Journal of Approximation Theory **98**, 203–222 (1999)
Article ID jath.1998.3285, available online at http://www.idealibrary.com on IDEAL®

On the Rate of Almost Everywhere Convergence of Certain Classical Integral Means

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Communicated by Rolf J. Nessel

Received October 15, 1997; accepted in revised form April 30, 1998

The main purpose of this article is to establish nearly optimal results concerning the rate of almost everywhere convergence of the Gauss–Weierstrass, Abel–Poisson, and Bochner–Riesz means of the one-dimensional Fourier integral. A typical result for these means is the following: *If the function f belongs to the Besov space* $B^s_{p,p}$, $1<p<\infty$, $0<s<1$, *then* $T_{m_t} f(x)-f(x)=o_x(t^s)$ *a.e. as* $t\to 0+$.

Let $m\in L^\infty(0,+\infty)$ and denote by m_t, $t>0$, the function $m_t(u)=m(tu)$, $u>0$; define operators $T_{m_t}(t>0)$ on $L^2(R^n)$ via their Fourier transform

$$(\widehat{T_{m_t} f})(\xi)=m_t(|\xi|)\,\hat{f}(\xi).$$

Note that the Abel–Poisson means $\{P_t\}$ are defined by $m(u)=e^{-u}$, the Gauss–Weierstrass means $\{W_t\}$ by $m(u)=e^{-u^2}$ (note that with the normalization of the kernel the family $\{W_t\}$ does not form a one-parameter semi-group in $t>0$), the Bochner–Riesz means by $m(u)=(1-u^2)^\alpha_+$. These means are very important in the problems of harmonic analysis, partial differential equations, theory of probability, etc.—convergence a.e. of the Poisson and the Gauss–Weierstrass means is one of the basic facts of harmonic

* During the fall of 1996 this author visited TH Darmstadt as a DAAD fellow. The paper is the result of the authors' collaboration initiated during this period.

0021-9045/99 $30.00

analysis. We mention that the important problem of a.e. convergence of the Bochner–Riesz means is only partially solved when $n \geqslant 2$, see [4].

Consider the following problem: *Under which smoothness conditions* (*in the L^p-norm*) *on f does $T_{m_t} f$ converge a.e. towards f with a prescribed rate $w(t)$ of convergence*?

In this paper we will discuss the one-dimensional case $n=1$ and in particular are interested in results of the following type:

$$T_{m_t} f(x) - f(x) = o_x(w(t)) \text{ a.e.} \qquad \text{when} \quad t \to 0+. \tag{1}$$

In a sequel to this paper we discuss the sharpness of the results obtained here.

1. POTENTIAL SPACES

The problem about the a.e. convergence rate of $T_{m_t} f$ was first investigated by Carbery [2] in the case that the function f belongs to some potential space over the n-dimensional Euclidean space R^n. To describe Carbery's and related results, following [6] (see also [7]), we first define a fractional integral of order α, $0 < \alpha \leqslant 1$, of $m \in L^\infty(0, \infty)$ by

$$I_\omega^\alpha(m)\,(u) = 1/\Gamma(\alpha) \int_u^\omega (v-u)^{\alpha-1}\, m(v)\, dv,$$

when $0 < u < \omega$, and $I_\omega^\alpha(m)\,(u) = 0$ when $u \geqslant \omega$. If $0 < \alpha < 1$ and $I_\omega^{1-\alpha}(m)$ is locally absolutely continuous for every $\omega > 0$, we define the fractional derivative $m^{(\alpha)}$ by

$$m^{(\alpha)}(u) = \lim_{\omega \to \infty} -\frac{d}{du} I_\omega^{1-\alpha} m(u).$$

Moreover, by induction on the integer part $[\alpha]$ of α, we define for arbitrary $0 < \alpha = [\alpha] + \delta$

$$m^{(\alpha)}(u) = \left(\frac{d}{du}\right)^{[\alpha]} m^{(\delta)}(u),$$

provided $m^{(\delta)}, \ldots, m^{(\alpha-1)}$ are locally absolutely continuous. Notice that for m with compact support in R^+

$$\widehat{m^{(\alpha)}}(\tau) = (-i\tau)^\alpha\, \hat{m}(\tau),$$

where $(-i\tau)^\alpha$ is defined by the principal branch. As in Carbery [2] we introduce the global Bessel potential space $\mathscr{L}^2_\alpha$ as the completion of the C^∞-functions with compact support in $(0; +\infty)$ with respect to the norm

$$\|m\|_{\mathscr{L}^2_\alpha} = \left\{ \int_0^{+\infty} \left| u^{\alpha+1} \left(\frac{d}{du}\right)^\alpha \left(\frac{m(u)}{u}\right) \right|^2 \frac{du}{u} \right\}^{1/2}.$$

THEOREM A (*see* [2]). *If* $\alpha > n(1/p - 1/2) + 1/2$ *when* $1 < p \leqslant 2$, *or* $\alpha > n(1/2 - 1/p) + 1/p$ *when* $2 \leqslant p < \infty$, *then*

$$\|\sup_{t>0} t^{-s} |T_{m_t} f| \|_{L^p(R^n)} \leqslant c_\alpha \| |\cdot|^{-s} m(\cdot) \|_{\mathscr{L}^2_\alpha} \|D^s f\|_{L^p(R^n)},$$

where D^s *is explained by* $(\widehat{D^s f}(\xi) = |\xi|^s \hat{f}(\xi)$, $\xi \in R^n$. *Furthermore, if* $n = 1$ *or* $n = 2$, *the result can be improved to* $\alpha > \max\{1/2, n(1/2 - 1/p)\}$ *provided* $2 \leqslant p < \infty$.

In the case of the real line, Theorem A immediately implies the following result.

COROLLARY 1.1. *Let* $1 < p < \infty$ *and* $\mu(u) = (m(u) - 1)\, u^{-s}$ *satisfy* $\mu \in \mathscr{L}^2_\alpha$ *with* $\alpha > \max\{1/p, 1/2\}$. *Then for all functions* f *with* $\|D^s f\|_p < \infty$ *there holds*

$$T_{m_t} f(x) - f(x) = O_x(t^s), \qquad t \to 0+ \tag{2}$$

a.e. *on* R.

The following result closely related to Theorem A is contained in Dappa and Trebels [7].

THEOREM B (*see* [7]). *Let* m *be a measurable bounded function on* $(0, +\infty)$ *which vanishes at infinity and which satisfies for* $\lambda > n\,|1/p - 1/2| + 1/2$

$$B_\lambda(m) \equiv \left(\int_0^\infty |u^\lambda m^{(\lambda)}(u)|^2 \frac{du}{u} \right)^{1/2} + \int_0^\infty u^{\lambda-1} |m^{(\lambda)}(u)|\, du < \infty. \tag{3}$$

Define the maximal operator T^*_m *on* $L^2(R^n)$ *by*

$$T^*_m f(x) = \sup_{t>0} |T_{m_t} f(x)|.$$

Then T^*_m *is of strong type* (p, p), $1 < p < \infty$, *with operator norm* $\|T^*_m\|_{p\to p} \leqslant CB_\lambda(m)$; *also* T^*_m *is of weak type* $(1, 1)$.

Here and in the following we denote generic constants that are independent of the functions (and sequences) by C.

Remark 1.1. (a) Corollary 1.1 remains valid if there the hypothesis $\mu \in \mathscr{L}^2_\alpha$ is replaced by $B_1(\mu) < \infty$ since

$$m(t|\xi|)\, \hat{f}(\xi) - \hat{f}(\xi) = \frac{m(t\,|\xi|) - 1}{(t\,|\xi|)^s}\,(t\,|\xi|)^s \hat{f}(\xi) = \mu(t\,|\xi|)\, t^s \widehat{D^s f}(\xi),$$

so

$$|T_{m_t} f(x) - f(x)|\, t^{-s} \leqslant T^*_\mu(D^s f)\,(x).$$

(b) The difference in the conditions of Theorems A and B may be illustrated at the example of the Abel–Poisson means $\{P_t\}$: Theorem A gives boundedness of the maximal function generated by $\mu(u) = (e^{-u} - 1)\, u^{-s}$ on $L^p(R)$, $1 < p < \infty$, only for $0 < s < 1$, whereas Theorem B (with $\lambda = 1$) yields boundedness for $0 < s \leqslant 1$. Thus Theorem B leads to the following result.

Corollary 1.2. *Let $f \in L^p(R)$, $1 < p < \infty$, satisfy $\|f(\cdot + h) - f(\cdot)\|_p = O(|h|)$. Then*

$$P_t f(x) - f(x) = O_x(t), \qquad t \to 0+,$$

a.e. on R.

For the proof we have only to note that by [1, p. 386] the hypothesis implies f to be locally absolutely continuous with $f' \in L^p(R)$, thus Hf, $Hf' \in L^p(R)$ since the Hilbert transform H is continuous on $L^p(R)$, $1 < p < \infty$, and that the symbol of Hd/dx is $|\xi|$. Now the argument in (a) leads to the assertion.

(c) Theorem A is generalized in Seeger [10].

Working on $L^2(R^n)$, Müller and Wang [8] have weakened the global conditions (3) to local ones. To become more precise introduce the localized Riemann–Liouville spaces $RL(2, \alpha)$ (see [3]) by

$$RL(2, \alpha) = \{m \in L^\infty(0, \infty) \colon \|m\|_{RL(2,\alpha)} = \sup_{t>0} \|(\chi m_t)^{(\alpha)}\|_2\},$$

with $\chi \in C_0^\infty(0, \infty)$ being an arbitrary non-negative and non-trivial bump function.

THEOREM C (see [8]). *Assume m is a continuous function on R^+ and $m \in RL(2, \alpha)$ for some $\alpha > 1/2$. Let ϕ be a non-decreasing function on $[0, \infty)$ satisfying the condition*

$$1 \leqslant \phi(2t) \leqslant \lambda \phi(t) \qquad \textit{for some} \quad \lambda \geqslant 1;$$

set $v = \log_2 \lambda$, where λ is smallest possible to satisfy the preceding condition. Set

$$\psi(t) = \phi(1) + \left(\int_1^{t+1} \frac{\phi^2(u)}{u}\, du \right)^{1/2},$$

$$L^2_\psi = \left\{ f \in L^2(R^n) \colon \int_{R^n} |\hat{f}(\xi)|^2\, |\psi(|\xi|)|^2\, d\xi < \infty \right\}.$$

If

$$\|(\chi(1 - m_t))^{(\alpha)}\|_2 = O(t^\beta) \qquad \textit{as} \quad t \to 0+, \quad \beta > v,$$

then, for every $f \in L^2_\psi$, there holds

$$T_{m_t} f(x) - f(x) = o\left(\frac{1}{\phi(1/t)} \right) \quad \textit{a.e.} \qquad \textit{as} \quad t \to 0+.$$

Theorem C applied to the general Riesz means recovers the following result of Chen [5] which we only state for the real line ($n = 1$).

THEOREM D (see [5]). *Let*

$$L^2_s(R) = \left\{ f \in L^2(R) \colon \left(\int_R |\hat{f}(\xi)|^2\, (1 + |\xi|^2)^s\, ds \right)^{1/2} < \infty \right\}$$

be the classical Bessel potential space. Let $\mu(\xi) = (1 - |\xi|^\gamma)^\alpha_+$, $\gamma > 0$, $0 < \alpha \leqslant 1$. Then, for every $f \in L^2_s(R)$,

$$T_{\mu_t} f(x) - f(x) = \begin{cases} o_x(t^s), & 0 < s < \gamma, \\ O_x(t^s), & s = \gamma, \end{cases} \qquad t \to 0+,$$

holds almost everywhere in R.

One can characterize (see, e.g., [13, p. 139]) the space $L^2_s(R)$ in terms of moduli of continuity $\omega(f, t)_2$, where

$$\omega(f, t)_p = \sup_{|h| < t} \|f(\cdot + h) - f(\cdot)\|_{L^p(R)}.$$

PROPOSITION. *Let $0<s<1$. Then the function $f \in L^2(R)$ belongs to $L^2_s(R)$ if and only if*

$$\int_0^\infty \left(\frac{\omega(f,t)_2}{t^s}\right)^2 \frac{dt}{t} < \infty.$$

The combination of Theorem D with the proposition gives a statement of the type: There is a certain rate of a.e. pointwise convergence of the general Riesz means (namely $o_x(t^s)$) if $\omega(f,t)_2$, the L^2-modulus of continuity of f, tends sufficiently fast to 0 when $t \to 0+$. In the following we want to replace

(i) $f \in L^2_s(R)$ by the condition that $f \in H^\omega_p$, where $\omega = \omega(t)$ is a given modulus of continuity and

$$H^\omega_p = \{f \in L^p(R) \colon \omega(f,t)_p \leqslant C\omega(t)\};$$

(ii) the rate t^s of convergence by some increasing function $w(t)$ which is only admitted when essentially $\sum_{k=1}^\infty (\omega(\theta_k)/w(\theta_k))^p < \infty$ for certain $\theta_k \to 0$;

(iii) the general Riesz means by a class of approximation processes.

Concerning the methods of proof we give a contribution to a programmatic remark by Shapiro [11, p. 120]: "Another interesting area for study is how far *pointwise* (rather than norm) approximation theorems can be inferred from the Fourier transform of the kernel." We mention [14], where first results for moduli of smoothness of higher order in R^n are outlined at the example of the Abel–Cartwright means.

Concerning the modulus of continuity $\omega(t)$ we distinguish the two cases: $\omega(t)/t \uparrow \infty$, $t \to 0+$ and $\omega(t) = t$. As Remark 1.1(b) already indicates, the condition $\omega(f,t) \leqslant Ct$ actually is a potential type condition we deal with first. Recall that in this case $f, f' \in L^p$ provided $1<p<\infty$ so that, by the Hardy–Littlewood theorem (see, e.g., [13, p. 5]),

$$|f(x+h)-f(x)| = \left|\int_x^{x+h} f'\right| \leqslant |h| \left||h|^{-1}\int_x^{x+h} |f'|\right| \leqslant |h|\, \mathscr{M}(f')\,(x)$$

a.e. on R. Then, if $K \in L^1(R)$ with $\int_R K = 1$ and $\int_R |hK(h)|\, dh < \infty$, we obtain, using the notation $K_t(h) = t^{-1}K(h/t)$,

$$|K_t * f(x) - f(x)| \leqslant \int_R |f(x+h)-f(x)|\, |K_t(h)|\, dh$$

$$\leqslant t\mathscr{M}(f')(x) \int_R |h/t|\, |K_t(h)|\, dh = C(x)\, t \int_R |hK(h)|\, dh,$$

thus an a.e. convergence rate $O_x(t)$. In the case $p=1$ we have $|f(x+h)-f(x)| \leqslant |h|\, \mathcal{M}(df)\,(x)$ (observe that now f is at least of bounded variation and that in this situation a Hardy–Littlewood theorem also holds—see, e.g, [11, p. 39, Theorem 17]). Hence, also in the case $p=1$ there holds the a.e. convergence rate $O_x(t)$.

Let us give a supplementary result for the Abel–Poisson means in the case $p=1$. Clearly its kernel $p(h)=c(1+h^2)^{-1}$ does not satisfy all of the preceding conditions. Also, on account of the unboundedness of the Hilbert transform on $L^1(R)$, one cannot expect an a.e. convergence rate $O_x(t)$ for these means. But

$$|P_t f(x)-f(x)| \leqslant \left(\int_{|h|\leqslant t} + \int_{|h|\geqslant t}\right) |f(x+h)-f(x)|\, p_t(h)\, dh = O_x(t\,|\log t|)$$

a.e. on R, where for the estimate of the first integral the pointwise Lipschitz condition, for the estimate of the second the boundedness of $|f(x+h)- f(x)| \leqslant \int_R |df| \leqslant C$ is used.

An analogous argument gives the estimate $\|P_t f - f\|_{L^1(R)} = O(t\,|\log t|)$. So the Abel–Poisson means and the Gauss–Weierstrass means have the same order of pointwise and norm estimates for elements from H_p^ω, $\omega(t)=t$.

2. HÖLDER SPACES

From now on we consider functions $f\in H_p^\omega$ with $\omega(t)/t \uparrow \infty$, $t\to 0+$. Usually the degree of approximation improves with the smoothness of the function. But there may be a critical convergence rate which cannot be improved even by ("non-trivial") C^∞-functions. This is described by the idea of saturation (see [1, p. 434]).

Definition. The strong approximation process $\{T_t\}_{t>0}$ on the Banach space X possesses the saturation property if there exists a positive function $\varphi(t) \downarrow 0$, $t\to 0+$, such that every $f\in X$ for which

$$\|T_t f - f\|_X = o(\varphi(t)), \qquad t\to 0+,$$

is an invariant element of $\{T_t\}_{t>0}$, i.e., $T_t f = f$ for all $t>0$, and there exists at least one noninvariant element $g\in X$ with $\|T_t g - g\|_X = O(\varphi(t))$. In this event, $\{T_t\}_{t>0}$ is said to be saturated in X with order $O(\varphi(t))$.

Since it may happen that the smoothness of f is not reflected in the rate of convergence we have to distinguish between approximation processes with saturation order worse than $O(t)$ and those which admit (for smooth functions) at least $O(t)$ as degree of convergence.

2.1. *Convolution Means with Well-Decreasing Kernels*

THEOREM 2.1. *Let $K \in L^1(R)$, satisfy the following properties.*

(a) $$\int_R K(x)\,dx = 1.$$

(b) *The radial majorant $\bar{K}(x) \equiv \sup_{|y| \geqslant |x|} |K(y)|$ belongs to $L(R)$.*

(c) $$\int_R |xK(x)|\,dx < \infty.$$

Let $\omega(t)$ be a modulus of continuity such that $\omega(t)/t \uparrow \infty$, $t \to 0+$. Define δ_k in the following way

$$\delta_0 = 1, \qquad \delta_{k+1} = \min \left\{ \delta \colon \max \left(\frac{\omega(\delta)}{\omega(\delta_k)}; \frac{\delta \omega(\delta_k)}{\delta_k \omega(\delta)} \right) = \frac{1}{2} \right\}, \qquad k = 0, 1, ... \tag{4}$$

Let $w(t)$ be a nondecreasing function such that $\omega(t)/w(t)$ is nondecreasing and

$$\sum_{k=1}^{\infty} \left(\frac{\omega(\delta_k)}{w(\delta_k)} \right)^p < \infty. \tag{5}$$

Then, for every function $f \in H_p^\omega(R)$, $1 < p < \infty$, there holds

$$K_t * f(x) - f(x) = o_x(w(t)) \quad \text{a.e.} \qquad \text{when} \quad t \to 0+. \tag{1}$$

Remark 2.1.1. If one defines a sequence $\{\delta_k\}$ via $\omega(\delta_k) = 2^{-k}$, then this choice is more restrictive than that in Theorem 2.1 as may be seen at the example of $\omega(t) = t \log 1/t$.

Proof. Recall that $K_t(x) = (1/t)\,K(x/t)$, hence $K_t \in L(R)$ uniformly in $t > 0$. The hypotheses (a) and (b) guarantee convergence a.e. of $K_t * f(x)$ to $f(x)$ when $t \to 0+$ (see, e.g., [13, p. 62]), in particular, $|f(x) - g(x)| \leqslant T_m^*(f-g)\,(x)$ a.e.

The argument which gives the asserted rate of convergence is based on the following lemma of Oskolkov [9].

LEMMA E (see [9]). *Let $\omega(\delta)$ be a modulus of continuity with the property that $\omega(\delta)/\delta \uparrow \infty$, $\delta \to 0+$, and define $\{\delta_k\}$ by (4).*

Then there holds $2\delta_{k+1} \leqslant \delta_k$, $k = 0, 1, \ldots$ *and for any* $k = 0, 1, \ldots$ *either*

$$2\omega(\delta_{k+1}) = \omega(\delta_k) \tag{6}$$

or

$$\frac{\omega(\delta_k)}{\delta_k} = \frac{\omega(\delta_{k+1})}{2\delta_{k+1}}. \tag{7}$$

Also, *with some* $A > 0$

$$A^{-1}\omega(\delta) \leqslant \sum_{k=0}^{\infty} \omega(\delta_k) \min\left(1, \frac{\delta}{\delta_k}\right) \leqslant A\omega(\delta), \qquad 0 \leqslant \delta \leqslant 1.$$

Denote by

$$f_\delta(x) = \frac{1}{\delta} \int_x^{x+\delta} f(t)\, dt$$

the Steklov means of f. The following two estimates are easily verified:

$$\| f - f_\delta \|_p \leqslant \omega(f, \delta)_p \leqslant C\omega(\delta), \tag{8}$$

$$\| (f_\delta)' \|_p \leqslant \frac{\omega(f, \delta)_p}{\delta} \leqslant C\frac{\omega(\delta)}{\delta}. \tag{9}$$

Now, if we define

$$\Phi_k(x) = \begin{cases} f_{\delta_k}(x), & \text{if (6) holds,} \\ f_{\delta_{k+1}}(x), & \text{otherwise,} \end{cases}$$

we have

$$\| f - \Phi_k \|_p \leqslant C\omega(\delta_{k+1}) \tag{10}$$

and

$$\| (\Phi_k)' \|_p \leqslant C\frac{\omega(\delta_k)}{\delta_k}. \tag{11}$$

Further, for $\delta_{k+1} < t \leqslant \delta_k$, there follows

$$
\begin{aligned}
&|K_t * f(x) - f(x)| \\
&\qquad \leqslant |K_t * (f - \Phi_k)(x)| + \int_R |\Phi_k(x+h) - \Phi_k(x)|\, |K_t(h)|\, dh \\
&\qquad\quad + |f(x) - \Phi_k(x)| \\
&\qquad \leqslant 2T_m^*(f - \Phi_k)(x) + \int_R \left| \int_x^{x+h} \Phi_k'(y)\, dy \right| |K_t(h)|\, dh =: I + II.
\end{aligned}
$$

Now

$$
\begin{aligned}
II &\leqslant \int_R \left| |h|^{-1} \int_x^{x+h} |\Phi_k'(y)|\, dy \right| |h|\, |K_t(h)|\, dh \\
&\leqslant \mathscr{M}(\Phi_k')(x)\, t \int_R (|h|/t)\, |K_t(h)|\, dh \\
&= t\mathscr{M}(\Phi_k')(x) \int_R |h|\, |K(h)|\, dh,
\end{aligned}
$$

where $\mathscr{M}$ denotes the Hardy–Littlewood maximal function. Hence (see [13, p. 62])

$$
\begin{aligned}
|K_t * f(x) - f(x)| &\leqslant C(T_m^*(f - \Phi_k)(x) + t\mathscr{M}(\Phi_k')(x)) \\
&\leqslant C\left(\frac{\mathscr{M}(f - \Phi_k)(x)}{w(\delta_{k+1})} + \frac{\mathscr{M}(\Phi_k')(x)\, \delta_k}{w(\delta_k)} \right) w(t) \\
&\leqslant C\left(\sup_{k \geqslant 0} \frac{\mathscr{M}(f - \Phi_k)(x)}{w(\delta_{k+1})} + \sup_{k \geqslant 0} \frac{\mathscr{M}(\Phi_k')(x)\, \delta_k}{w(\delta_k)} \right) w(t).
\end{aligned}
$$

Thus we only need to estimate the terms in brackets, i.e., to find conditions on w that guarantee the finiteness a.e. for all functions $f \in H_p^\omega(R)$.

Oskolkov [9] was the first to give conditions with this property; these were rewritten in appropriate form by Soljanik [12]. By (10) we have

$$
\begin{aligned}
\left\| \sup_{k \geqslant 0} \frac{\mathscr{M}(f - \Phi_k)(\cdot)}{w(\delta_{k+1})} \right\|_p^p &\leqslant \sum_{k \geqslant 0} \frac{\|\mathscr{M}(f - \Phi_k)\|_p^p}{w^p(\delta_{k+1})} \\
&\leqslant C_p \sum_{k \geqslant 0} \frac{\|f - \Phi_k\|_p^p}{w^p(\delta_{k+1})} \leqslant C_p \sum_{k \geqslant 0} \left(\frac{\omega(\delta_{k+1})}{w(\delta_{k+1})} \right)^p,
\end{aligned}
$$

and by (11)

$$\left\| \sup_{k \geq 0} \frac{\mathscr{M}(\Phi'_k)(\cdot)\, \delta_k}{w(\delta_k)} \right\|_p^p \leqslant \sum_{k \geq 0} \frac{\|\mathscr{M}(\Phi'_k)\|_p^p\, \delta_k^p}{w^p(\delta_k)}$$

$$\leqslant C_p \sum_{k \geq 0} \frac{\|\Phi'_k\|_p^p\, \delta_k^p}{w^p(\delta_k)} \leqslant C_p \sum_{k \geq 0} \left(\frac{\omega(\delta_k)}{w(\delta_k)}\right)^p.$$

Thus, by hypothesis, we obtain that

$$\left(\sup_{k \geq 0} \frac{\mathscr{M}(f - \Phi_k)(x)}{w(\delta_{k+1})} + \sup_{k \geq 0} \frac{\mathscr{M}(\Phi'_k)(x)\, \delta_k}{w(\delta_k)} \right) \in L^p(R)$$

and hence finiteness a.e. But this proves Theorem 2.1 since to every $w(t)$ satisfying (5) it is possible to find $\tilde{w}(t) = o(w(t))$ which also satisfies (5).

For applications, first consider the kernel K_γ given by its Fourier transform $\hat{K}_\gamma(|\xi|) = m(|\xi|)$, $m(u) = e^{-\gamma^\gamma}$, which generates the Abel–Cartwright means $\{W_t^\gamma\}$. By [15], the kernel K_γ has at least a decrease of $C\,|x|^{-2-(\gamma-1)/2}$ at infinity, so for $\gamma > 1$ these means satisfy the conditions of Theorem 2.1.

Next discuss the general Riesz means $\{R_t^{\alpha,\gamma}\}$, generated by $m(u) = (1 - u^\gamma)_+^\alpha$, $\gamma > 0$, $\alpha > 0$, with $K_{\alpha,\gamma}$ as associated kernel. By [15] the following estimate holds for large $|x|$: $K_{\alpha,\gamma}(x) \leqslant C(|x|^{-1-\gamma} + |x|^{-1-\alpha})$. Thus, for $\gamma, \alpha > 1$, the hypotheses of Theorem 2.1 are satisfied.

Summarizing, we have established the following

Corollary 2.1.1. *Let $\omega(t)$ be a modulus of continuity with $\omega(t)/t \uparrow \infty$, $t \to 0+$; let the sequence $\{\delta_k\}$ be defined by* (4) *and $w(t)$ be a nondecreasing function which satisfies* (5) *and has the property that $\omega(t)/w(t)$ is nondecreasing. Then for every $f \in H_p^\omega(R)$, $1 < p < \infty$, the estimate* (1) *is true for the general Riesz means $\{R_t^{\alpha,\gamma}\}$ with parameters $\alpha > 1, \gamma > 1$, and for the Abel–Cartwright means $\{W_t^\gamma\}$ of order $\gamma > 1$. In particular,* (1) *is valid for the Gauss–Weierstrass means.*

Remark 2.1.2. Though the Abel–Poisson means $\{P_t\}$ (i.e., $\{W_t^\gamma\}$ with $\gamma = 1$) are not admitted in Corollary 2.1.1, we can derive an analogous result.

Corollary 2.1.2 (see [12]). *Assume that the hypotheses of Corollary* 2.1.1 *hold. Then for $f \in H_p^\omega(R)$, $1 < p < \infty$,*

$$P_t f(x) - f(x) = o_x(w(t)) \qquad \text{a.e.}$$

Proof. First assume f to be real-valued. Since the Hilbert transformation is continuous on $L^p(R)$, $1<p<\infty$, we have $F:=f+iHf\in H_p^\omega(R)$. Define a kernel K by its Fourier transform $\hat{K}(\xi)=\Lambda(\xi)\,e^{-\xi}$ where the C^∞-function $\Lambda(\xi)=1$, $\xi\geqslant 0$ and $=0$ for $\xi\leqslant -1$. Since the (distributional) Fourier transform of F vanishes on the negative half-line one has $P_tF(x)=K_t * F(x)$ a.e. But $K\in S$, the set of rapidly decreasing, smooth functions, certainly satisfies the hypotheses of Theorem 2.1. Thus, with $w(t)$ as in Theorem 2.1,

$$P_tF(x)-F(x)=o_x(w(t)) \qquad \text{a.e.}$$

Obviously, this also holds for the real part giving the assertion for real-valued functions. The case of complex-valued f follows by applying this procedure separately to the real part of f and to its imaginary part.

2.2. *Convolution Means with Slowly Decreasing Kernels and Good Approximation Behavior*

THEOREM 2.2. *Let $m\in L^\infty(0,\infty)$ be such a function that for some smooth cut-off function ψ, $\psi(u)=1$ for $u\leqslant 1/2$ and $=0$ for $u\geqslant 3/4$ the function $(m\psi)^\wedge$ satisfies the conditions of the previous theorem and the Fourier transform of the function $m(1-\psi)$ has a summable radial majorant. Let $\omega(t)$ be a modulus of continuity and define θ_k such that $\omega(\theta_k)=2^{-k}$. Let $w(t)$ be a nondecreasing function with the properties that $\omega(t)/w(t)$ is nondecreasing and*

$$\sum_{k=1}^{\infty}\left(\frac{\omega(\theta_k)}{w(\theta_k)}\right)^p<\infty. \tag{12}$$

Then, for every function $f\in H_p^\omega(R)$, $1<p<\infty$, the estimate (1) *is true, i.e.,*

$$T_{m_t}f(x)-f(x)=o_x(w(t)) \text{ a.e.} \qquad \text{when} \quad t\to 0+. \tag{1}$$

It will become clear from the proof that Theorem 2.2 remains valid, if the function ψ is replaced by an arbitrary C_0^∞-bump-function which is equal to 1 on some neighborhood of the origin.

Proof. We represent

$$T_{m_t}f-f=T_{(m_t-1)\,\psi_t}f+T_{(m_t-1)(1-\psi_t)}f \tag{13}$$

and first consider the contribution given by the multiplier $(m-1)(1-\psi)$. By the choice of ψ we have

$$\operatorname{supp}(m-1)(1-\psi)\subset\{\xi\colon |\xi|\geqslant 1/2\}.$$

Introduce $\chi \in C_0^\infty$ such that $\chi(\xi)=1$ for $|\xi| \leqslant 1/2$ and $\chi(\xi)=0$ for $|\xi| \geqslant 1$. Then, using the properties of the well-known de la Vallée Poussin sums,

$$\|T_{(1-\chi_t)} f\|_p = \|f - T_{\chi_t} f\|_p \leqslant c_p(\chi)\, \omega(f, t)_p, \qquad 0 < t < 1. \tag{14}$$

Now assume $\theta_{k+1} < t \leqslant \theta_k$. Since

$$\operatorname{supp}(m_t - 1)(1 - \psi_t) \subset \left\{\xi \colon |\xi| \geqslant \frac{1}{2\theta_k}\right\},$$

$$1 - \chi_{2\theta_k}(\xi) = 1 \qquad \text{for} \quad |\xi| \geqslant \frac{1}{2\theta_k},$$

we have

$$T_{(m_t - 1)(1-\psi_t)} f = T_{(m_t - 1)(1-\psi_t)(1-\chi_{2\theta_k})} f = T_{(m_t-1)(1-\psi_t)}(T_{1-\chi_{2\theta_k}} f).$$

This implies the key relation

$$|T_{(m_t-1)(1-\psi_t)} f(x)| \leqslant C \sup_{k \geqslant 0} \frac{T^*_{(m-1)(1-\psi)}(T_{1-\chi_{2\theta_k}} f)(x)}{w(\theta_{k+1})}\, w(t), \qquad t > 0, \tag{15}$$

for almost all $x \in R$. But

$$T_{(m_t-1)(1-\psi_t)} f = T_{m_t(1-\psi_t)} f + T_{\psi_t} f - f,$$

so

$$T^*_{(m-1)\,1-\psi)} f \leqslant T^*_{m(1-\psi)} f + T^*_\psi f + |f|.$$

Now $T^*_{m(1-\psi)} \colon L^p \to L^p$ is a bounded maximal operator since it is generated via convolution by a kernel which by hypothesis has a summable radial majorant. By the same argument, $T^*_\psi \colon L^p \to L^p$ is bounded since ψ is a smooth function with compact support. So we may conclude that $\|T^*_{(m-1)\,(1-\psi)}\|_{p \to p} < \infty$ and, therefore,

$$\left\| \sup_{k \geqslant 0} \frac{T^*_{(m-1)(1-\psi)}(T_{1-\chi_{2\theta_k}} f)}{w(\theta_{k+1})} \right\|_p^p \leqslant C \sum_{k \geqslant 0} \frac{\|T^*_{(m-1)(1-\psi)}(T_{1-\chi_{2\theta_k}} f)\|_p^p}{w(\theta_{k+1})^p}$$

$$\leqslant C\, \|T^*_{(m-1)\,(1-\psi)}\|_{p \to p}^p \sum_{k \geqslant 0} \frac{\|T_{1-\chi_{2\theta_k}} f\|_p}{w(\theta_{k+1})^p}.$$

The estimate (14) gives $\| T_{1-\chi_{2\theta_k}} f \|_p \leqslant C\omega(\theta_k) \leqslant C\omega(\theta_{k+1})$, thus

$$\left\| \sup_{k \geqslant 0} \frac{T^*_{(m-1)(1-\psi)}(T_{1-\chi_{2\theta_k}} f)}{w(\theta_{k+1})} \right\|_p^p \leqslant C_p(m, \psi) \sum_{k=1}^{\infty} \left(\frac{\omega(\theta_k)}{w(\theta_k)} \right)^p,$$

which is finite by the assumption (12) of the theorem.

Moreover, since $\omega(t)/w(t)$ is increasing and $\delta_k \leqslant \theta_k$, δ_k given by (4), the hypothesis (12) implies (5) and, therefore, the first term of the right side in the decomposition (13) can be estimated by Theorem 2.1 which completes the proof of Theorem 2.2.

Note that the multiplier of the general Riesz means $\{R_t^{\alpha, \gamma}\}$, namely $(1-|\xi|^{\gamma})_+^{\alpha}$, satisfies the conditions of Theorem 2.2 with $\gamma > 1, \alpha > 0$. In particular we obtain the following estimates for the Bochner–Riesz means $\{R_t^{\alpha, 2}\}$ of order $\alpha > 0$.

COROLLARY 2.2.1. *Let $\omega(t)\, t^{-s}$, $s > 0$, be an increasing function satisfying*

$$\int_0^1 (t^{-s}\omega(t))^p \frac{dt}{t} < \infty$$

for some p, $1 < p < \infty$. Then, for every $f \in H_p^{\omega}$ and for almost all $x \in R$,

$$R_t^{\alpha, 2} f(x) - f(x) = o_x(t^s), \qquad t \to 0+.$$

COROLLARY 2.2.2. *If $\omega(t) = t^{\lambda}$, $0 < \lambda \leqslant 1$, then for every $f \in H_p^{\omega}(R)$ and for any $\varepsilon > 0$*

$$R_t^{\alpha, 2} f(x) - f(x) = o_x\left(t^{\lambda} \left(\log \frac{1}{t} \right)^{1/p + \varepsilon} \right), \qquad t \to 0+,$$

a.e. on R.

COROLLARY 2.2.3. *If $\omega(t) = (\log 1/t)^{-\lambda}$, $\lambda > 0$, then for every $f \in H_p^{\omega}(R)$ and for any $\varepsilon > 0$*

$$R_t^{\alpha, 2} f(x) - f(x) = o_x\left(\left(\log \frac{1}{t} \right)^{-\lambda} \left(\log\log \frac{1}{t} \right)^{1/p + \varepsilon} \right), \qquad t \to 0+,$$

a.e. on R.

2.3. *Fourier Multiplier Means with Saturation Order Worse than or Equal to $O(t)$*

THEOREM 2.3. *Let ψ be the cut-off function from Theorem 2.2; let $m(u)$ be such a function that the Fourier transform of $(1-\psi)m$ has a summable* *radial majorant. Let $\omega(t)$ be a modulus of continuity and define θ_k such that $\omega(\theta_k)=2^{-k}$. Also, for every $j=0, 1, ...,$ set*

$$A_j \equiv 2^{-j}\, \|m'(2^{-j}\cdot)\chi_{[1/2;\,2]}(\cdot)\|_\infty + \|(1-m(2^{-j}\cdot))\chi_{[1/2;\,2]}(\cdot)\|_\infty .$$

Then, for every nondecreasing function $w(t)$ with $\omega(t)/w(t)$ nondecreasing and

$$\sum_{j=1}^{\infty} A_j \left(\sum_{k=1}^{\infty}\left(\frac{\omega(\theta_k)}{w(2^{-j}\theta_k)}\right)^p\right)^{1/p} < \infty \tag{16}$$

and for every function $f \in H_p^\omega(R)$, $1<p<\infty$, the estimate (1) *holds.*

Proof. We start with the decomposition (13). To estimate its first term $T_{(m_t-1)\psi_t} f$ we make use of the idea of "changing variables" which has been used by Müller and Wang [8].

Set $\mu \equiv (m-1)\psi$ and remember that $\mu=0$ for $|\xi| \geqslant 3/4$ and $\mu(\xi)=m(\xi)-1$ for $|\xi|\leqslant 1/2$. Choose $h \in C_0^\infty$ such that $\operatorname{supp}(h) \subset [1/2; 2]$ and

$$\sum_{j=0}^{\infty} h(2^j t)=1 \qquad \text{for} \quad |t|<1.$$

Without loss of generality we may assume $\theta_0=1$ and then obtain

$$\sup_{0<t<1} \frac{|T_{\mu_t} f|}{w(t)} \leqslant \sum_{j=0}^{\infty} \sup_{0<t<1} \frac{|T_{\mu_t h_{t2^j}} f|}{w(t)} = \sum_{j=0}^{\infty} \sup_{0<\tau\leqslant 2j} \frac{|T_{h_\tau \mu_{\tau 2^{-j}}} f|}{w(\tau 2^{-j})}$$

$$\leqslant \sum_{j=0}^{\infty} \sup_{k\geqslant 0} \sup_{\theta_{k+1}<\tau\leqslant\theta_k} \frac{|T_{h_\tau \mu_{\tau 2^{-j}}} f|}{w(\tau 2^{-j})} + \sum_{j=0}^{\infty} \sup_{1<\tau\leqslant 2^j} \frac{|T_{h_\tau \mu_{\tau 2^{-j}}} f|}{w(2^{-j})}. \tag{17}$$

We begin to discuss the critical first sum on the right side of (17). It is clear that $\theta_{k+1}<\tau\leqslant\theta_k$ implies $\operatorname{supp}(h_\tau)\subset[(2\theta_k)^{-1}, 2(\theta_{k+1})^{-1}]$. Now choose a non-decreasing function $\chi\in C^\infty$ such that $\chi(\xi)=1$ for $|\xi|\geqslant 1/2$ and $\chi(\xi)=0$ for $|\xi|\leqslant 1/4$; set $\chi_k(\xi)=\chi(\theta_k\xi)$. Hence $\chi_k(\xi)=1$ for $|\xi|\geqslant(2\theta_k)^{-1}$ and $\chi_k(\xi)=0$ for $|\xi|\leqslant(4\theta_k)^{-1}$.

Then we have $\chi_k h_\tau \equiv h_\tau$ for $\theta_{k+1}<\tau\leqslant\theta_k$ and thus

$$\left\| \sum_{j=0}^{\infty} \sup_{k\geqslant 0} \frac{\sup_{\theta_{k+1}<\tau\leqslant\theta_k} |T_{h_\tau\mu_{\tau 2^{-j}}} f|}{w(2^{-j}\theta_{k+1})} \right\|_p$$

$$\leqslant \sum_{j=0}^{\infty} \left\| \sup_{k\geqslant 0} \frac{T^*_{h\mu_{2^{-j}}}(T_{\chi_k} f)}{w(2^{-j}\theta_{k+1})} \right\|_p$$

$$\leqslant \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} \left(\frac{\|T^*_{h\mu_{2^{-j}}}(T_{\chi_k} f)\|_p}{w(2^{-j}\theta_{k+1})} \right)^p \right)^{1/p}$$

$$\leqslant \sum_{j=0}^{\infty} \|T^*_{h\mu_{2^{-j}}}\|_{p\to p} \left(\sum_{k=0}^{\infty} \left(\frac{\|T_{\chi_k} f\|_p}{w(2^{-j}\theta_{k+1})} \right)^p \right)^{1/p}$$

$$\leqslant C \sum_{j=0}^{\infty} \|T^*_{h\mu_{2^{-j}}}\|_{p\to p} \left(\sum_{k=0}^{\infty} \left(\frac{\omega(\theta_k)}{w(2^{-j}\theta_{k+1})} \right)^p \right)^{1/p}$$

since $\|T_{\chi_k} f\|_p = \|T_{1-\chi_k} f - f\|_p \leqslant C_\chi \omega(f, \theta_k)_p$ and the symbol of the involved approximate identity, namely $(1-\chi_k)$, is a smooth function with compact support. Thus the first series on the right hand side of (17) is finite almost everywhere provided that $\|T^*_{h\mu_{2^{-j}}}\|_{p\to p} \leqslant CA_j$. The same holds true for the second series on the right side of (17), since certainly

$$\frac{1}{w(2^{-j})} \leqslant C \left(\sum_{k=0}^{\infty} \left(\frac{\omega(\theta_k)}{w(2^{-j}\theta_{k+1})} \right)^p \right)^{1/p}$$

uniformly in j.

Thus there remains to estimate $\|T^*_{h\mu_{2^{-j}}}\|_{p\to p}$. Note that $\psi_{2^{-j}}(x) = 1$ for $0 \leqslant |x| \leqslant 2^{j-1}$, $j \geqslant 2$, and, therefore, $\mu_{2^{-j}} h = (m_{2^{-j}} - 1)\, \psi_{2^{-j}} h = (m_{2^{-j}} - 1) h$, $j \geqslant 2$, thus

$$T^*_{\mu_{2^{-j}} h} f = T^*_{(m_{2^{-j}} - 1) h} f, \qquad j \geqslant 2.$$

For the estimation of $\|T^*_{h(1-m_{2^{-j}})}\|^p_{p\to p}$ we use the Dappa–Trebels estimate (3). Choose in (3) as differentiation order $\lambda = 1$ and exchange the function m by $(1-m_{2^{-j}})h$. Then $\operatorname{supp}((1-m_{2^{-j}})\, h) \subset [1/2; 2]$ and, therefore,

$$\begin{aligned}\int_0^\infty u|m'(u)|^2\, du &\leqslant \int_{1/2}^2 u\, |m'(2^{-j}u)\, 2^{-j}|^2\, du\, \|h\|^2_\infty \\ &\quad + \int_{1/2}^2 u\, |1 - m(2^{-j}u)|^2\, du\, \|h'\|^2_\infty \\ &\leqslant C (2^{-j})^2\, \|m'(2^{-j}\cdot)\, \chi_{[1/2;\, 2]}\|^2_\infty \\ &\quad + C\, \|(1 - m(2^{-j}\cdot))\, \chi_{[1/2;\, 2]}\|^2_\infty \leqslant CA_j^2.\end{aligned}$$

Further,

$$\begin{aligned}\int_0^\infty |m'(u)|\, du &\leqslant \int_{1/2}^2 |m'(2^{-j}u)\, 2^{-j} h(u)|\, du + \int_{1/2}^2 |(1 - m(2^{-j}u))\, h'(u)|\, du \\ &\leqslant C\, \|h\|_\infty\, 2^{-j}\, \|m'(2^{-j}\cdot)\, \chi_{[1/2;\, 2]}\|_\infty \\ &\quad + C\, \|h'\|_\infty\, \|(1 - m(2^{-j}\cdot))\, \chi_{[1/2;\, 2]}\|_\infty \leqslant CA_j.\end{aligned}$$

Thus the condition (3) from Theorem B, namely $B_1((1-m_{2^{-j}})h) \leqslant CA_j$ is satisfied and Theorem B gives the required estimate. Theorem 2.3 is hence established.

Remark 2.3.1. If one takes $m(u)=e^{-u^\gamma}$ one can easily calculate $A_j \leqslant C_\gamma 2^{-j\gamma}$. Hence, for the Abel–Cartwright means the condition (16) specializes to

$$\sum_{j=2}^{\infty} 2^{-j\gamma} \left(\sum_{k=0}^{\infty} (2^{-k}/w(2^{-j}\theta_k))^p \right)^{1/p} < \infty. \tag{18}$$

For the general Riesz means $\{R_t^{\alpha,\gamma}\}$, generated by $m(u)=(1-u^\gamma)_+^\alpha$, one can show the same estimate $A_j \leqslant C2^{-j\gamma}$ provided $0<\gamma\leqslant 1$, $\alpha>0$.

Remark 2.3.2. If we choose $w(t)=t^s$, $0<s<\gamma$ in (18), then

$$\begin{aligned}\sum_{j=2}^{\infty} 2^{-j\gamma} \left(\sum_{k=0}^{\infty} \left(\frac{\omega(\theta_k)}{(2^{-j}\theta_k)^s} \right)^p \right)^{1/p} &\leqslant C_{p,s} \left(\sum_{k=0}^{\infty} \omega(\theta_k)^p\, \theta_k^{-sp} \right)^{1/p} \\ &\leqslant C_{p,s} \left(\int_0^1 \left(\frac{\omega(t)}{t^s} \right)^p \frac{dt}{t} \right)^{1/p}.\end{aligned}$$

So the left hand side of the condition (16) for $w(t)=t^s$ in fact is the semi-norm of a function f in the Besov space $B^s_{p,p}$ (see [16]) which illuminates the nature of (16). In particular, Corollary 2.2.1 is also true for the Abel–Cartwright means and the general Riesz means with parameter $\gamma \leqslant 1$.

We formulate some special cases for the Abel–Cartwright means $\{W_t^\gamma\}$ in the case $0<\gamma<1$.

Corollary 2.3.1. *If* $\omega(t)=t^\lambda$, $0<\lambda\leqslant 1$, *then, for every* $f \in H_p^\omega(R)$, $1<p<\infty$, *and for any* $\varepsilon>0$, *there holds a.e. on* R

$$W_t^\gamma f(x) - f(x) = \begin{cases} o_x(t^\lambda(\log(1/t))^{1/p+\varepsilon}), & 0<\lambda<\gamma, \\ o_x(t^\gamma(\log(1/t))^{1+1/p+\varepsilon}), & \lambda=\gamma, \\ O_x(t^\gamma), & \lambda>\gamma, \quad t\to 0+. \end{cases}$$

Indeed, if we choose $w(t)=t^\lambda(\log(1/t))^\delta$ with δ to be specified later, the left hand side of (18) can be rewritten as

$$\sum_{j=2}^{\infty} 2^{-j\gamma} \left(\sum_{k=0}^{\infty} \left(\frac{2^{-k}}{(2^{-j}2^{-k/\lambda})^{\lambda} (j+k/\lambda)^{\delta}} \right)^{p} \right)^{1/p}$$
$$\leqslant \sum_{j=2}^{\infty} 2^{-j(\gamma-\lambda)} \left(\sum_{k=0}^{\infty} (j+k/\lambda)^{-\delta p} \right)^{1/p}.$$

The double series converges if $\lambda < \gamma$, $\delta > 1/p$ or $\lambda = \gamma$, $\delta > 1 + 1/p$.

The last claim of Corollary 2.3.1 does not follow from Theorem 2.3 since (18) cannot be verified. Intuitively it is clear that one cannot expect a better convergence rate than $O(t^{\gamma})$ since the optimal norm-approximation order of the Abel–Cartwright means $\{W_t^{\gamma}\}$ is $O(t^{\gamma})$ (see [1, p. 466]). Now observe that $f \in H_p^{t^{\lambda}}(R)$ implies $D^{\gamma} f \in L^p(R)$, $\gamma < \lambda$, which follows, e.g., from [1, Theorem 12.3.11] (or see [16]). Then it is easy to check that $m(u) = (1 - e^{-u^{\gamma}})\, u^{-\gamma}$ satisfies (3) and hence, by Remark 1.1(a), the assertion.

By an argument, similar to that showing the first two assertions of Corollary 2.3.1, we obtain

COROLLARY 2.3.2. *If* $\omega(t) = (\log(1/t))^{-\lambda}$, $\lambda > 0$, *then for every* $f \in H_p^{\omega}(R)$ *and for any* $\varepsilon > 0$

$$W_t^{\gamma} f(x) - f(x) = o_x \left(\left(\log \frac{1}{t} \right)^{-\lambda} (\log \log(1/t))^{1/p+\varepsilon} \right), \qquad t \to 0+,$$

a.e. on R.

Finally let us indicate the L^1-case. As the maximal function arises from a convolution with a kernel which has a summable radial majorant it is of weak type (1,1). Therefore, instead of the norm estimates in the proofs of Theorem 2.1 and Theorem 2.2 we can apply the weak type estimates. So Theorems 2.1 and 2.2 are still true also for $p = 1$. The difference between the case $p = 1$ and the case $p > 1$ is only that the constant o_x on the right hand side of (1) as a function of x may be chosen as an L^p-function for $p > 1$, but for $p = 1$ only as a function from weak L^1.

Moreover, by the Dappa–Trebels Theorem B we have (1,1) weak type estimates for the corresponding maximal operators; in particular, we can estimate the Lebesgue measure of

$$E_{\alpha} \equiv \left\{ x \in R \colon \sum_{j=2}^{\infty} \sup_{k \geqslant 0} \frac{T^{*}_{h_{\mu_{2^{-j}}}}(T_{\chi_k} f)}{w(2^{-j}\theta_{k+1})} > \alpha \right\}.$$

To this end, choose $c_j > 0$ such that

$$\sum_{j=2}^{\infty} \frac{1}{c_j} = 1. \tag{19}$$

Then

$$|E_\alpha| \leqslant \sum_{j=2}^{\infty} \left| \left\{ x\colon \sup_{k \geqslant 0} \frac{T^*_{h\mu_{2^{-j}}}(T_{\chi_k} f)}{w(2^{-j}\theta_{k+1})} > \frac{\alpha}{c_j} \right\} \right|$$
$$\leqslant \sum_{j=2}^{\infty} \sum_{k=0}^{\infty} \left| \left\{ x\colon \frac{T^*_{h\mu_{2^{-j}}}(T_{\chi_k} f)}{w(2^{-j}\theta_{k+1})} > \frac{\alpha}{c_j} \right\} \right|$$
$$\leqslant \frac{C}{\alpha} \sum_{j=2}^{\infty} c_j \, \|T^*_{h\mu_{2^{-j}}}\|_{L \to L_{1,\infty}} \sum_{k=0}^{\infty} \frac{\omega(\theta_k)}{w(2^{-j}\theta_{k+1})}.$$

Thus, Theorem 2.3 remains true also for the case $p=1$ when we replace (16) by the condition (for the definition of $B_\lambda(m)$ see (3))

$$\sum_{j=1}^{\infty} c_j B_\lambda((1-m_{2^{-j}})\,h)) \sum_{k=0}^{\infty} \frac{\omega(\theta_k)}{w(2^{-j}\theta_{k+1})} < \infty, \qquad \text{some} \quad \lambda > 1, \tag{20}$$

which of course is more restrictive than (18). If we consider the Abel–Cartwright means $\{W^\gamma_t\}$, for which $m(t) = e^{-t^\gamma}$, $\gamma > 0$, and take $\lambda = 2$, then $B_2(\cdot) \leqslant C2^{-j\gamma}$. With $c_j = Cj^2$ we thus obtain

Corollary 2.3.3. *If* $\omega(t) = t^\lambda$, $0 < \lambda < \gamma < 1$, *then, for every* $f \in H^\omega_1(R)$ *and for any* $\varepsilon > 0$, *there holds*

$$W^\gamma_t f(x) - f(x) = o_x(t^\lambda(\log(1/t))^{1+\varepsilon}), \qquad t \to 0+,$$

a.e. on R.

ACKNOWLEDGMENTS

The authors thank I. Shevchuk for helpful discussions around the case $\omega(t) = t$ and the referees for their careful reading of the paper.

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