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Blow up of solutions of pseudoparabolic equations

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A B S T R A C T

We obtain sufficient conditions for the blow up of solutions of the initial-boundary value problem for nonlinear pseudoparabolic equation involving nonlinear convective term.

1. Introduction

We study the following initial-boundary value problem:

$$u_t - \Delta u_t - \Delta u - u^p u_{x_1} = |u|^{2m} u, \quad x \in \Omega,\ t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{2}$$

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t \geqslant 0. \tag{3}$$

Here $\Omega \in \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $p \geqslant 1$ is a given integer and $m \geqslant 1$ is a given number. Eq. (1) with $m = 1$, $p = 2$ models nonstationary processes in semiconductors in the presence of a nonlinear force and a constant homogeneous external electric field.

Nonlinear pseudoparabolic equations of the form

$$u_t - \Delta u_t - \nu \Delta u = f(x, u, \nabla u), \quad \nu > 0, \tag{4}$$

appear in the study of various problems of hydrodynamics, thermodynamics and filtration theory (see [2,4,14]). The linear version of (4) was first studied by S.L. Sobolev [14] in 1954. Thus the equation of the form (4) is also called a Sobolev type equation. S.A. Galpern [6] studied the Cauchy problem for the equation of the form

$$Mu_t + Lu = f, \tag{5}$$

where M and L are linear elliptic operators. R.E. Showalter [11] investigated a linear pseudoparabolic equation (5), where M and L are second order elliptic operators. In this paper and in [13] existence, uniqueness and regularity of a weak solution of the initial-boundary value problem for (5) is established. Actually [13] is the first paper called (5) pseudoparabolic equation.

The first paper on nonlinear pseudoparabolic equation is the paper [12], where it is established existence and uniqueness of a weak solution of the initial value problem for the differential operator equation of the form

$$M(t)u_t + L(t)u = F(t,u). \tag{6}$$

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A systematic study of global existence and uniqueness of the Cauchy problem for the nonlinear differential operator equations covering a wide class of nonlinear pseudoparabolic equations was done in the paper of Showalter and Ting [13] and in the book of Gajewski, Gröger and Zacharias [5].

One of the important representatives of (4) is the Benjamin–Bona–Mahony–Bürgers (BBMB) equation

$$u_t - \nu u_{xx} - u_{xxt} - u_x + u u_x = 0. \tag{7}$$

Amick, Bona and Schonbeck [1] studied the asymptotic behavior of solutions in $L^2(R)$ and $L^\infty(R)$ of the Cauchy problem for this equation. The results obtained here were developed [17] for equations of the form

$$u_t - \nu u_{xx} - u_{xxt} - u_x + u^m u_x = 0,$$

where $m \geqslant 0$. Karch [8] investigated asymptotic behavior of solutions of the Cauchy problem for the multidimensional BBMB equation, that is Eq. (4) when f has the form $f = (\vec{b}, \nabla u) + \nabla \cdot \vec{F}(u)$. Wang and Yang [16] proved existence of a finite dimensional global attractor of the semigroup generated by the periodic initial-boundary value problem for the one dimensional BBMB equation. Çelebi, Kalantarov and Polat [3] studied the problem of existence of a global attractor and the exponential attractor of the semigroup generated by the periodic initial-boundary value problem for Eq. (4) with $f = (\vec{b}, \nabla u) + \nabla \cdot \vec{F}(u) + h(x)$. Stanislavova, Stefanov and Wang [15] studied the problem of existence of a global attractor for multidimensional BBMB equation in $H^1(R^3)$.

The first result on blow up of solutions for nonlinear pseudoparabolic equation was obtained Levine [10]. Levine studied the Cauchy problem for the following nonlinear differential operator equation

$$Pu_t + Au = F(u),$$

where P, A are linear positive operators and $F(u)$ is a potential operator in a Hilbert space H. This result gives sufficient conditions of the blow up of solutions to the Cauchy problem and initial-boundary value problems for equations of the form

$$u_t - \Delta u - \Delta u_t = f(u),$$

where f satisfies

$$f(s)s - k\int_0^s f(\tau)\,d\tau \geqslant 0, \quad k > 2. \tag{8}$$

The concavity method invented by Levine in [10] was generalized in Kalantarov and Ladyzhenskaya [7]. The result obtained in [7] can be applied to pseudoparabolic equations of the form

$$u_t - \Delta u - \Delta u_t + b(x, t, u, \nabla u) = f(u),$$

where f satisfies (8) and b has a linear growth with respect to u and ∇u.

Korpusov and Sveshnikov [9] established sufficient conditions for global nonexistence of solutions of initial-boundary problem for the following Benjamin–Bona–Mahony–Bürgers equation

$$u_t - \Delta u_t - \Delta u - u u_{x_1} - u^3 = 0.$$

In what follows we are using the following notations:

$$\|v\| := \|v\|_{L^2(\Omega)}, \qquad (u, v) := \int_\Omega uv\,dx, \qquad \|v\|_p := \|v\|_{L^p(\Omega)}.$$

We will need the standard Cauchy and Young inequalities.

For each $a, b, \epsilon > 0$, and $q = p/(p-1)$, $1 < p < \infty$ the following inequality holds true

$$ab \leqslant \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2, \qquad ab \leqslant \frac{\epsilon}{p}a^p + \frac{1}{q\epsilon^{1/(p-1)}}b^q. \tag{9}$$

We will use also the following proposition established in [7].

Lemma 1.1. *Suppose that a positive, twice differentiable function $\Psi(t)$ satisfies the inequality*

$$\Psi''(t)\Psi(t) - (1+\alpha)\big[\Psi'(t)\big]^2 \geqslant -2M_1\Psi'(t)\Psi(t) - M_2\big[\Psi(t)\big]^2, \quad \text{for all } t > 0,$$
$$\Psi(0) > 0, \qquad \Psi'(0) > -\gamma_2\alpha^{-1}\Psi(0) \quad \text{and} \quad M_1 + M_2 > 0, \tag{10}$$

where $\alpha > 0$, $M_1, M_2 \geqslant 0$, $M_1 + M_2 > 0$. Then $\Psi(t)$ tends to infinity as

$$t \to t_1 \leqslant t_2 = \frac{1}{2\sqrt{M_1^2 + \alpha M_2}} \ln \frac{\gamma_1\Psi(0) + \alpha\Psi'(0)}{\gamma_2\Psi(0) + \alpha\Psi'(0)}.$$

Here $\gamma_1 = -M_1 + \sqrt{M_1^2 + \alpha M_2}$ and $\gamma_2 = -M_1 - \sqrt{M_1^2 + \alpha M_2}$.

2. Blow up of solutions

Theorem 2.1. *Suppose that $1 < p < m$, and the initial function u_0 satisfies the following condition*:

$$\|u_0\|_{2(m+1)}^{2(m+1)} > \|\nabla u_0\|^2 + \left[\|u_0\|^2 + \|\nabla u_0\|^2 + \frac{(m-p)2^{(m+1)/(m-p)}|\Omega|}{(p+1)(2m+1)}\right] \times \frac{\sqrt{8(p+1)}}{m(6p+5)-1}\left[\sqrt{2(p+1)} + \sqrt{m(m+1)(6p+5)+2p+1-m}\right].$$

Then the solution of the problem (1)–(3) *blows up in a finite time.*

Proof. Multiplying Eq. (1) by u and integrating over Ω we get

$$\frac{1}{2}\frac{d}{dt}\left[\|u\|^2 + \|\nabla u\|^2\right] = -\|\nabla u\|^2 + \|u\|_{2(m+1)}^{2(m+1)}. \tag{11}$$

Next we multiply (1) by u_t and integrate over Ω:

$$\|u_t\|^2 + \|\nabla u_t\|^2 = \frac{1}{2(m+1)}\frac{d}{dt}\|u\|_{2(m+1)}^{2(m+1)} - \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 - \frac{1}{p+1}\left(u^{p+1}, u_{tx_1}\right). \tag{12}$$

Assume that $p < m$, and consider the following function

$$\Psi(t) := \left\|u(t)\right\|^2 + \left\|\nabla u(t)\right\|^2 + C_0,$$

where C_0 is a nonnegative parameter to be chosen below. It is clear that

$$\Psi'(t) = 2(u, u_t) + (\nabla u, \nabla u_t).$$

Due to the Cauchy–Schwarz inequality we have

$$\left[\Psi'(t)\right]^2 = 4\left[(u, u_t) + (\nabla u, \nabla u_t)\right]^2 \leqslant 4\left(\|u\|^2 + \|\nabla u\|^2\right)\left(\|u_t\|^2 + \|\nabla u_t\|^2\right).$$

Hence

$$\left[\Psi'(t)\right]^2 \leqslant 4\Psi(t)\left(\|u_t\|^2 + \|\nabla u_t\|^2\right). \tag{13}$$

By using the Cauchy–Schwarz inequality and the Young inequality we obtain:

$$\left|\frac{d}{dt}\|\nabla u\|^2\right| \leqslant \frac{1}{\epsilon_0}\|\nabla u\|^2 + \epsilon_0\|\nabla u_t\|^2, \tag{14}$$

$$\left|\left(u^{p+1}, u_{tx_1}\right)\right| \leqslant \frac{1}{2\epsilon_1}\|u\|_{2(p+1)}^{2(p+1)} + \frac{\epsilon_1}{2}\|\nabla u_t\|^2, \tag{15}$$

$$\|u\|_{2(p+1)}^{2(p+1)} \leqslant \frac{p+1}{m+1}\epsilon_2^{(m+1)/(p+1)}\|u\|_{2(m+1)}^{2(m+1)} + \frac{m-p}{m+1}\epsilon_2^{(m+1)/(p-m)}|\Omega|. \tag{16}$$

Here ϵ_0, ϵ_1 and ϵ_2 are positive parameters. By using (11) we obtain from (12):

$$\|\nabla u_t\|^2 + \|u_t\|^2 = \frac{1}{4(m+1)}\Psi''(t) - \frac{m}{2(m+1)}\frac{d}{dt}\|\nabla u\|^2 - \frac{1}{p+1}\left(u^{p+1}, u_{tx_1}\right).$$

Employing (14) and (15) we obtain

$$\|\nabla u_t\|^2 + \|u_t\|^2 \leqslant \frac{1}{4(m+1)}\Psi''(t) + \frac{m}{2\epsilon_0(m+1)}\|\nabla u\|^2 + \frac{1}{2\epsilon_1(p+1)}\|u\|_{2(p+1)}^{2(p+1)} + \left[\frac{m\epsilon_0}{2(m+1)} + \frac{\epsilon_1}{2(p+1)}\right]\|\nabla u_t\|^2. \tag{17}$$

Next we use the estimate (16) for $\|u\|_{2(p+1)}^{2(p+1)}$ in (17) and obtain

$$\|\nabla u_t\|^2 + \|u_t\|^2 \leqslant \frac{1}{4(m+1)}\Psi''(t) + \frac{m}{2\epsilon_0(m+1)}\|\nabla u\|^2 + \frac{\epsilon_2^{(m+1)/(p+1)}}{2\epsilon_1(m+1)}\|u\|_{2(m+1)}^{2(m+1)} + C_1 + \left[\frac{m\epsilon_0}{2(m+1)} + \frac{\epsilon_1}{2(p+1)}\right]\|\nabla u_t\|^2 \tag{18}$$

where $C_1 = \frac{(m-p)|\Omega|}{2\epsilon_1(p+1)(m+1)\epsilon_2^{(m+1)/(m-p)}}$. It follows from (11) that

$$\frac{\epsilon_2^{(m+1)/(p+1)}}{2\epsilon_1(m+1)}\|u\|_{2(m+1)}^{2(m+1)} = \frac{\epsilon_2^{(m+1)/(p+1)}}{4(m+1)\epsilon_1}\Psi'(t) + \frac{\epsilon_2^{(m+1)/(p+1)}}{2\epsilon_1(m+1)}\|\nabla u\|^2.$$

Thus (18) implies

$$\|\nabla u_t\|^2 + \|u_t\|^2 \leqslant \frac{1}{4(m+1)}\Psi''(t) + \left[\frac{m}{2\epsilon_0(m+1)} + \frac{\epsilon_2^{(m+1)/(p+1)}}{2\epsilon_1(m+1)}\right]\|\nabla u\|^2 + \left[\frac{m\epsilon_0}{2(m+1)} + \frac{\epsilon_1}{2(p+1)}\right]\|\nabla u_t\|^2 + \frac{\epsilon_2^{(m+1)/(p+1)}}{4(m+1)\epsilon_1}\Psi'(t) + C_1. \tag{19}$$

By using (13) and the inequality $\|\nabla u(t)\|^2 \leqslant \Psi(t) - C_0$ we obtain from (19) the estimate

$$\frac{1}{4\Psi(t)}\left[\Psi'(t)\right]^2\left(1 - \frac{m\epsilon_0}{2(m+1)} - \frac{\epsilon_1}{2(p+1)}\right) \leqslant \frac{1}{4(m+1)}\Psi''(t) + \frac{\epsilon_2^{(m+1)/(p+1)}}{4(m+1)\epsilon_1}\Psi'(t)\left[\frac{m}{2\epsilon_0(m+1)} + \frac{\epsilon_2^{(m+1)/(p+1)}}{2\epsilon_1(m+1)}\right]\Psi(t) + C_1 - C_0\left[\frac{m}{2\epsilon_0(m+1)} + \frac{\epsilon_2^{(m+1)/(p+1)}}{2\epsilon_1(m+1)}\right].$$

We choose in the last inequality $C_0 = \frac{(m-p)2^{(2m-p+1)/(m-p)}|\Omega|}{(p+1)(4m+1)}$, $\epsilon_0 = \frac{1}{2}$, $\epsilon_1 = \frac{1}{4}$, $\epsilon_2 = 2^{-(p+1)/(m+1)}$. Multiplication of both sides of the obtained inequality by $4(m+1)\Psi(t)$ gives

$$\Psi(t)\Psi''(t) - \left(1 + \frac{m(6p+5)-1}{8(p+1)}\right)\left[\Psi'(t)\right]^2 \geqslant -2\Psi(t)\Psi'(t) - 4(m+1)\Psi^2(t).$$

So the inequality (10) is satisfied with $\alpha = \frac{m(6p+5)-1}{8(p+1)} > 0$, $M_1 = 1$ and $M_2 = 4(m+1)$. Thus we can apply Lemma 1.1 and get the desired result. □

Theorem 2.2. *Suppose that $p = m$, $m \geqslant 1$, and the initial function u_0 satisfies the following condition*:

$$\|u_0\|_{2(m+1)}^{2(m+1)} > \left(1 + \frac{1}{m^2}\right)\|u_0\|^2 + \left(2 + \frac{1}{m^2}\right)\|\nabla u_0\|^2.$$

Then the solution of the problem (1)–(3) *blows up in a finite time.*

Proof. Under the transformation $u(t) = e^{-t}v(t)$ Eq. (1) takes the form

$$v_t - \Delta v_t - v - e^{-mt}v^m v_{x_1} = e^{-2mt}|v|^{2m}v. \tag{20}$$

Multiplying (20) by v and v_t, and integrating over Ω we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\|v\|^2 + \|\nabla v\|^2\right] = \|v\|^2 + e^{-2mt}\|v\|_{2m+2}^{2m+2}, \tag{21}$$

$$\|v_t\|^2 + \|\nabla v_t\|^2 = \frac{1}{2}\frac{d}{dt}\|v\|^2 + \frac{e^{-2mt}}{2(m+1)}\frac{d}{dt}\|v\|_{2(m+1)}^{2(m+1)} - \frac{e^{-mt}}{m+1}\left(v^{m+1}, v_{tx_1}\right). \tag{22}$$

Now we are going to prove the blow up theorem by using the function

$$\Phi(t) := \|v\|^2 + \|\nabla v\|^2.$$

Similar to (13), (14) and (15) we have

$$\left[\Phi'(t)\right]^2 \leqslant 4\Phi(t)\left(\|v_t\|^2 + \|\nabla v_t\|^2\right), \tag{23}$$

$$\left|\frac{d}{dt}\|v(t)\|^2\right| \leqslant \|v_t\|^2 + \|v\|^2, \tag{24}$$

and

$$\left|\left(v^{m+1}, v_{tx_1}\right)\right| \leqslant \frac{1}{2\epsilon(t)}\|v\|_{2(m+1)}^{2(m+1)} + \frac{\epsilon(t)}{2}\|\nabla v_t\|^2. \tag{25}$$

Here $\epsilon(t)$, $t \geqslant 0$ is a positive continuous function. Employing (21) and (22) we obtain

$$\|v_t\|^2 + \|\nabla v_t\|^2 = \frac{m}{2(m+1)}\Phi'(t) + \frac{1}{4(m+1)}\Phi''(t) - \frac{m}{m+1}\|v\|^2 + \frac{m}{2(m+1)}\frac{d}{dt}\|v\|^2 - \frac{e^{-mt}}{m+1}\left(v^{m+1}, v_{tx_1}\right). \tag{26}$$

By using (24) and (25) we obtain

$$\|v_t\|^2 + \|\nabla v_t\|^2 \leqslant \frac{1}{2(m+1)}\left[m\Phi'(t) + \frac{1}{2}\Phi''(t) + m\|v_t\|^2 + \epsilon(t)e^{-mt}\|\nabla v_t\|^2 + e^{-mt}\epsilon^{-1}(t)\|v\|_{2m+2}^{2m+2}\right]. \tag{27}$$

We use the inequality $e^{-2mt}\|v\|_{2m+2}^{2m+2} \leqslant \frac{1}{2}\Phi'(t)$, take $\epsilon(t) = me^{mt}$ in (27), and obtain

$$\frac{m+2}{2(m+1)}\left(\|v_t\|^2 + \|\nabla v_t\|^2\right) \leqslant \frac{2m^2+1}{4m(m+1)}\Phi'(t) + \frac{1}{4(m+1)}\Phi''(t). \tag{28}$$

By using (23) in (28) we get

$$\frac{m+2}{2(m+1)}\frac{1}{4\Phi(t)}\left[\Phi'(t)\right]^2 \leqslant \frac{2m^2+1}{4m(m+1)}\Phi'(t) + \frac{1}{4(m+1)}\Phi''(t).$$

We multiply both sides of the obtained inequality by $4(m+1)\Phi(t)$

$$\Phi''(t)\Phi(t) - \left(1 + \frac{m}{2}\right)\left[\Phi'(t)\right]^2 \geqslant -\frac{2m^2+1}{m}\Phi'(t)\Phi(t).$$

Thus the inequality (10) is satisfied for $\alpha = m/2$, $M_1 = (2m^2+1)/(2m)$, and the conclusion follows from Lemma 1.1. □

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