \in \Gamma$, define

$$\mathcal{X}^1_\gamma(I) := \{x \in C^1(I) : x(\gamma t) = \gamma x(t),\ \forall\, t \in I \cap \gamma^{-1} I\}. \tag{4}$$

We claim that

(i) $\mathcal{X}^1_\gamma(I)$ is convex and closed;
(ii) $\mathcal{X}^1_{\gamma^{-1}}(I) = \mathcal{X}^1_\gamma(I)$;
(iii) $\mathcal{X}^1_\gamma(I) \cap \mathcal{X}^1_\sigma(I) \subset \mathcal{X}^1_{\gamma\sigma}(I)$;
(iv) $\mathcal{X}^1_\Gamma(I) = \bigcap_{i=1}^{k} \mathcal{X}^1_{\gamma_i}(I)$, where $V = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ is a finite set of (topological) generator of Γ.

It suffices to prove these results for Γ to be finitely generated, since we can then pass to the closure of any such group by continuity of x. For this purpose, we suppose that $\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle$.

For (i), we can prove that $\mathcal{X}^1_\gamma(I)$ is a closed linear subspace indeed. Let $x, y \in \mathcal{X}^1_\gamma(I)$ and $a, b \in \mathbf{R}$ and consider $z = ax + by$. Obviously, $z \in C^1(I)$. Further, for all $t \in I \cap \gamma^{-1} I$, we get $x(\gamma t) = \gamma x(t)$ and $y(\gamma t) = \gamma y(t)$. So $z(\gamma t) = \gamma z(t)$, i.e., $z \in \mathcal{X}^1_\gamma(I)$. Of course, $\mathcal{X}^1_\gamma(I)$ is convex. Closure is also clear. Thus, (i) is proved. Claim (ii) is shown through the change of variables $s = \gamma t$ since $\gamma \neq 0$.

For (iii), it suffices to give its proof when $|\gamma\sigma| \leq 1$. Otherwise, we can work with $\mathcal{X}^1_{\sigma^{-1}\gamma^{-1}}(I)$ instead because of (ii). If $x \in \mathcal{X}^1_\gamma(I) \cap \mathcal{X}^1_\sigma(I)$, then

$$x(\gamma t) = \gamma x(t), \qquad \forall\, t \in I \cap \gamma^{-1} I \tag{5}$$

and

$$x(\sigma t) = \sigma x(t), \qquad \forall t \in I \cap \sigma^{-1} I. \tag{6}$$

For $s \in I \cap (\gamma\sigma)^{-1}(I)$, we get

$$s \in I \qquad \text{and} \qquad (\gamma\sigma)s \in I. \tag{7}$$

There is no loss of generality in assuming that $|\sigma| \leq 1$ because $|\gamma\sigma| \leq 1$ and Γ is Abelian. Put $t = \sigma s$. We get from (7) that $t, \sigma t, s, \sigma s \in I$. Then (5) and (6) imply that $x(\gamma\sigma s) = \gamma x(t) = \gamma x(\sigma s) = \gamma\sigma x(s)$, which means that $x \in \mathcal{X}^1_{\gamma\sigma}(I)$.

To prove (iv), note that $\mathcal{X}^1_\Gamma(I) = \bigcap_{\gamma\in\Gamma} \mathcal{X}^1_\gamma(I)$ follows directly from the definitions of $\mathcal{X}^1_\Gamma(I)$ and $\mathcal{X}^1_\gamma(I)$. By (ii) and (iii), it is equal to $\bigcap_{\gamma\in V} \mathcal{X}^1_\gamma(I)$, proving (iv). Since the set of generators V is finite, (i) and (iv) imply that $\mathcal{X}^1_\Gamma(I)$ is closed and convex. This completes the proof. ∎

LEMMA 2. *Subset $\mathcal{X} \subset C^1(I)$ is sequentially compact if and only if $\mathcal{X}$ is uniformly bounded and the set $\mathcal{X}'$ consisting of derivatives of functions in $\mathcal{X}$ is equicontinuous.*

This is a version of Ascoli-Arzela's lemma for C^1 functions. Its proof can be found in [14].

LEMMA 3. *If $\Gamma \subset GL(\mathbf{R})$ is topologically finitely generated, then $\mathcal{X}^1_\Gamma(I; K_0, K_1)$ is a compact convex subset of $C^1(I)$.*

PROOF. Consider $\mathcal{X}^1_\Gamma(I; K_0, K_1)$ in the topology induced by $\|\cdot\|_1$. It is easy to check that $\mathcal{X}^1(I; K_0, K_1)$ is a convex closed subset of $C^1(I)$. For all $x \in \mathcal{X}^1(I; K_0, K_1)$, we have

$$\|x\|_1 = \|x\| + \|x'\| \leq \eta + K_0,$$

i.e., $\mathcal{X}^1(I; K_0, K_1)$ is uniformly bounded. Moreover, the set of derivatives of all elements for $\mathcal{X}^1(I; K_0, K_1)$ is equicontinuous since $|x'(t) - x'(s)| \leq K_1|t - s|$. By Lemma 2, $\mathcal{X}^1(I; K_0, K_1)$ is a compact convex subset of $C^1(I)$. However, Lemma 1 tells that $\mathcal{X}^1_\Gamma(I)$ is a convex closed subset, so $\mathcal{X}^1_\Gamma(I; K_0, K_1)$, being an intersection of two convex closed subsets, is also a convex closed subset of $\mathcal{X}^1(I; K_0, K_1)$. Hence, $\mathcal{X}^1_\Gamma(I; K_0, K_1)$ is compact. ∎

4. PROOFS OF THEOREMS

Proofs of Theorem 1

Define a mapping $T : \mathcal{X}^1_\Gamma(I; 1, \sum_{i=1}^k C_i) \to C^1(I)$ by

$$(Tx)(t) = \int_0^t G\left(x^{n_1}(\xi), x^{n_2}(\xi), \dots, x^{n_k}(\xi)\right) d\xi, \qquad x \in \mathcal{X}^1_\Gamma\left(I; 1, \sum_{i=1}^k C_i\right). \tag{8}$$

We claim that T maps $\mathcal{X}^1_\Gamma(I; 1, \sum_{i=1}^k C_i)$ into itself. Actually,

$$\left|x^j(t) - x^j(s)\right| \leq |t - s|, \qquad t, s \in I, \quad j \in \mathbf{N}, \tag{9}$$

since $|x(t) - x(s)| \leq |t - s|$ for $x \in \mathcal{X}^1_\Gamma(I; 1, \sum_{i=1}^k C_i)$. Moreover,

$$(Tx)(0) = 0,$$
$$(Tx)'(0) = G\left(x^{n_1}(0), x^{n_2}(0), \dots, x^{n_k}(0)\right) = 0.$$

By Hypothesis (A_1) and (9), we see

$$\begin{aligned}
|(Tx)(t) - (Tx)(s)| &= \left|\int_s^t G\left(x^{n_1}(\xi), x^{n_2}(\xi), \dots, x^{n_k}(\xi)\right) d\xi\right| \\
&\leq \left|\int_s^t \left|G\left(x^{n_1}(\xi), \dots, x^{n_k}(\xi)\right) - G\left(x^{n_1}(0), \dots, x^{n_k}(0)\right)\right| d\xi\right| \\
&\leq \sum_{i=1}^k C_i \left|\int_s^t \left|x^{n_i}(\xi) - x^{n_i}(0)\right| d\xi\right| \leq \sum_{i=1}^k C_i \left|\int_s^t |\xi|\, d\xi\right| \\
&\leq \eta \sum_{i=1}^k C_i |t - s| \leq |t - s|,
\end{aligned} \tag{10}$$

for all $t, s \in I$. Similarly,

$$
\begin{aligned}
|(Tx)'(t) - (Tx)'(s)| &= |G(x^{n_1}(t), \dots, x^{n_k}(t)) - G(x^{n_1}(s), \dots, x^{n_k}(s))| \\
&\le \sum_{i=1}^{k} C_i |x^{n_i}(t) - x^{n_i}(s)| \\
&\le \sum_{i=1}^{k} C_i |t - s|, \qquad \forall\, t, s \in I.
\end{aligned}
\tag{11}
$$

Furthermore, by (A_2), we have

$$
\begin{aligned}
(Tx)(\gamma t) &= \int_0^{\gamma t} G(x^{n_1}(\xi), x^{n_2}(\xi), \dots, x^{n_k}(\xi))\, d\xi \\
&= \gamma \int_0^{t} G(x^{n_1}(\gamma\tau), x^{n_2}(\gamma\tau), \dots, x^{n_k}(\gamma\tau))\, d\tau \\
&= \gamma \int_0^{t} G(x^{n_1}(\tau), x^{n_2}(\tau), \dots, x^{n_k}(\tau))\, d\tau \\
&= \gamma (Tx)(t),
\end{aligned}
\tag{12}
$$

for all $\gamma \in \Gamma$ and $t \in I$ with $\gamma t \in I$. Thus, relations (10)–(12) imply that $(Tx)(t) \in \mathcal{X}_\Gamma^1(I; 1, \sum_{i=1}^k C_i)$ and so our claim is proved.

For $x, y \in \mathcal{X}_\Gamma^1(I; 1, \sum_{i=1}^k C_i)$, we can prove by induction that

$$
\|x^n - y^n\| \le n \|x - y\|. \tag{13}
$$

Thus,

$$
\begin{aligned}
&|G(x^{n_1}(t), x^{n_2}(t), \dots, x^{n_k}(t)) - G(y^{n_1}(t), y^{n_2}(t), \dots, y^{n_k}(t))| \\
&\quad \le \sum_{i=1}^{k} C_i |x^{n_i}(t) - y^{n_i}(t)| \le \left(\sum_{i=1}^{k} n_i C_i \right) \|x - y\| \\
&\quad \le \left(\sum_{i=1}^{k} n_i C_i \right) \|x - y\|_1, \qquad \forall\, t \in I.
\end{aligned}
\tag{14}
$$

It follows that

$$
\begin{aligned}
\|Tx - Ty\|_1 &= \|Tx - Ty\| + \|(Tx)' - (Ty)'\| \\
&= \max_{x \in I} \left| \int_0^x \{G(x^{n_1}(\xi), \dots, x^{n_k}(\xi)) - G(y^{n_1}(\xi), \dots, y^{n_k}(\xi))\}\, d\xi \right| \\
&\quad + \max_{x \in I} |G(x^{n_1}(t), \dots, x^{n_k}(t)) - G(y^{n_1}(t), \dots, y^{n_k}(t))| \\
&\le (\eta + 1) \left(\sum_{i=1}^{k} n_i C_i \right) \|x - y\|_1,
\end{aligned}
\tag{15}
$$

implying that T is a continuous mapping. Since $\mathcal{X}_\Gamma^1(I; 1, \sum_{i=1}^k C_i)$ is a compact convex subset of $C^1(I)$ by Lemma 3, by Schauder's fixed-point theorem, there is a mapping $z(t)$ in $\mathcal{X}_\Gamma^1(I; 1, \sum_{i=1}^k C_i)$ such that

$$
z(t) = \int_0^t G(z^{n_1}(\xi), z^{n_2}(\xi), \dots, z^{n_k}(\xi))\, d\xi.
$$

By differentiating both sides of the above equality, we can check that z is the desired solution of (1). This completes the proof. ∎

Remark that the class $\mathcal{X}_\Gamma^1(I; 1, \sum_{i=1}^k C_i)$ confines Lipschitzian constants of those functions are not greater than 1, so as to guarantee iteration of $x(t)$ is meaningful.

Proof of Theorem 2

By (15) and (3), the mapping T, defined in (8), is a contraction on the closed subset $\mathcal{X}_\Gamma^1(I;1,\sum_{i=1}^k C_i)$ of $C^1(I)$. Thus, T has a unique fixed point and equation (1) has a unique solution in $\mathcal{X}_\Gamma^1(I;1,\sum_{i=1}^k C_i)$.

For arbitrarily given functions G_1 and G_2 both satisfying (A_1) and (A_2), by the uniqueness just proved, there exist uniquely functions x_1 and x_2 of Γ-equivariance such that

$$x_i'(t) = G_i\left(x_i^{n_1}(t), x_i^{n_2}(t), \dots, x_i^{n_k}(t)\right), \qquad i = 1,2. \tag{16}$$

Observe that

$$\begin{aligned}
&\left|G_1\left(x_1^{n_1}(t),\dots,x_1^{n_k}(t)\right) - G_2\left(x_2^{n_1}(t),\dots,x_2^{n_k}(t)\right)\right| \\
&\le \left|G_1\left(x_1^{n_1}(t),\dots,x_1^{n_k}(t)\right) - G_2\left(x_1^{n_1}(t),\dots,x_1^{n_k}(t)\right)\right| \\
&+ \left|G_2\left(x_1^{n_1}(t),\dots,x_1^{n_k}(t)\right) - G_2\left(x_2^{n_1}(t),\dots,x_2^{n_k}(t)\right)\right| \\
&\le \|G_1 - G_2\| + \left(\sum_{i=1}^k n_i C_i\right) \|x_1 - x_2\|.
\end{aligned}$$

Then,

$$\begin{aligned}
\|x_1 - x_2\|_1 &= \left\| \int_0^t \left\{G_1\left(x_1^{n_1}(\xi),\dots,x_1^{n_k}(\xi)\right) - G_2\left(x_2^{n_1}(\xi),\dots,x_2^{n_k}(\xi)\right)\right\} d\xi \right\|_1 \\
&= \left\| \int_0^t \left\{G_1\left(x_1^{n_1}(\xi),\dots,x_1^{n_k}(\xi)\right) - G_2\left(x_2^{n_1}(\xi),\dots,x_2^{n_k}(\xi)\right)\right\} d\xi \right\| \\
&\quad + \left\|G_1\left(x_1^{n_1}(t),\dots,x_1^{n_k}(t)\right) - G_2\left(x_2^{n_1}(t),\dots,x_2^{n_k}(t)\right)\right\| \\
&\le (\eta+1)\left\{\|G_1 - G_2\| + \left(\sum_{i=1}^k n_i C_i\right)\|x_1 - x_2\|_1\right\}
\end{aligned}$$

since $|t| \le \eta$ and $\|x_1 - x_2\| \le \|x_1 - x_2\|_1$. Consequently, by (3),

$$\|x_1 - x_2\|_1 \le \frac{\eta + 1}{1 - (\eta+1)\sum\limits_{i=1}^k n_i C_i} \|G_1 - G_2\|.$$

This proves the C^1 dependence of solutions on the given function G, which ends the proof. ∎

5. EXAMPLES AND REMARKS

Consider the iterative differential equation

$$x'(t) = \frac{1}{10}\left(x^2(t)\right)^2, \qquad t \in I = [-1,1], \tag{17}$$

a special case of Feckăn's consideration [5]. Clearly $G(t) = (1/10)t^2$ satisfies (A_2), that is, $G(\gamma t) = G(t)$, for all $\gamma \in \mathbf{Z}_2$ and for all $t \in I$ with $\gamma t \in I$. Moreover,

$$|G(t) - G(s)| = \frac{1}{10}\left|t^2 - s^2\right| \le \frac{1}{5}|t - s|,$$

implying that (A_1) is fulfilled with $C = 1/5$ and $\eta = 1$. By Theorem 1, equation (17) has a $\mathbf{Z}_2$-equivariant solution x, which satisfies $x(0) = x'(0) = 0$, $|x(t) - x(s)| \le |t - s|$, and $|x'(t) - x'(s)| \le (1/5)|t - s|$, for all $t, s \in I$. Notice that $(\eta + 1)nC = 4/5 < 1$. By Theorem 2, such a solution x is unique in $\mathcal{X}_\Gamma^1(I;1,1/5)$ and C^1 smoothly dependent on G.

Another iterative differential equation

$$x'(t) = \frac{1}{73}\left(x^2(t)\right)^2 + \frac{1}{37}\left(x^4(t)\right)^4, \qquad t \in I = [-1,1], \tag{18}$$

is not of Feckăn's form [5]. Observe that the function $G(t_1,t_2) = (1/73)t_1^2 + (1/37)t_2^4$ satisfies (A_2), i.e., $G(\gamma t_1, \gamma t_2) = G(t_1,t_2)$, for all $\gamma \in \mathbf{Z}_2$ and for all $t_i \in I$ with $\gamma t_i \in I$, $i = 1,2$. Moreover,

$$\begin{aligned} |G(t_1,t_2) - G(s_1,s_2)| &= \left|\frac{1}{73}\left(t_1^2 - s_1^2\right) + \frac{1}{37}\left(t_2^4 - s_2^4\right)\right| \\ &\le \frac{2}{73}|t_1 - s_1| + \frac{4}{37}|t_2 - s_2|, \end{aligned}$$

that is, (A_1) is fulfilled with $C_1 = 2/73$, $C_2 = 4/37$, and $\eta = 1$ since

$$\eta \sum_{i=1}^{k} C_i = \frac{366}{2701} < 1.$$

By Theorem 1, equation (18) has a $\mathbf{Z}_2$-equivariant solution x, which satisfies $x(0) = x'(0) = 0$, $|x(t) - x(s)| \le |t - s|$, and $|x'(t) - x'(s)| \le (366/2701)|t - s|$ for all $t, s \in I$. Because $(\eta + 1)\sum_{i=1}^{k} n_i C_i = 2632/2701 < 1$, by Theorem 2, this solution x is unique in $\mathcal{X}_\Gamma^1(I; 1, 366/2701)$ and C^1 smoothly dependent on G.

As a remark, Eder's equation $x' = x(x(t))$ as discussed in [4] does not satisfy Assumption (A_2), although its right-hand side is in the form of linear iteration. Consistent with our theorems, $G(t) = t$ for this equation. Obviously, $G(\gamma t) = \gamma t = t$, for all $t \in I$ with $\gamma t \in I$ if and only if $\gamma = 1$. That is, except the trivial one consisting of only the unity, no subgroups can fit for the equation with (A_2). More generally, solutions of equivariance for the equation $x' = x^n(t)$ are still a problem.

With a slight modification, consider the equation $x' = (x^n(t))^m$, where n is the index of iteration and m is the degree of power. Then the corresponding function $G(t) = t^m$ satisfies (A_2) with a subgroup $\langle\gamma\rangle$ if $\gamma^m = 1$. This reduces to a discussion on roots of unity in the complex field $\mathbf{C}$ and suggests a further investigation of equation (1) in $\mathbf{C}$ in the future.

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