Distance-regular graphs, MH-colourings and MLD-colourings

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Abstract

It is shown that the only bipartite distance regular graphs of diameter 3 which are MLD-colourable are the incidence graphs of complementary Hadamard designs.

1. Introduction

Let $G = (V, E)$ denote a graph of diameter $d$ having vertex set $V$ and edge set $E$. In what follows, we use $n = |V|$ and $m = |E|$. By a line-distinguishing colouring of $G$ we mean a vertex colouring in which each pair of colours occurs at most once on an edge and when each pair of colours occurs exactly once on an edge the colouring is said to be minimal; it is then referred to as a minimal line-distinguishing colouring or MLD-colouring. A line-distinguishing colouring of $G$ which is also a proper vertex colouring is called a harmonious colouring of $G$; when each pair of distinct colours occurs exactly once on an edge the colouring is said to be a minimal harmonious colouring or MH-colouring \cite{3}. It is easy to see that if a regular graph is either MLD-colourable using $l$ colours or MH-colourable using $h$ colours then each colour occurs the same number $t$ of times. Thus, we have $n = tl$ or $n = th$, respectively. In previous work we have considered strongly regular MLD-colourable graphs \cite{3}, $n/2$-MLD-colourable graphs \cite{1} and $n/2$-MH-colourable graphs which are the incidence graphs of 2-designs \cite{2}. In this note we consider some extensions of each of these ideas.

Let $v_0 \in V$. We use $V_i$ to denote the set of vertices distance $i$ from $v_0$. Suppose that for each $i$, where $1 \leq i \leq d - 1$, for any vertex $v_i \in V_i$, there are $a_i$ vertices adjacent to $v_i$ in $V_i$, $b_i$ vertices adjacent to $v_i$ in $V_{i+1}$ and $c_i$ vertices adjacent to $v_i$ in $V_{i-1}$. Further, for

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a vertex $v_d$ such that $\text{dist}(v_d, v_1) = d$ there are $a_d$ vertices adjacent to $v_d$ in $V_{d-1}$ and $c_d$ vertices adjacent to $v_d$ in $V_{d-1}$. For completeness we define $a_0 = c_0 = b_d = 0$. The graph $G$ is said to be distance-regular if all the $a_i, b_i$ and $c_i$ are fixed independently of $v_0$ and $v_i$. We also put $b_0 = |\{\text{vertices distance 1 from } v_0\}| = \rho$ so that $G$ is regular. Generally, for each $i$, $a_i + b_i + c_i = \rho$.

**Lemma 1.** If $G$ is any distance-regular graph which is not a complete graph and if $G$ is MLD-colourable then $a_1 = 0$.

**Proof.** Suppose $G$ is MLD-colourable using $l$ colours $1, 2, \ldots, l$. $G$ contains at least one pair of adjacent vertices $v_0, v_1$ coloured $i, i$. Let $v_1'$, having colour $j$, be any other vertex adjacent to $v_0$. Then $\{v_1', v_1\}$ cannot be an edge of $G$ for the pair of colours $\{i, j\}$ would occur twice, contrary to the definition of a MLD-colouring. Thus $a_i = 0$. \[\square\]

Distance regular graphs for which $d = 1$ are the complete graphs which are trivially MH-colourable and MLD-colourable. Distance regular graphs for which $d = 2$ are the strongly regular graphs which have the property that given distinct vertices $u, v$ the number of vertices adjacent to both $u$ and $v$ depends only upon whether $u$ and $v$ are adjacent. For these there is no MH-colourable example and the only MLD-colourable example is the Petersen graph [3].

We hence consider the next case $d = 3$. Here we can specify the 12 values of $a_i, b_i$ and $c_i, i = 0, 1, 2, 3$ in terms of five parameters $\rho, \lambda, \mu, \nu$ and the distance from a given vertex according to the following Table 1.

Then

$$|V_i|b_i = |V_{i+1}|c_{i+1}$$

so, since $|V_0| = 1$ and $|V_1| = \rho$, \[\rho(\rho - \lambda - 1) = |V_2|\mu. \tag{1.1}\]

Also, $|V_2|((\rho - \mu - \nu) = |V_3|\delta$.

Hence,

$$n = |V| = 1 + \rho + \frac{\rho(\rho - \lambda - 1)}{\mu} + \frac{\rho(\rho - \lambda - 1)(\rho - \mu - \nu)}{\mu\delta}. \tag{1.2}$$

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<th>Table 1</th>
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<td>Distance</td>
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</table>
2. Bipartite distance-regular graphs of diameter 3

The formula (1.2) is simplified in the case of an MLD-colourable graph since, by Lemma 1, we can put $\lambda = 0$. If we look at the case of bipartite graphs it becomes even simpler since we then also have $v = 0$ and $\rho = 0$. Table 1 may now be modified to Table 2. (1.2) gives in this case

$$n = 1 + \rho + \frac{\rho(\rho - 1)}{\mu} + \frac{(\rho - 1)(\rho - \mu)}{\mu}.$$  \hspace{1cm} (2.1)

Suppose that $G$ has an MLD-colouring using colours $1, 2, \ldots, l$ each $t$ times and that $v_0$ is a vertex which has colour 1. Further, suppose that $v_1$, adjacent to $v_0$, also has colour 1. Then the other $t - 2$ vertices, (if $t > 2$), which are assigned colour 1 are all members of $V_3$. Without loss of generality we may suppose that the other $\rho - 1$ vertices of $V_1$ are assigned the colours $2, 3, \ldots, \rho$. Further, the $\rho - 1$ vertices of $V_2$ which are adjacent to $v_1$ must be assigned colours $\rho + 1, \ldots, 2\rho - 1$. Then there are precisely $\rho(t - 1)$ vertices of $V_2$ which are adjacent to those $t - 2$ vertices of $V_3$ which have colour 1 and these must be assigned further distinct colours. Thus, the total number of colours is

$$l = 2\rho - 1 + \rho(t - 2) = \rho t - 1.$$ 

Hence using (2.1) we have

$$t(\rho t - 1) = 1 + \rho + \frac{(\rho - 1)(2\rho - \mu)}{\mu}. \hspace{1cm} (2.2)$$

We consider the cases of $t \geq 4$, $t = 3$ and $t = 2$ separately.

**Case $t \geq 4$:** From (2.2) we get

$$4(4\rho - 1) \geq 1 + \rho + \frac{(\rho - 1)(2\rho - \mu)}{\mu}.$$ 

From this

$$\frac{8\mu + 1 - \sqrt{64\mu^2 + 4\mu + 1}}{2} \leq \rho \leq \frac{8\mu + 1 + \sqrt{64\mu^2 + 4\mu + 1}}{2},$$
which cannot be an integer since

\[(8\mu)^2 < 64\mu^2 + 4\mu + 1 < (8\mu + 1)^2.\]

Case \(t = 3\): Putting \(t = 3\) in (2.2) gives \(2\rho^2 - (9\mu + 2)\rho + 5\mu = 0\). Hence, \(\rho = \frac{[(9\mu + 2) \pm \sqrt{81\mu^2 - 4\mu + 4}]}{4}\).

This has to be a nonnegative integer for an integer \(\mu \geq 1\). Hence, \(81\mu^2 - 18\mu + 1 < 81\mu^2 - 4\mu + 4 < 81\mu^2 + 18\mu + 1\), i.e. \((9\mu - 1)^2 < 81\mu^2 - 4\mu + 4 < (9\mu + 1)^2\), so the only solution is \(81\mu^2 - 4\mu + 4 = (9\mu)^2\), i.e. \(\mu = 1\) whence \(\rho = (11 \pm 9)/4\) and since \(\rho > 1\) we can only have \(\rho = 5\).

Correspondingly, \(n = 42\), \(|V_1| = 5\), \(|V_2| = 20\) and \(|V_3| = 16\). It follows that the 20 vertices of \(V_2\) are each adjacent to only one vertex of \(V_1\) and hence the graph has no quadrangles. Being a bipartite graph it has no 5-cycles, hence it is of girth 6 and thus it can only be the unique (to within isomorphism) \((5,6)\)-cage. This graph is known to be the incidence graph of the unique 21 point plane \(\text{PG}(2, 2^2)\). To see that it has no 14-MLD-colourings suppose it has one; let \(P\) and \(l\) be an incident pair of point \(P\) and line \(l\) of \(\text{PG}(2, 2^2)\) corresponding to vertices both assigned colour \(i\). Then if the third vertex with colour \(i\) corresponds to a point \(P'\) there is a unique line \(l'\) of the plane incident with both \(P\) and \(P'\). Suppose that the corresponding vertex of the graph is assigned colour \(j\). The colour pair \(\{i,j\}\) occurs twice on an edge contrary to the definition of MLD-colourable, whence we have a contradiction. The dual argument deals with the case of the third element with colour \(i\) corresponding to a line \(P')\) of the plane incident both \(P\) and \(P'\). Suppose that the corresponding vertex of the graph is assigned colour \(j\). The colour pair \(\{i,j\}\) occurs twice on an edge contrary to the definition of MLD-colourable, whence we have a contradiction. The dual argument deals with the case of the third element with colour \(i\) corresponding to a line \(P')\) of the plane incident both \(P\) and \(P'\).

Case \(t = 2\): Putting \(t = 2\) in (2.1) gives, since \(\rho \neq 1\), \(2\mu = \rho\); then \(|V_2| = 2(\rho - 1)\) and \(|V_3| = \rho - 1\). Table 2 may be modified to give Table 3.

One each of the two vertices assigned any given colour occurs in each of the classes of the bipartition of the vertices. Thus the vertex classes may be labelled \(X = \{x_0, \ldots, x_{w-1}\}\) and \(Y = \{y_0, \ldots, y_{w-1}\}\) where \(u = n/2\). Each vertex in \(X\) is adjacent to a vertex in \(Y\) of the same colour. Thus, without loss of generality, we may assume that \(x_i, y_i\) are assigned colour \(i\) for \(i = 0, 1, \ldots, u - 1\), that \(x_i\) is adjacent to \(y_i\) and that \(x_0\) is adjacent to \(y_0, \ldots, y_{\rho-1}\). The vertex \(y_0\) is necessarily adjacent to \(x_0, x_\rho, \ldots, x_{2\rho-2}\).

The graph \(G\) has diameter 3, therefore the vertices \(x_1, x_2, \ldots, x_{\rho-1}\) are at distance 2 from \(x_0\) and the vertices \(y_\rho, \ldots, y_{2\rho-2}\) are at distance 3 from \(x_0\). Consider the vertex \(x_1\). From Table 3, this is adjacent to \(\rho/2\) of the vertices \(y_1, \ldots, y_{\rho-1}\) and to \(\rho/2\) of the vertices \(y_\rho, \ldots, y_{2\rho-2}\). Clearly, we have the incidence graph of a symmetric 2-design \((2\rho - 1, \rho, \frac{\rho}{2})\). This is the complement of a Hadamard design, being complementary to

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<th>Table 3</th>
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the MH-colourable designs discussed in [2]. By arguments similar to those used in [2] it may be seen that these are the only symmetric designs which have MLD-colourable incidence graphs. We thus have the following result.

**Theorem.** A bipartite distance regular graph of diameter 3 is MLD-colourable if and only if it is the complementary design of a skew-Hadamard design.

**References**

