Blow up of solutions of pseudoparabolic equations

M. Meyvaci

Department of Mathematics, Mimar Sinan University of Fine Arts, Beşiktas, Istanbul, Turkey

ARTICLE INFO

Article history:
Received 30 June 2008
Available online 12 November 2008
Submitted by T. Witelski

Keywords:
Pseudoparabolic equation
Sobolev equation
Blow up of solution

ABSTRACT

We obtain sufficient conditions for the blow up of solutions of the initial-boundary value problem for nonlinear pseudoparabolic equation involving nonlinear convective term.

1. Introduction

We study the following initial-boundary value problem:

$$u_t - \Delta u_t - \Delta u - u^p u_{xx} = |u|^{2m}u, \quad x \in \Omega, \quad t > 0,$$

$$u(x, 0) = u_0(x), \quad x \in \Omega,$$

$$u(x, t) = 0, \quad x \in \partial \Omega, \quad t \geq 0.$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial \Omega$, $p \geq 1$ is a given integer and $m \geq 1$ is a given number. Eq. (1) with $m = 1, p = 2$ models nonstationary processes in semiconductors in the presence of a nonlinear force and a constant homogeneous external electric field.

Nonlinear pseudoparabolic equations of the form

$$u_t - \Delta u_t - \nu \Delta u = f(x, u, \nabla u), \quad \nu > 0,$$

appear in the study of various problems of hydrodynamics, thermodynamics and filtration theory (see [2,4,14]). The linear version of (4) was first studied by S.L. Sobolev [14] in 1954. Thus the equation of the form (4) is also called a Sobolev type equation. S.A. Galpern [6] studied the Cauchy problem for the equation of the form

$$Mu_t + Lu = f,$$


The first paper on nonlinear pseudoparabolic equation is the paper [12], where it is established existence and uniqueness of a weak solution of the initial value problem for the differential operator equation of the form

$$M(t)u_t + L(t)u = F(t, u).$$
A systematic study of global existence and uniqueness of the Cauchy problem for the nonlinear differential operator equations covering a wide class of nonlinear pseudoparabolic equations was done in the paper of Showalter and Ting [13] and in the book of Gajewski, Gröger and Zacharias [5].

One of the important representatives of (4) is the Benjamin–Bona–Mahony–Bürgers (BBMB) equation

$$u_t - v u_{xx} - u_{xxt} - u_x + uu_x = 0. \quad (7)$$

Amick, Bona and Schonbeck [1] studied the asymptotic behavior of solutions in $L^2(R)$ and $L^\infty(R)$ of the Cauchy problem for this equation. The results obtained here were developed [17] for equations of the form

$$u_t - v u_{xx} - u_{xxt} - u_x + uu_x + u^m u_x = 0,$$

where $m \geq 0$. Karch [8] investigated asymptotic behavior of solutions of the Cauchy problem for the multidimensional BBMB equation, that is Eq. (4) when $f$ has the form $f = (b, \nabla u) + \nabla \cdot \tilde{F}(u)$. Wang and Yang [16] proved existence of a finite dimensional global attractor of the semigroup generated by the periodic initial-boundary value problem for the one dimensional BBMB equation. Çelebi, Kalantarov and Polat [3] studied the problem of existence of a global attractor and the exponential attractor of the semigroup generated by the periodic initial-boundary value problem for Eq. (4) with $f = (b, \nabla u) + \nabla \cdot \tilde{F}(u) + h(x)$. Stanislavova, Stefanov and Wang [15] studied the problem of existence of a global attractor for multidimensional BBMB equation in $H^1(R^3)$.

The first result on blow up of solutions for nonlinear pseudoparabolic equation was obtained Levine [10]. Levine studied the Cauchy problem for the following nonlinear differential operator equation

$$P u_t + A u = F(u),$$

where $P$, $A$ are linear positive operators and $F(u)$ is a potential operator in a Hilbert space $H$. This result gives sufficient conditions of the blow up of solutions to the Cauchy problem and initial-boundary value problems for equations of the form

$$u_t - \Delta u - \Delta u_t = f(u),$$

where $f$ satisfies

$$f(s) = k \int_0^s f(\tau) d\tau \geq 0, \quad k > 2. \quad (8)$$

The concavity method invented by Levine in [10] was generalized in Kalantarov and Ladyzhenskaya [7]. The result obtained in [7] can be applied to pseudoparabolic equations of the form

$$u_t - \Delta u - \Delta u_t + b(x, t, u, \nabla u) = f(u),$$

where $f$ satisfies (8) and $b$ has a linear growth with respect to $u$ and $\nabla u$.

Korpusov and Sveshnikov [9] established sufficient conditions for global nonexistence of solutions of initial-boundary problem for the following Benjamin–Bona–Mahony–Bürgers equation

$$u_t - \Delta u_t - \Delta u - uu_{x_1} - u^2 = 0.$$

In what follows we are using the following notations:

$$\|v\| := \|v\|_{L^1(\Omega)}, \quad (u, v) := \int_\Omega uv dx, \quad \|v\|_p := \|v\|_{L^p(\Omega)}.$$

We will use the standard Cauchy and Young inequalities.

For each $a, b, \epsilon > 0$, and $q = p/(p - 1)$, $1 < p < \infty$ the following inequality holds true

$$ab \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2, \quad ab \leq \frac{\epsilon}{p} a^p + \frac{1}{q\epsilon^{1/(p-1)}} b^q. \quad (9)$$

We will use also the following proposition established in [7].

**Lemma 1.1.** Suppose that a positive, twice differentiable function $\Psi(t)$ satisfies the inequality

$$\Psi''(t) \Psi(t) - (1 + \alpha) \left[ \Psi'(t) \right]^2 \geq -2M_1 \Psi'(t) \Psi(t) - M_2 \left[ \Psi(t) \right]^2, \quad \text{for all } t > 0,$$

$$\Psi(0) > 0, \quad \Psi'(0) > -\gamma_2 \alpha^{-1} \Psi(0) \quad \text{and} \quad M_1 + M_2 > 0,$$

where $\alpha > 0$, $M_1, M_2 \geq 0$, $M_1 + M_2 > 0$. Then $\Psi(t)$ tends to infinity as

$$t \to t_1 \leq t_2 = \frac{1}{2 \sqrt{M_1^2 + \alpha M_2}} \ln \frac{\gamma_1 \Psi(0) + \alpha \Psi'(0)}{\gamma_2 \Psi(0) + \alpha \Psi'(0)}.$$  

Here $\gamma_1 = -M_1 + \sqrt{M_1^2 + \alpha M_2}$ and $\gamma_2 = -M_1 - \sqrt{M_1^2 + \alpha M_2}.$
2. Blow up of solutions

**Theorem 2.1.** Suppose that $1 < p < m$, and the initial function $u_0$ satisfies the following condition:

\[
\|u_0\|_{2(m+1)}^2 > \|\nabla u_0\|^2 + \left[\|u_0\|^2 + \|\nabla u_0\|^2 + \frac{(m-p)2^{(m+1)/(m-p)}|\Omega|}{(p+1)(2m+1)}\right] \times \frac{\sqrt{8(p+1)}}{m(6p+5)-1} \left[\sqrt{2(p+1)} + \sqrt{m(m+1)(6p+5)+2p+1-m}\right].
\]

Then the solution of the problem (1)–(3) blows up in a finite time.

**Proof.** Multiplying Eq. (1) by $u$ and integrating over $\Omega$ we get

\[
\frac{1}{2} \frac{d}{dt} \left[\|u\|^2 + \|\nabla u\|^2\right] = -\|\nabla u\|^2 + \|u\|_{2(m+1)}^2.
\]

Next we multiply (1) by $\epsilon u$ and integrate over $\Omega$:

\[
\|u_t\|^2 + \|\nabla u_t\|^2 = \frac{1}{2(m+1)} \frac{d}{dt} \|u\|_{2(m+1)}^2 - \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 - \frac{1}{p+1} (u^{p+1}, u_{tx}).
\]

Assume that $p < m$, and consider the following function

\[
\Psi(t) := \|u(t)\|^2 + \|\nabla u(t)\|^2 + C_0,
\]

where $C_0$ is a nonnegative parameter to be chosen below. It is clear that

\[
\Psi'(t) = 2(u, u_t) + (\nabla u, \nabla u_t).
\]

Due to the Cauchy–Schwarz inequality we have

\[
\left[\Psi'(t)\right]^2 = 4[(u, u_t) + (\nabla u, \nabla u_t)]^2 \leq 4\|u\|^2 + \|\nabla u\|^2)\|u_t\|^2 + \|\nabla u_t\|^2).
\]

Hence

\[
\left[\Psi'(t)\right]^2 \leq 4\Psi(t)\left[\|u_t\|^2 + \|\nabla u_t\|^2\right]
\]

By using the Cauchy–Schwarz inequality and the Young inequality we obtain:

\[
\left|\frac{d}{dt} \|\nabla u\|^2\right| \leq \frac{1}{\epsilon_0} \|\nabla u\|^2 + \epsilon_0 \|\nabla u_t\|^2,
\]

\[
|u^{p+1}, u_{tx}| \leq \frac{1}{2\epsilon_1} \|u\|_{2(p+1)}^2 + \frac{\epsilon_1}{2} \|\nabla u_t\|^2,
\]

\[
\|u\|_{2(p+1)}^2 \leq \frac{p+1}{m+1} \epsilon_2 \|u\|_{2(m+1)}^2 + \frac{m-p}{m+1} \epsilon_2 \|u\|_{2(m+1)}^2 \|\nabla u\|^2.
\]

Here $\epsilon_0, \epsilon_1$ and $\epsilon_2$ are positive parameters. By using (11) we obtain from (12):

\[
\|\nabla u_t\|^2 + \|u_t\|^2 = \frac{1}{4(m+1)} \Psi''(t) - \frac{m}{2(m+1)} \frac{d}{dt} \|\nabla u\|^2 - \frac{1}{p+1} (u^{p+1}, u_{tx}).
\]

Employing (14) and (15) we obtain

\[
\|\nabla u_t\|^2 + \|u_t\|^2 \leq \frac{1}{4(m+1)} \Psi''(t) + \frac{m}{2\epsilon_0(m+1)} \|\nabla u\|^2 + \frac{1}{2\epsilon_1(p+1)} \|u\|_{2(p+1)}^2 + \frac{m\epsilon_0}{2(m+1)} \frac{\epsilon_1}{2(p+1)} \|\nabla u_t\|^2.
\]

Next we use the estimate (16) for $\|u\|_{2(p+1)}^2$ in (17) and obtain

\[
\|\nabla u_t\|^2 + \|u_t\|^2 \leq \frac{1}{4(m+1)} \Psi''(t) + \frac{m}{2\epsilon_0(m+1)} \|\nabla u\|^2 + \frac{\epsilon_2}{2\epsilon_1(m+1)} \|u\|_{2(m+1)}^2 + C_1 \frac{m\epsilon_0}{2(m+1)} \frac{\epsilon_1}{2(p+1)} \|\nabla u_t\|^2.
\]
where $C_1 = \frac{(m-p)|\Omega|}{2e_1(p+1)(m+1)e_2(m^3-1)/(m-p)}$. It follows from (11) that

$$\frac{\epsilon_2}{2e_1(m+1)}\|u\|_{2(m+1)}^{2(m+1)/(p+1)} = \frac{\epsilon_2}{4(m+1)}\Psi'(t) + \frac{\epsilon_2}{2e_1(m+1)}\|\nabla u\|^2.$$  

Thus (18) implies

$$\|\nabla u_t\|^2 + \|u_t\|^2 \leq \frac{1}{4(m+1)}\psi''(t) + \left(\frac{m}{2e_0(m+1)} + \frac{\epsilon_1}{2(m+1)}\right)\|\nabla u\|^2 + \left(\frac{m}{2e_0(m+1)} + \frac{\epsilon_1}{2(m+1)}\right)\|\|u_t\|^2$$

$$+ \frac{\epsilon_2}{4(m+1)}\psi'(t) + C_1.$$  

By using (13) and the inequality $\|\nabla u(t)\|^2 \leq \psi(t) - C_0$ we obtain from (19) the estimate

$$\frac{1}{4\psi(t)}\psi'(t)^2 \left(1 - \frac{m_0}{2(m+1)} - \frac{\epsilon_1}{2(m+1)}\right) \leq \frac{1}{4(m+1)}\psi''(t) + \frac{\epsilon_2}{4(m+1)}\psi'(t) \left(\frac{m}{2e_0(m+1)} + \frac{\epsilon_1}{2(m+1)}\right)\psi(t)$$

$$+ C_1 - C_0 \left[\frac{m}{2e_0(m+1)} + \frac{\epsilon_2}{2(m+1)}\right].$$  

We choose in the last inequality $C_0 = \frac{(m-p)m(m^2-1)}{2(m+1)}$, $\epsilon_0 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{4}$, $\epsilon_2 = 2^{(m+1)/(m+1)}$. Multiplication of both sides of the obtained inequality by $4(m+1)|\psi(t)|$ gives

$$\psi(t)\psi''(t) - \left(1 + \frac{6p+5}{8(m+1)}\right)\psi'(t)^2 \geq -2\psi(t)\psi'(t) - 4(m+1)|\psi|^2(t).$$  

So the inequality (10) is satisfied with $\alpha = \frac{m^2+5}{4(m+1)} > 0$, $M_1 = 1$ and $M_2 = 4(m+1)$. Thus we can apply Lemma 1.1 and get the desired result.

**Theorem 2.2.** Suppose that $p = m, m \geq 1$, and the initial function $u_0$ satisfies the following condition:

$$\|u_0\|_{2(m+1)}^2 > \left(1 + \frac{1}{m}\right)\|u_0\|^2 + \left(2 + \frac{1}{m}\right)\|\nabla u_0\|^2.$$  

Then the solution of the problem (1)-(3) blows up in a finite time.

**Proof.** Under the transformation $u(t) = e^{-t}v(t)$ Eq. (1) takes the form

$$v_t - \Delta v - v - e^{-2t}v^m v_{x_1} = e^{-2t}v^2 v_{x_1}.\quad (20)$$  

Multiplying (20) by $v$ and $v_t$, and integrating over $\Omega$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|\nabla v\|^2 = \|v\|^2 + e^{-2t}\|v_t\|_{2(m+1)}^2.$$  

$$\|v_t\|^2 + \|\nabla v_t\|^2 = \frac{1}{2}\frac{d}{dt}\|v\|^2 + \frac{e^{-2t}}{2(m+1)}\frac{d}{dt}\|v\|_{2(m+1)}^2 - \frac{e^{-2t}}{m+1}(v^{m+1}, v_{x_1}).\quad (22)$$  

Now we are going to prove the blow up theorem by using the function

$$\Phi(t) = \|v\|^2 + \|\nabla v\|^2.$$  

Similar to (13), (14) and (15) we have

$$\left[\Phi'(t)^2\right] \leq 4\Phi(t)(\|v_t\|^2 + \|\nabla v_t\|^2),$$  

$$\left[\frac{d}{dt}\|v(t)\|_{2(m+1)}^2\right] \leq \|v_t\|^2 + \|v\|^2,$$  

and

$$\left|v^{m+1}, (v, v_{x_1})\right| \leq \frac{1}{2e_0(t)}\|v\|_{2(m+1)}^2 + \frac{\epsilon(t)}{2}\|v_t\|^2.$$  

Here $\epsilon(t), t \geq 0$ is a positive continuous function. Employing (21) and (22) we obtain

$$\|v_t\|^2 + \|\nabla v_t\|^2 = \frac{m}{2(m+1)}\Phi'(t) + \frac{\epsilon(t)}{4(m+1)}\Phi'(t) - \frac{m}{m+1}\|v\|^2 + \frac{m}{2(m+1)}\frac{d}{dt}\|v\|^2 - \frac{e^{-2t}}{m+1}(v^{m+1}, v_{x_1}).$$
By using (24) and (25) we obtain
\[
\|v_t\|^2 + \|\nabla v_t\|^2 \leq \frac{1}{2(m+1)} \left[ m\Phi'(t) + \frac{1}{2} \Phi''(t) + m\|v_t\|^2 + \epsilon(t)e^{-mt}\|\nabla v_t\|^2 + e^{-mt}\epsilon^{-1}(t)\|v\|^{2m+2} \right].
\] (27)

We use the inequality \(e^{-2mt}\|v\|^{2m+2} \leq \frac{1}{2}\Phi'(t)\), take \(\epsilon(t) = me^{mt}\) in (27), and obtain
\[
\frac{m + 2}{2(m+1)} (\|v_t\|^2 + \|\nabla v_t\|^2) \leq \frac{2m^2 + 1}{4m(m+1)} \Phi'(t) + \frac{1}{4(m+1)} \Phi''(t).
\] (28)

By using (23) in (28) we get
\[
\frac{m + 2}{2(m+1)} \frac{1}{4\Phi(t)}[\Phi'(t)]^2 \leq \frac{2m^2 + 1}{4m(m+1)} \Phi'(t) + \frac{1}{4(m+1)} \Phi''(t).
\]
We multiply both sides of the obtained inequality by \(4(m+1)\Phi(t)\)
\[
\Phi''(t)\Phi(t) - \left(1 + \frac{m}{2}\right)[\Phi'(t)]^2 \geq -\frac{2m^2 + 1}{m} \Phi'(t)\Phi(t).
\]
Thus the inequality (10) is satisfied for \(\alpha = m/2, M_1 = (2m^2 + 1)/(2m)\), and the conclusion follows from Lemma 1.1. □

References