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Structure of derivations on various algebras of measurable operators for type I von Neumann algebras

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Abstract

Given a von Neumann algebra M denote by $S(M)$ and $LS(M)$ respectively the algebras of all measurable and locally measurable operators affiliated with M. For a faithful normal semi-finite trace τ on M let $S(M,\tau)$ be the algebra of all τ-measurable operators from $S(M)$. We give a complete description of all derivations on the above algebras of operators in the case of type I von Neumann algebra M. In particular, we prove that if M is of type I_∞ then every derivation on $LS(M)$ (resp. $S(M)$ and $S(M,\tau)$) is inner.

Keywords: Von Neumann algebras; Noncommutative integration; Measurable operator; Locally measurable operator; τ-Measurable operator; Type I von Neumann algebra; Derivation; Inner derivation

Introduction

Derivations on unbounded operator algebras, in particular on various algebras of measurable operators affiliated with von Neumann algebras, appear to be a very attractive special case of the general theory of unbounded derivations on operator algebras. The present paper continues the

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series of papers of the authors [1,2] devoted to the study and a description of derivations on the algebra $LS(M)$ of locally measurable operators with respect to a von Neumann algebra M and on various subalgebras of $LS(M)$.

Let A be an algebra over the complex number. A linear operator $D : A \to A$ is called a *derivation* if it satisfies the identity $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$ (Leibniz rule). Each element $a \in A$ defines a derivation D_a on A given as $D_a(x) = ax - xa$, $x \in A$. Such derivations D_a are said to be *inner derivations*. If the element a implementing the derivation D_a on A, belongs to a larger algebra B, containing A (as a proper ideal as usual) then D_a is called a *spatial derivation*.

In the particular case where A is commutative, inner derivations are identically zero, i.e. trivial. One of the main problems in the theory of derivations is automatic innerness or spatialness of derivations and the existence of noninner derivations (in particular, nontrivial derivations on commutative algebras).

On this way A.F. Ber, F.A. Sukochev, V.I. Chilin [3] obtained necessary and sufficient conditions for the existence of nontrivial derivations on commutative regular algebras. In particular they have proved that the algebra $L^0(0, 1)$ of all (classes of equivalence of) complex measurable functions on the interval $(0, 1)$ admits nontrivial derivations. Independently A.G. Kusraev [13] by means of Boolean-valued analysis has established necessary and sufficient conditions for the existence of nontrivial derivations and automorphisms on universally complete complex f-algebras. In particular he has also proved the existence of nontrivial derivations and automorphisms on $L^0(0, 1)$. It is clear that these derivations are discontinuous in the measure topology, and therefore they are neither inner nor spatial. It seems that the existence of such pathological example of derivations deeply depends on the commutativity of the underlying von Neumann algebra M. In this connection the present authors have initiated the study of the above problems in the noncommutative case [1,2], by considering derivations on the algebra $LS(M)$ of all locally measurable operators with respect to a semi-finite von Neumann algebra M and on various subalgebras of $LS(M)$. Recently another approach to similar problems in the framework of type I AW^*-algebras has been outlined in [7].

The main result of the paper [2] states that if M is a type I von Neumann algebra, then every derivation D on $LS(M)$ which is identically zero on the center Z of the von Neumann algebra M (i.e. which is Z-linear) is automatically inner, i.e. $D(x) = ax - xa$ for an appropriate $a \in LS(M)$. In [2, Example 3.8] we also gave a construction of noninner derivations D_δ on the algebra $LS(M)$ for type $\mathrm{I_{fin}}$ von Neumann algebra M with nonatomic center Z, where δ is a nontrivial derivation on the algebra $LS(Z)$ (i.e. on the center of $LS(M)$) which is isomorphic with the algebra $L^0(\Omega, \Sigma, \mu)$ of all measurable functions on a nonatomic measure space (Ω, Σ, μ).

The main idea of the mentioned construction is the following.

Let A be a commutative algebra and let $M_n(A)$ be the algebra of $n \times n$ matrices over A. If $e_{i,j}$, $i, j = \overline{1, n}$, are the matrix units in $M_n(A)$, then each element $x \in M_n(A)$ has the form

$$x = \sum_{i,j=1}^{n} \lambda_{ij} e_{ij}, \quad \lambda_{i,j} \in A, \quad i, j = \overline{1, n}.$$

Let $\delta : A \to A$ be a derivation. Setting

$$D_\delta\left(\sum_{i,j=1}^{n} \lambda_{ij} e_{ij}\right) = \sum_{i,j=1}^{n} \delta(\lambda_{ij}) e_{ij} \tag{1}$$

we obtain a well-defined linear operator D_δ on the algebra $M_n(A)$. Moreover D_δ is a derivation on the algebra $M_n(A)$ and its restriction onto the center of the algebra $M_n(A)$ coincides with the given δ.

In [1] we have proved spatialness of all derivations on the noncommutative Arens algebra $L^\omega(M,\tau)$ associated with an arbitrary von Neumann algebra M and a faithful normal semi-finite trace τ. Moreover if the trace τ is finite then every derivation on $L^\omega(M,\tau)$ is inner.

In the present paper we give a complete description of all derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a type I von Neumann algebra M, and also on its subalgebras $S(M)$—of measurable operators and $S(M,\tau)$ of τ-measurable operators, where τ is a faithful normal semi-finite trace on M. We prove that the above mentioned construction of derivations D_δ from [2] gives the general form of pathological derivations on these algebras and these exist only in the type $\mathrm{I}_{\mathrm{fin}}$ case, while for type I_∞ von Neumann algebras M all derivations on $LS(M)$, $S(M)$ and $S(M,\tau)$ are inner. Moreover we prove that an arbitrary derivation D on each of these algebras can be uniquely decomposed into the sum $D = D_a + D_\delta$ where the derivation D_a is inner (for $LS(M)$, $S(M)$ and $S(M,\tau)$) while the derivation D_δ is constructed in the above mentioned manner from a nontrivial derivation δ on the center of the corresponding algebra.

In Section 1 we give necessary definition and preliminaries from the theory of measurable operators, Hilbert–Kaplansky modules and also prove some key results concerning the structure of the algebra of locally measurable operators affiliated with a type I von Neumann algebra.

In Section 2 we describe derivations on the algebra $LS(M)$ of all locally measurable operators for a type I von Neumann algebra M.

Sections 3 and 4 are devoted to derivation respectively on the algebra $S(M)$ of all measurable operators and on the algebra $S(M,\tau)$ of all τ-measurable operators with respect to M, where M is a type I von Neumann algebra and τ is a faithful normal semi-finite trace on M.

Finally, Section 5 contains an application of the above results to the description of the first cohomology group for the considered algebras.

1. Locally measurable operators affiliated with type I von Neumann algebras

Let H be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. Consider a von Neumann algebra M in $B(H)$ with the operator norm $\|\cdot\|_M$. Denote by $P(M)$ the lattice of projections in M.

A linear subspace $\mathcal{D}$ in H is said to be *affiliated* with M (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for every unitary u from the commutant

$$M' = \{y \in B(H):\ xy = yx,\ \forall x \in M\}$$

of the von Neumann algebra M.

A linear operator x on H with the domain $\mathcal{D}(x)$ is said to be *affiliated* with M (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$.

A linear subspace $\mathcal{D}$ in H is said to be *strongly dense* in H with respect to the von Neumann algebra M, if

(1) $\mathcal{D}\eta M$;
(2) there exists a sequence of projections $\{p_n\}_{n=1}^\infty$ in $P(M)$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathcal{D}$ and $p_n^\perp = \mathbf{1} - p_n$ is finite in M for all $n \in \mathbb{N}$, where $\mathbf{1}$ is the identity in M.

A closed linear operator x acting in the Hilbert space H is said to be *measurable* with respect to the von Neumann algebra M, if $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in H. Denote by $S(M)$ the set of all measurable operators with respect to M.

A closed linear operator x in H is said to be *locally measurable* with respect to the von Neumann algebra M, if $x\eta M$ and there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in M such that $z_n \uparrow \mathbf{1}$ and $z_n x \in S(M)$ for all $n \in \mathbb{N}$.

It is well known [14] that the set $LS(M)$ of all locally measurable operators with respect to M is a unital $*$-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator.

Let τ be a faithful normal semi-finite trace on M. We recall that a closed linear operator x is said to be τ*-measurable* with respect to the von Neumann algebra M, if $x\eta M$ and $\mathcal{D}(x)$ is τ-dense in H, i.e. $\mathcal{D}(x)\eta M$ and given $\varepsilon > 0$ there exists a projection $p \in M$ such that $p(H) \subset \mathcal{D}(x)$ and $\tau(p^{\perp}) < \varepsilon$. The set $S(M,\tau)$ of all τ-measurable operators with respect to M is a solid $*$-subalgebra in $S(M)$ (see [15]).

Consider the topology t_τ of convergence in measure or *measure topology* on $S(M,\tau)$, which is defined by the following neighborhoods of zero:

$$V(\varepsilon,\delta) = \left\{x \in S(M,\tau)\colon \exists e \in P(M),\ \tau\left(e^{\perp}\right) \leqslant \delta,\ xe \in M,\ \|xe\|_M \leqslant \varepsilon\right\},$$

where ε, δ are positive numbers.

It is well known [15] that $S(M,\tau)$ equipped with the measure topology is a complete metrizable topological $*$-algebra.

Note that if the trace τ is a finite then

$$S(M,\tau) = S(M) = LS(M).$$

The following result describes one of the most important properties of the algebra $LS(M)$ (see [14,16]).

Proposition 1.1. *Suppose that the von Neumann algebra M is the C^*-product of the von Neumann algebras M_i, $i \in I$, where I is an arbitrary set of indices, i.e.*

$$M = \bigoplus_{i\in I} M_i = \left\{\{x_i\}_{i\in I}\colon x_i \in M_i,\ i \in I,\ \sup_{i\in I} \|x_i\|_{M_i} < \infty\right\}$$

with coordinate-wise algebraic operations and involution and with the C^*-norm $\|\{x_i\}_{i\in I}\|_M = \sup_{i\in I} \|x_i\|_{M_i}$. Then the algebra $LS(M)$ is $*$-isomorphic to the algebra $\prod_{i\in I} LS(M_i)$ (with the coordinate-wise operations and involution), i.e.

$$LS(M) \cong \prod_{i\in I} LS(M_i)$$

($\cong$ *denoting $*$-isomorphism of algebras*). *In particular, if M is a finite, then*

$$S(M) \cong \prod_{i\in I} S(M_i).$$

It should be noted that similar isomorphisms are not valid in general for the algebras $S(M)$, $S(M,\tau)$ (see [14]).

Proposition 1.1 implies that given any family $\{z_i\}_{i\in I}$ of mutually orthogonal central projections in M with $\bigvee_{i\in I} z_i = \mathbf{1}$ and a family of elements $\{x_i\}_{i\in I}$ in $LS(M)$ there exists a unique element $x \in LS(M)$ such that $z_i x = z_i x_i$ for all $i \in I$. This element is denoted by $x = \sum_{i\in I} z_i x_i$.

Further in this section we shall prove several crucial results concerning the properties of algebras of measurable operators for type I von Neumann algebras. In particular we present an alternative and shorter proof of the statement that the algebra of locally measurable operators in this case is isomorphic to the algebra of bounded operators acting on a Hilbert–Kaplansky module (cf. [2]).

It is well known [18] that every commutative von Neumann algebra M is $*$-isomorphic to the algebra $L^\infty(\Omega) = L^\infty(\Omega, \Sigma, \mu)$ of all (classes of equivalence of) complex essentially bounded measurable functions on a measure space (Ω, Σ, μ) and in this case $LS(M) = S(M) \cong L^0(\Omega)$, where $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ the algebra of all (classes of equivalence of) complex measurable functions on (Ω, Σ, μ).

Further we shall need the description of the centers of the algebras $S(M)$ and $S(M,\tau)$ for type I von Neumann algebras.

It should be noted, that if M is a finite von Neumann algebra with a faithful normal semi-finite trace τ, then the restriction τ_Z of the trace τ onto the center Z of M is also semi-finite.

Indeed by [19, Chapter V, Theorem 2.6] M admits the canonical center valued trace $T: M \to Z$. It is known that $T(x) = \overline{co}\{uxu^*,\ u \in U\} \cap Z$, where $\overline{co}\{uxu^*,\ u \in U\}$ denotes the norm closure in M of the convex hull of the set $\{uxu^*,\ u \in U\}$ and U is the set of all unitaries from M. Therefore given any finite trace ρ (since it is norm-continuous and linear on M) one has $\rho(Tx) = \rho(x)$ for all $x \in M$. Given a normal semi-finite trace τ on M there exists a monotone increasing net $\{e_\alpha\}$ of projection in M with $\tau(e_\alpha) < \infty$ and $e_\alpha \uparrow \mathbf{1}$. The trace $\rho_\alpha(x) = \tau(e_\alpha x)$, $x \geqslant 0$, $x \in M$, is finite for any α and therefore for all $x \in M$, $x \geqslant 0$, we have $\tau(Tx) = \lim_\alpha \tau(e_\alpha Tx) = \lim_\alpha \tau(e_\alpha x) = \tau(x)$. Now given any projection $z \in Z$ there exists a non-zero projection $p \in M$ such that $p \leqslant z$ and $\tau(p) < \infty$. Consider the element $T(p) \in Z$. From properties of T it follows that $T(p)$ is a non-zero positive element in Z with $\tau(T(p)) = \tau(p) < \infty$ and $T(p) \leqslant T(z) = z$. By the spectral theorem there exists a non-zero projection z_0 in Z such that $z_0 \leqslant \lambda T(p)$ for an appropriate positive number λ. Therefore $\tau(z_0) \leqslant \tau(\lambda T(p)) = \lambda\tau(p) < \infty$ and $z_0 \leqslant \lambda z$, i.e. $z_0 \leqslant z$, and thus the restriction of τ onto Z is also semi-finite.

Proposition 1.2. *If M is finite von Neumann algebra of type I with the center Z and a faithful normal semi-finite trace τ, then $Z(S(M)) = S(Z)$ and $Z(S(M,\tau)) = S(Z,\tau_Z)$, where τ_Z is the restriction of the trace τ on Z.*

Proof. Given $x \in S(Z)$ take a sequence of orthogonal projections $\{z_n\}$ in Z such that $z_n x \in Z$ for all n. Since M is finite, one has that $LS(M) = S(M)$ and therefore $x = \sum_n z_n x \in LS(M) = S(M)$, i.e. $x \in Z(S(M))$.

Conversely, let $x \in Z(S(M))$, $x \geqslant 0$ and let $x = \int_0^\infty \lambda\, d\lambda$ be it spectral resolution. Put $z_1 = e_1$ and $z_k = e_k - e_{k-1}$, $k \geqslant 2$. Then $\{z_k\}$ is a family of mutually orthogonal central projections with $\bigvee_k z_k = \mathbf{1}$. It is clear that $z_k x \in Z$ for all k. Therefore $x = \sum_n z_n x \in S(Z)$, and thus $Z(S(M)) = S(Z)$. In a similar way we obtain that $Z(S(M,\tau)) = S(Z,\tau_Z)$. The proof is complete. □

Recall that M is a type I_∞ if M is of type I and does not have non-zero finite central projections.

Proposition 1.3. *Let M be a type I_∞ von Neumann algebra with the center Z. Then the centers of the algebras $S(M)$ and $S(M,\tau)$ coincide with Z.*

Proof. Suppose that $z \in S(M)$, $z \geqslant 0$, is a central element and let $z = \int_0^\infty \lambda\, de_\lambda$ be its spectral resolution. Then $e_\lambda \in Z$ for all $\lambda > 0$. Assume that $e_n^\perp \neq 0$ for all $n \in \mathbb{N}$. Since M is of type I_∞, Z does not contain non-zero finite projections. Thus $e_n^\perp$ is infinite for all $n \in \mathbb{N}$, which contradicts the condition $z \in S(M)$. Therefore there exists $n_0 \in \mathbb{N}$ such that $e_n^\perp = 0$ for all $n \geqslant n_0$, i.e. $z \leqslant n_0\mathbf{1}$. This means that $z \in Z$, i.e. $Z(S(M)) = Z$. Similarly $Z(S(M,\tau)) = Z$. The proof is complete. □

Let M be a von Neumann algebra of type I_n $(n \in \mathbb{N})$ with the center Z. Then M is $*$-isomorphic to the algebra $M_n(Z)$ of $n \times n$ matrices over Z (see [17, Theorem 2.3.3]).

In this case the algebras $S(M,\tau)$ and $S(M)$ can be described in the following way.

Proposition 1.4. *Let M be a von Neumann algebra of type I_n, $n \in \mathbb{N}$, with a faithful normal semi-finite trace τ and let $Z(S(M,\tau))$ denote the center of the algebra $S(M,\tau)$. Then $S(M,\tau) \cong M_n(Z(S(M,\tau)))$.*

Proof. Let $\{e_{ij}\colon i,j \in \overline{1,n}\}$ be matrix units in $M_n(Z)$. Consider the $*$-subalgebra in $S(M,\tau)$ generated by the set

$$Z\big(S(M,\tau)\big) \cup \{e_{ij}\colon i,j \in \overline{1,n}\}.$$

This $*$-subalgebra consists of elements of the form

$$\sum_{i,j=1}^{n} \lambda_{ij} e_{ij}, \quad \lambda_{i,j} \in Z\big(S(M,\tau)\big), \quad i,j = \overline{1,n}$$

and it is $*$-isomorphic with $M_n(Z(S(M,\tau))) \subseteq S(M,\tau)$. Since M is t_τ-dense in $S(M,\tau)$, it is sufficient to show that the subalgebra $M_n(Z(S(M,\tau)))$ is closed in $S(M,\tau)$ with respect to the topology t_τ. The center $Z(S(M,\tau))$ is t_τ-closed in $S(M,\tau)$ and therefore the subalgebra

$$e_{11} Z\big(S(M,\tau)\big) e_{11} = Z\big(S(M,\tau)\big) e_{11},$$

is also t_τ-closed in $S(M,\tau)$.

Consider a sequence $x_m = \sum_{i,j=1}^n \lambda_{ij}^{(m)} e_{ij}$ in $M_n(Z(S(M,\tau)))$ such that $x_m \to x \in S(M,\tau)$ in the topology t_τ. Fixing $k,l \in \overline{1,n}$ we have that $e_{1k} x_m e_{l1} \to e_{1k} x e_{l1}$. Since $e_{1k} x_m e_{l1} = \lambda_{kl}^{(m)} e_{11}$ one has $\lambda_{kl}^{(m)} e_{11} \to e_{1k} x e_{l1}$. The t_τ-closedness of $Z(S(M,\tau)) e_{11}$ in $S(M,\tau)$ implies that

$$\lambda_{kl}^{(m)} e_{11} \to \lambda_{kl} e_{11} \tag{2}$$

for an appropriate $\lambda_{kl} \in Z(S(M,\tau))$. Multiplying (2) by e_{k1} from the left side and by e_{1l} from the right-hand side we obtain that $\lambda_{kl}^{(m)} e_{kl} \to \lambda_{kl} e_{kl}$. Therefore $x_m \to \sum_{i,j=1}^n \lambda_{ij} e_{ij}$ and thus $x = \sum_{i,j=1}^n \lambda_{ij} e_{ij}$. This implies that $S(M,\tau) \cong M_n(Z(S(M,\tau)))$. The proof is complete. □

Proposition 1.5. *Let M be a von Neumann algebra of type I_n, $n \in \mathbb{N}$, and let $Z(S(M))$ denote the center of $S(M)$. Then $S(M) \cong M_n(Z(S(M)))$.*

Proof. Let τ be a faithful normal semi-finite trace on M and consider a family $\{z_\alpha\}$ of mutually orthogonal central projections in M with $\bigvee_\alpha z_\alpha = \mathbf{1}$ and such that $\tau_\alpha = \tau|_{z_\alpha M}$ is finite for every α (such family exists because M is of type I_n, $n < \infty$). Then

$$M = \bigoplus_\alpha z_\alpha M.$$

Since each trace τ_α is finite one has

$$S(z_\alpha M) = S(z_\alpha M, \tau_\alpha) = M_n\big(Z\big(S(z_\alpha M, \tau_\alpha)\big)\big) = M_n\big(Z\big(S(z_\alpha M)\big)\big),$$

i.e.

$$S(z_\alpha M) = M_n\big(Z\big(S(z_\alpha M)\big)\big).$$

This and Proposition 1.1 imply that

$$\begin{aligned} S(M) &\cong \prod_\alpha S(z_\alpha M) = \prod_\alpha M_n\big(Z\big(S(z_\alpha M)\big)\big) \\ &= M_n\left(\prod_\alpha \big(z_\alpha Z\big(S(M)\big)\big)\right) = M_n\big(Z\big(S(M)\big)\big), \end{aligned}$$

i.e.

$$S(M) \cong M_n\big(Z\big(S(M)\big)\big).$$

The proof is complete. □

The last proposition enables us to obtain the following important property of the algebra $LS(M)$ in the case of type I von Neumann algebra M.

Proposition 1.6. *Let M be a type I von Neumann algebra. Then for any element $x \in LS(M)$ there exists a countable family of mutually orthogonal central projections $\{z_k\}_{k\in\mathbb{N}}$ in M such that $\bigvee_k z_k = \mathbf{1}$ and $z_k x \in M$ for all k.*

Proof. *Case 1*. The algebra M has type I_n, $n \in \mathbb{N}$. In this case $LS(M) = S(M)$ and Proposition 1.5 implies that $S(M) \cong M_n(Z(S(M)))$. Consider $x = \sum_{i,j=1}^n \lambda_{ij} e_{ij} \in M_n(Z(S(M)))$. Put $c = \bigvee_{i,j=1}^n |\lambda_{ij}|$. Then $c \in Z(S(M))$ and if $c = \int_0^\infty \lambda\, d\lambda$ is its spectral resolution, put $z_1 = e_1$ and $z_k = e_k - e_{k-1}$, $k \geqslant 2$. Then $\{z_k\}$ is the family of mutually orthogonal central projections with $\bigvee_k z_k = \mathbf{1}$ and by definition $z_k c \in Z$ for all k. Therefore $z_k|\lambda_{ij}| \in Z$ for every $k \in \mathbb{N}$, $1 \leqslant i, j \leqslant n$. Thus $z_k x \in M$ for all k.

Case 2. M is a finite von Neumann algebra of type I. Then there exists a family $\{q_n\}_{n\in F}$, $F\subseteq\mathbb{N}$, of central projections from M with $\sup_{n\in F} q_n=\mathbf{1}$ such that the algebra M is $*$-isomorphic with the C^*-product of von Neumann algebras q_nM of type I_n respectively, $n\in F$, i.e.

$$M\cong\bigoplus_{n\in F} q_nM.$$

By Proposition 1.1 we have that

$$S(M)\cong\prod_{n\in F} S(q_nM).$$

Take $x=\{x_n\}_{n\in F}\in\prod_{n\in F} S(q_nM)$. The case 1 implies that for every $n\in F$ there exists a family $\{z_{n,m}\}$ of mutually orthogonal central projections with $\bigvee_m z_{n,m}=q_n$ and $z_{n,m}x_n\in q_nM$ for all $m\in\mathbb{N}$.

In this case we have the countable family $\{z_{n,k}\}_{(n,k)\in F\times\mathbb{N}}$ of mutually orthogonal central projections with $\bigvee_{(n,k)\in F\times\mathbb{N}} z_{n,k}=\mathbf{1}$ and $z_{n,k}x\in M$ for all $(n,k)\in F\times\mathbb{N}$.

Case 3. M is an arbitrary von Neumann algebra of type I and $x\in S(M)$. Without loss of generality we may assume that $x\geqslant 0$.

Let $x=\int_0^\infty\lambda\, d\lambda$ be the spectral resolution of x. Since $x\in S(M)$ by the definition there exists $\lambda_0>0$ such that $e_{\lambda_0}^{\perp}$ is a finite projection. Thus $e_{\lambda_0}^{\perp}Me_{\lambda_0}^{\perp}$ is a finite von Neumann algebra of type I and $xe_{\lambda_0}^{\perp}\in S(e_{\lambda_0}^{\perp}Me_{\lambda_0}^{\perp})$. From the case 2 we have that there exists a family of mutually orthogonal projections $\{z'_m\}_{m\in\mathbb{N}}$ in $e_{\lambda_0}^{\perp}Me_{\lambda_0}^{\perp}$ such that $\bigvee_{m\geqslant 1} z'_m=e_{\lambda_0}^{\perp}$ and $z'_mxe_{\lambda_0}^{\perp}\in e_{\lambda_0}^{\perp}Me_{\lambda_0}^{\perp}$ for all $m\in\mathbb{N}$. Each central projection z' in $e_{\lambda_0}^{\perp}Me_{\lambda_0}^{\perp}$ has the form $z'=e_{\lambda_0}^{\perp}ze_{\lambda_0}^{\perp}$ for an appropriate central projection $z\in M$. Moreover passing if necessary to $z(e_{\lambda_0}^{\perp})z$ one may chose z with $z\leqslant z(e_{\lambda_0}^{\perp})$, where $z(e_{\lambda_0}^{\perp})$ is the central cover of the projection $e_{\lambda_0}^{\perp}$ in M. Let $z'_m=e_{\lambda_0}^{\perp}z_me_{\lambda_0}^{\perp}$, $m\in\mathbb{N}$. Mutually orthogonality of the family $\{z'_m\}$ then implies the similar property of the corresponding $\{z_m\}$. Denote $z_0=z(e_{\lambda_0}^{\perp})^{\perp}$. Then $\bigvee_{m\geqslant 0} z_m=\mathbf{1}$ and

$$z_0x=z_0xe_{\lambda_0}+z_0xe_{\lambda_0}^{\perp}=z_0xe_{\lambda_0}\in M,$$
$$z_mx=z_mxe_{\lambda_0}+z_mxe_{\lambda_0}^{\perp}=z_mxe_{\lambda_0}+z'_mxe_{\lambda_0}^{\perp}\in M$$

for all $m\in\mathbb{N}$, i.e. $z_mx\in M$ for all $m\geqslant 0$.

Case 4. The general case, i.e. M is a type I von Neumann algebra and $x\in LS(M)$. By the definition there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of mutually orthogonal central projection with $\bigvee_n f_n=\mathbf{1}$ and $f_nx\in S(M)$ for all $n\in\mathbb{N}$. Then the case 3 implies that for each $n\in\mathbb{N}$ there exists a sequence $\{z_{n,m}\}$ of mutually orthogonal central projections with $\bigvee_m z_{n,m}=f_n$ and $z_{n,m}x_n\in f_nM$ for all $m\in\mathbb{N}$.

Now we have that $\{z_{n,k}\}_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ is a countable family of mutually orthogonal central projections with $\bigvee_{(n,k)\in\mathbb{N}\times\mathbb{N}} z_{n,k}=\mathbf{1}$ and $z_{n,k}x\in M$ for all $(n,k)\in\mathbb{N}\times\mathbb{N}$. The proof is complete. □

Now let us recall some notions and results from the theory of Hilbert–Kaplansky modules.

Let (Ω,Σ,μ) be a measure space and let H be a Hilbert space. A map $s:\Omega\to H$ is said to be simple, if $s(\omega)=\sum_{k=1}^n\chi_{A_k}(\omega)c_k$, where $A_k\in\Sigma$, $A_i\cap A_j=\emptyset$, $i\neq j$, $c_k\in H$,

$k = \overline{1, n}$, $n \in \mathbb{N}$. A map $u : \Omega \to H$ is said to be measurable, if for any $A \in \sum$ with $\mu(A) < \infty$ there is a sequence (s_n) of simple maps such that $\|s_n(\omega) - u(\omega)\| \to 0$ almost everywhere on A.

Let $\mathcal{L}(\Omega, H)$ be the set of all measurable maps from Ω into H, and let $L^0(\Omega, H)$ denote the space of all equivalence classes with respect to the equality almost everywhere. Denote by $\hat{u}$ the equivalence class from $L^0(\Omega, H)$ which contains the measurable map $u \in \mathcal{L}(\Omega, H)$. Further we shall identify the element $u \in \mathcal{L}(\Omega, H)$ and the class $\hat{u}$. Note that the function $\omega \to \|u(\omega)\|$ is measurable for any $u \in \mathcal{L}(\Omega, H)$. The equivalence class containing the function $\|u(\omega)\|$ is denoted by $\|\hat{u}\|$. For $\hat{u}, \hat{v} \in L^0(\Omega, H), \lambda \in L^0(\Omega)$ put $\hat{u} + \hat{v} = \widehat{u(\omega) + v(\omega)}$, $\lambda\hat{u} = \widehat{\lambda(\omega)u(\omega)}$. Equipped with the $L^0(\Omega)$-valued inner product

$$\langle x, y\rangle = \bigl\langle x(\omega), y(\omega)\bigr\rangle_H,$$

where $\langle\cdot,\cdot\rangle_H$ in the inner product in H, $L^0(\Omega, H)$ becomes a Hilbert–Kaplansky module over $L^0(\Omega)$. The space

$$L^\infty(\Omega, H) = \bigl\{x \in L^0(\Omega, H) \colon \langle x, x\rangle \in L^\infty(\Omega)\bigr\}$$

is a Hilbert–Kaplansky module over $L^\infty(\Omega)$.

It should be noted that $L^\infty(\Omega, H)$ is a Banach space with respect to the norm $\|x\|_\infty = \|(x, x)^{\frac{1}{2}}\|_{L^\infty(\Omega)}$.

Let us show that if $\dim H = \alpha$ then the Hilbert–Kaplansky module $L^\infty(\Omega, H)$ is α-homogeneous.

Indeed, let $\{\varphi_i\}_{i\in J}$ be an orthonormal basis in H with the cardinality α, and consider the equivalence class $\hat{\varphi}_i$ from $L^\infty(\Omega, H)$ containing the constant vector-function

$$\omega \in \Omega \to \varphi_i.$$

From the definition of the inner-product it is clear that

$$\langle\hat{\varphi}_i, \hat{\varphi}_j\rangle = \delta_{ij}\mathbf{1},$$

where δ_{ij} is the Kroenecker symbol, $\mathbf{1}$ is the identity from $L^\infty(\Omega)$.

Let us show that if $y \in L^\infty(\Omega, H)$ and $\langle\hat{\varphi}_i, y\rangle = 0$ for all $i \in J$ then $y = 0$. Put

$$Sp\{\hat{\varphi}_i\} = \left\{\sum_{k=1}^{n} \lambda_k\hat{\varphi}_{i_k} \colon \lambda_k \in L^\infty(\Omega),\ i_k \in J,\ k = \overline{1, n},\ n \in \mathbb{N}\right\}.$$

Since the set of elements of the form $\sum_{k=1}^n t_k\varphi_{i_k}$, where $t_k \in \mathbb{C}$, $i_k \in J$, $k = \overline{1, n}$, $n \in \mathbb{N}$, is norm dense in H, we have that $\inf\{\|\psi - y\| \colon \psi \in Sp\{\hat{\varphi}_i\}\} = 0$, for each fixed $y \in L^\infty(\Omega, H)$. The set $Sp\{\hat{\varphi}_i\}$ is an $L^\infty(\Omega)$-submodule in $L^\infty(\Omega, H)$, and by [6, Proposition 4.1.5] there exists a sequence $\{\psi_k\}$ in $Sp\{\hat{\varphi}_i\}$ such that $\|\psi_k - y\| \downarrow 0$, i.e. the set $Sp\{\hat{\varphi}_i\}$ is (bo)-dence in $L^\infty(\Omega, H)$.

Now let $y \in L^\infty(\Omega, H)$ be such an element that $\langle\hat{\varphi}_i, y\rangle = 0$ for all $i \in J$. Then $\langle\xi, y\rangle = 0$ for all $\xi \in Sp\{\hat{\varphi}_i\}$. Since the set $Sp\{\hat{\varphi}_i\}$ is (bo)-dense in $L^\infty(\Omega, H)$, we have that $\langle\xi, y\rangle = 0$ for all $\xi \in L^\infty(\Omega, H)$. In particular $\langle y, y\rangle = 0$, i.e. $y = 0$.

Therefore $\{\hat{\varphi}_i\}_{i\in J}$ is an orthogonal basis in $L^\infty(\Omega, H)$ with the cardinality α, i.e. $L^\infty(\Omega, H)$ is α-homogeneous, where $\alpha = \dim H$.

Denote by $B(L^0(\Omega, H))$ the algebra of all $L^0(\Omega)$-bounded $L^0(\Omega)$-linear operators on $L^0(\Omega, H)$ and by $B(L^\infty(\Omega, H))$—the algebra of all $L^\infty(\Omega)$-bounded $L^\infty(\Omega)$-linear operators on $L^\infty(\Omega, H)$.

In [4] it was proved that $B(L^0(\Omega, H))$ is a C^*-algebra over $L^0(\Omega)$.

Put

$$B\big(L^0(\Omega, H)_b\big) = \big\{x \in B\big(L^0(\Omega, H)\big) \colon \|x\| \in L^\infty(\Omega)\big\}.$$

Note that the correspondence

$$x \mapsto x|_{L^\infty(\Omega,H)}$$

gives a $*$-isomorphism between the $*$-algebras $B(L^0(\Omega, H)_b$ and $B(L^\infty(\Omega, H))$. We further shall identify $B(L^0(\Omega, H)_b)$ with $B(L^\infty(\Omega, H))$, i.e. the operator x from $B(L^0(\Omega, H)_b)$ is identified with its restriction $x|_{L^\infty(\Omega,H)}$.

Since $L^\infty(\Omega, H)$ is a Hilbert–Kaplansky module over $L^\infty(\Omega)$, [9, Theorem 7] implies that $B(L^\infty(\Omega, H))$ is an AW^*-algebra of type I and its center is $*$-isomorphic with $L^\infty(\Omega)$. If $\dim H = \alpha$, then $L^\infty(\Omega, H)$ is α-homogeneous and by [9, Theorem 7] the algebra $B(L^\infty(\Omega, H))$ has the type I_α. The center $Z(B(L^\infty(\Omega, H)))$ of this AW^*-algebra coincides with the von Neumann algebra $L^\infty(\Omega)$ and thus by [10, Theorem 2] $B(L^\infty(\Omega, H))$ is also a von Neumann algebra. Thus for $\dim H = \alpha$ we have that $B(L^\infty(\Omega, H))$ is a von Neumann algebra of type I_α.

Now let M be a homogeneous von Neumann algebra of type I_α with the center $L^\infty(\Omega)$. Since two von Neumann algebras of the same type I_α with isomorphic center are mutually $*$-isomorphic, it follows that the algebra M is $*$-isomorphic to the algebra $B(L^\infty(\Omega, H))$, where $\dim H = \alpha$.

It is well known [19] that given any type I von Neumann algebra M, there exists a (cardinal-indexed) system of central orthogonal projections $(q_\alpha)_{\alpha\in J} \subset \mathcal{P}(M)$ with $\sum_{\alpha\in J} q_\alpha = \mathbf{1}$ such that $q_\alpha M$ is a homogeneous von Neumann algebra of type I_α, i.e. $q_\alpha M \cong B(L^\infty(\Omega_\alpha, H_\alpha))$ with $\dim H_\alpha = \alpha$, and the algebra M is $*$-isomorphic to the C^*-product of the algebras M_α, i.e.

$$M \cong \bigoplus_{\alpha\in J} M_\alpha.$$

Note that if $L^\infty(\Omega)$ is the center of M then $q_\alpha L^\infty(\Omega) \cong L^\infty(\Omega_\alpha)$ for an appropriate Ω_α, $\alpha \in J$. Therefore

$$L^\infty(\Omega) \cong \bigoplus_{\alpha\in J} L^\infty(\Omega_\alpha).$$

The product

$$\bigoplus_{\alpha\in J} L^\infty(\Omega_\alpha, H_\alpha)$$

equipped with coordinate-wise algebraic operations and inner product becomes a Hilbert–Kaplansky module over $L^\infty(\Omega)$. The product

$$\bigoplus_{\alpha\in J} B\big(L^\infty(\Omega_\alpha, H_\alpha)\big)$$

equipped with coordinate-wise algebraic operations and involution becomes a $*$-algebra and moreover

$$\bigoplus_{\alpha\in J} B\big(L^\infty(\Omega_\alpha, H_\alpha)\big) \cong B\bigg(\bigoplus_{\alpha\in J} L^\infty(\Omega_\alpha, H_\alpha)\bigg).$$

Indeed, take $x \in B(\bigoplus_{\alpha\in J} L^\infty(\Omega_\alpha, H_\alpha))$. For each α define the operator x_α on $L^\infty(\Omega_\alpha, H_\alpha)$ by

$$x_\alpha(\varphi_\alpha) = q_\alpha x(\varphi_\alpha),\ \varphi_\alpha \in L^\infty(\Omega_\alpha, H_\alpha).$$

Then

$$\{x_\alpha\} \in \bigoplus_{\alpha\in J} B\big(L^\infty(\Omega_\alpha, H_\alpha)\big)$$

and the correspondence

$$x \mapsto \{x_\alpha\}$$

gives a $*$-homomorphism from $B(\bigoplus_{\alpha\in J} L^\infty(\Omega_\alpha, H_\alpha))$ into $\bigoplus_{\alpha\in J} B(L^\infty(\Omega_\alpha, H_\alpha))$.

Conversely, consider

$$\{x_\alpha\} \in \bigoplus_{\alpha\in J} B\big(L^\infty(\Omega_\alpha, H_\alpha)\big).$$

Define the operator x on $\bigoplus_{\alpha\in J} L^\infty(\Omega_\alpha, H_\alpha)$ by

$$x\big(\{\varphi_\alpha\}\big) = \big\{x_\alpha(\varphi_\alpha)\big\}, \qquad \{\varphi_\alpha\} \in \bigoplus_{\alpha\in J} L^\infty(\Omega_\alpha, H_\alpha).$$

Then $x \in B(\bigoplus_{\alpha\in J} L^\infty(\Omega_\alpha, H_\alpha))$ and therefore

$$M \cong \bigoplus_{\alpha\in J} B\big(L^\infty(\Omega_\alpha, H_\alpha)\big) \cong B\bigg(\bigoplus_{\alpha\in J} L^\infty(\Omega_\alpha, H_\alpha)\bigg).$$

The direct product

$$\prod_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha)$$

equipped with the coordinate-wise algebraic operations and inner product forms a Hilbert–Kaplansky module over $L^0(\Omega) \cong \prod_{\alpha\in J} L^0(\Omega_\alpha)$.

The proof of the following proposition in [2] has a small gap, therefore here we shall give an alternative proof for this result.

Proposition 1.7. *If von Neumann algebra M is $*$-isomorphic with $B(\bigoplus_{\alpha\in J} L^\infty(\Omega_\alpha, H_\alpha))$ then the algebra $LS(M)$ is $*$-isomorphic with $B(\prod_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha))$.*

Proof. Let Φ be a $*$-isomorphism between M and $B(\bigoplus_{\alpha\in J} L^\infty(\Omega_\alpha, H_\alpha))$. Take $x \in B(\prod_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha))$ and let $\|x\|$ be its $L^0(\Omega)$-valued norm. Consider a family of mutually orthogonal projections $\{z_n\}_{n\in\mathbb{N}}$ in $L^\infty(\Omega)$ with $\bigvee z_n = \mathbf{1}$ such that $z_n\|x\| \in L^\infty(\Omega)$ for all $n \in \mathbb{N}$. Then $z_n x \in M$ for all $n \in \mathbb{N}$ and $\sum_n z_n\Phi(z_n x) \in LS(M)$. Put

$$\Psi : x \to \sum_n z_n\Phi(z_n x).$$

It is clear that Ψ is a well-defined $*$-homomorphism from $B(\prod_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha))$ into $LS(M)$. Since given any element $x \in LS(M)$ there exists a sequence of mutually orthogonal central projections $\{z_n\}$ in M such that $z_n x \in M$ for all $n \in \mathbb{N}$ (Proposition 1.6) and $x = \sum_n z_n x$, this implies that Ψ is a $*$-isomorphism between $LS(M)$ and $B(\prod_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha))$. The proof is complete. □

It is known [4] that $B(\prod_{\alpha\in J} L^0(\Omega_\alpha, H_\alpha))$ is a C^*-algebra over $L^0(\Omega)$ and therefore there exists a map $\|\cdot\| : LS(M) \to L^0(\Omega)$ such that for all $x, y \in LS(M)$, $\lambda \in L^0(\Omega)$ one has

$$\|x\| \geqslant 0, \qquad \|x\| = 0 \quad \Leftrightarrow \quad x = 0;$$
$$\|\lambda x\| = |\lambda|\|x\|;$$
$$\|x + y\| \leqslant \|x\| + \|y\|;$$
$$\|xy\| \leqslant \|x\|\|y\|;$$
$$\|xx^*\| = \|x\|^2.$$

This map $\|\cdot\| : LS(M) \to L^0(\Omega)$ is called the *center-valued* norm on $LS(M)$.

2. Derivations on the algebra $LS(M)$

In this section we shall give a complete description of derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a type I von Neumann algebra M. It is clear that if a derivation D on $LS(M)$ is inner then it is Z-linear, i.e. $D(\lambda x) = \lambda D(x)$ for all $\lambda \in Z$, $x \in LS(M)$, where Z is the center of the von Neumann algebra M. The following main result of [2] asserts that the converse is also true.

Theorem 2.1. *Let M be a type I von Neumann algebra with the center Z. Then every Z-linear derivation D on the algebra $LS(M)$ is inner.*

Proof. (See [2, Theorem 3.2].) □

We are now in position to consider arbitrary (non-Z-linear, in general) derivations on $LS(M)$. The following simple but important remark is crucial in our further considerations.

Remark 1. Let A be an algebra with the center Z and let $D : A \to A$ be a derivation. Given any $x \in A$ and a central element $\lambda \in Z$ we have

$$D(\lambda x) = D(\lambda)x + \lambda D(x)$$

and

$$D(x\lambda) = D(x)\lambda + xD(\lambda).$$

Since $\lambda x = x\lambda$ and $\lambda D(x) = D(x)\lambda$, it follows that $D(\lambda)x = xD(\lambda)$ for any $\lambda \in A$. This means that $D(\lambda) \in Z$, i.e. $D(Z) \subseteq Z$. Therefore given any derivation D on the algebra A we can consider its restriction $\delta : Z \to Z$.

Now let M be a homogeneous von Neumann algebra of type $I_n, n \in \mathbb{N}$, with the center Z. Then the algebra M is $*$-isomorphic with the algebra $M_n(Z)$ of all $n \times n$-matrices over Z, and the algebra $LS(M) = S(M)$ is $*$-isomorphic with the algebra $M_n(S(Z))$ of all $n \times n$ matrices over $S(Z)$, where $S(Z)$ is the algebra of measurable operators for the commutative von Neumann algebra Z.

The algebra $LS(Z) = S(Z)$ is isomorphic to the algebra $L^0(\Omega) = L(\Omega, \Sigma, \mu)$ of all measurable functions on a measure space (see Section 2) and therefore it admits (in nonatomic cases) non-zero derivations (see [3,13]).

Let $\delta : S(Z) \to S(Z)$ be a derivation and D_δ be a derivation on the algebra $M_n(S(Z))$ defined by (1) in Introduction.

The following lemma describes the structure of an arbitrary derivation on the algebra of locally measurable operators for homogeneous type I_n, $n \in \mathbb{N}$, von Neumann algebras.

Lemma 2.2. *Let M be a homogenous von Neumann algebra of type $I_n, n \in \mathbb{N}$. Every derivation D on the algebra $LS(M)$ can be uniquely represented as a sum*

$$D = D_a + D_\delta,$$

where D_a is an inner derivation implemented by an element $a \in LS(M)$ while D_δ is the derivation of the form (1) *generated by a derivation δ on the center of $LS(M)$ identified with $S(Z)$.*

Proof. Let D be an arbitrary derivation on the algebra $LS(M) \cong M_n(S(Z))$. Consider its restriction δ onto the center $S(Z)$ of this algebra, and let D_δ be the derivation on the algebra $M_n(S(Z))$ constructed as in (1). Put $D_1 = D - D_\delta$. Given any $\lambda \in S(Z)$ we have

$$D_1(\lambda) = D(\lambda) - D_\delta(\lambda) = D(\lambda) - D(\lambda) = 0,$$

i.e. D_1 is identically zero on $S(Z)$. Therefore D_1 is Z-linear and by Theorem 2.1 we obtain that D_1 is inner derivation and thus $D_1 = D_a$ for an appropriate $a \in M_n(S(Z))$. Therefore $D = D_a + D_\delta$.

Suppose that

$$D = D_{a_1} + D_{\delta_1} = D_{a_2} + D_{\delta_2}.$$

Then $D_{a_1} - D_{a_2} = D_{\delta_2} - D_{\delta_1}$. Since $D_{a_1} - D_{a_2}$ is identically zero on the center of the algebra $M_n(S(Z))$ this implies that $D_{\delta_2} - D_{\delta_1}$ is also identically zero on the center of $M_n(S(Z))$. This means that $\delta_1 = \delta_2$, and therefore $D_{a_1} = D_{a_2}$, i.e. the decomposition of D is unique. The proof is complete. □

Now let M be an arbitrary finite von Neumann algebra of type I with the center Z. There exists a family $\{z_n\}_{n\in F}$, $F \subseteq \mathbb{N}$, of central projections from M with $\sup_{n\in F} z_n = \mathbf{1}$ such that the algebra M is $*$-isomorphic with the C^*-product of von Neumann algebras $z_n M$ of type I_n respectively, $n \in F$, i.e.

$$M \cong \bigoplus_{n\in F} z_n M.$$

By Proposition 1.1 we have that

$$LS(M) \cong \prod_{n\in F} LS(z_n M).$$

Suppose that D is a derivation on $LS(M)$, and δ is its restriction onto its center $S(Z)$. Since δ maps each $z_n S(Z) \cong Z(LS(z_n M))$ into itself, δ generates a derivation δ_n on $z_n S(Z)$ for each $n \in F$.

Let D_{δ_n} be the derivation on the matrix algebra $M_n(z_n Z(LS(M))) \cong LS(z_n M)$ defined as in (1). Put

$$D_\delta\big(\{x_n\}_{n\in F}\big) = \big\{D_{\delta_n}(x_n)\big\}, \quad \{x_n\}_{n\in F} \in LS(M). \tag{3}$$

Then the map D_δ is a derivation on $LS(M)$.

Now Lemma 2.2 implies the following result:

Lemma 2.3. *Let M be a finite von Neumann algebra of type I. Each derivation D on the algebra $LS(M)$ can be uniquely represented in the form*

$$D = D_a + D_\delta,$$

where D_a is an inner derivation implemented by an element $a \in LS(M)$, and D_δ is a derivation given as (3).

In order to consider the case of type I_∞ von Neumann algebra we need some auxiliary results concerning derivations on the algebra $L^0(\Omega) = L(\Omega, \Sigma, \mu)$.

Recall that a net $\{\lambda_\alpha\}$ in $L^0(\Omega)$ (o)-converges to $\lambda \in L^0(\Omega)$ if there exists a net $\{\xi_\alpha\}$ monotone decreasing to zero such that $|\lambda_\alpha - \lambda| \leqslant \xi_\alpha$ for all α.

Denote by ∇ the complete Boolean algebra of all idempotents from $L^0(\Omega)$, i.e. $\nabla = \{\tilde{\chi}_A\colon A \in \Sigma\}$, where $\tilde{\chi}_A$ is the element from $L^0(\Omega)$ which contains the characteristic function of the set A. A *partition of the unit* in ∇ is a family (π_α) of orthogonal idempotents from ∇ such that $\bigvee_\alpha \pi_\alpha = \mathbf{1}$.

Lemma 2.4. *Any derivation δ on the algebra $L^0(\Omega)$ commutes with the mixing operation on $L^0(\Omega)$, i.e.*

$$\delta\left(\sum_\alpha \pi_\alpha \lambda_\alpha\right) = \sum_\alpha \pi_\alpha \delta(\lambda_\alpha)$$

for an arbitrary family $(\lambda_\alpha) \subset L^0(\Omega)$ and any partition $\{\pi_\alpha\}$ of the unit in ∇.

Proof. Consider a family $\{\lambda_\alpha\}$ in $L^0(\Omega)$ and a partition of the unit $\{\pi_\alpha\}$ in $\nabla \subset L^0(\Omega)$. Since $\delta(\pi) = 0$ for any idempotent $\pi \in \nabla$, we have $\delta(\pi_\alpha) = 0$ for all α and thus $\delta(\pi_\alpha \lambda) = \pi_\alpha \delta(\lambda)$ for any $\lambda \in L^0(\Omega)$. Therefore for each π_{α_0} from the given partition of the unit we have

$$\pi_{\alpha_0}\delta\left(\sum_\alpha \pi_\alpha \lambda_\alpha\right) = \delta\left(\pi_{\alpha_0}\sum_\alpha \pi_\alpha \lambda_\alpha\right) = \delta(\pi_{\alpha_0}\lambda_{\alpha_0}) = \pi_{\alpha_0}\delta(\lambda_{\alpha_0}).$$

By taking the sum over all α_0 we obtain

$$\delta\left(\sum_\alpha \pi_\alpha \lambda_\alpha\right) = \sum_\alpha \pi_\alpha \delta(\lambda_\alpha).$$

The proof is complete. □

Recall [11] that a subset $K \subset L^0(\Omega)$ is called *cyclic*, if $\sum_{\alpha\in J} \pi_\alpha u_\alpha \in K$ for each family $(u_\alpha)_{\alpha\in J} \subset K$ and for any partition of the unit $(\pi_\alpha)_{\alpha\in J}$ in ∇.

We need the following technical result.

Lemma 2.5. *Let A be a cyclic subset in $L^0(\Omega)$. If the set πA is unbounded above for each non-zero $\pi \in \nabla$, then given any $n \in \mathbb{N}$ there exists $\lambda_n \in A$ such that $\lambda_n \geqslant n\mathbf{1}$.*

Proof. For fixed $n \in \mathbb{N}$ and an arbitrary $\lambda \in A$ denote

$$\pi_\lambda = \bigvee\{q \in \nabla:\ q\lambda \geqslant qn\}.$$

Then

$$\pi_\lambda \lambda \geqslant \pi_\lambda n \tag{4}$$

and

$$\pi_\lambda^\perp \lambda \leqslant \pi_\lambda^\perp n. \tag{5}$$

Put

$$\pi_0 = \bigvee\{\pi_\lambda:\ \lambda \in A\}.$$

Since

$$\pi_0^\perp = \bigwedge\left\{\pi_\lambda^\perp:\ \lambda \in A\right\}$$

from (5) we obtain

$$\pi_0^{\perp}\lambda \leqslant \pi_0^{\perp} n$$

for all $\lambda \in A$, i.e. $\pi_0^{\perp} A$ is bounded above. By the assumption of lemma $\pi_0^{\perp} = 0$, i.e.

$$\pi_0 = \bigvee\{\pi_\lambda\colon \lambda \in A\} = \mathbf{1}.$$

By [20, p. 111, Theorem 4] there exists a partition of unit $\{\pi_i\}$ in ∇ such that for any π_i there exists $\lambda_i \in A$ with $\pi_i \leqslant \pi_{\lambda_i}$. Put $\lambda_n = \sum_i \pi_i\lambda_i$. Since A is a cyclic we have $\lambda_n \in A$. From (4) one has $\pi_{\lambda_i}\lambda_i \geqslant \pi_{\lambda_i} n$ for all i. Thus $\pi_i\lambda_i \geqslant \pi_i n$ for all i, therefore $\lambda_n \geqslant n\mathbf{1}$. The proof is complete. □

Given an arbitrary derivation δ on L^0 the element

$$z_\delta = \inf\{\pi \in \nabla\colon \pi\delta = \delta\}$$

is called the support of the derivation δ.

Lemma 2.6. *Given any nontrivial derivation $\delta : L^0(\Omega) \to L^0(\Omega)$ there exist a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $L^\infty(\Omega)$ with $|\lambda_n| \leqslant \mathbf{1}$, $n \in \mathbb{N}$ such that*

$$\left|\delta(\lambda_n)\right| \geqslant n z_\delta$$

for all $n \in \mathbb{N}$.

Proof. Considering if necessary the algebra $z_\delta L^0(\Omega)$ instead $L^0(\Omega)$ and the derivation $z_\delta\delta$ instead δ, we may assume that $z_\delta = \mathbf{1}$.

Put $A = \{\delta(\lambda)\colon \lambda \in L^0(\Omega),\ |\lambda| \leqslant \mathbf{1}\}$ and let us prove that for any non-zero $\pi \in \nabla$ the set πA is unbounded from above. Suppose that the set $\pi\{\delta(\lambda)\colon \lambda \in L^0(\Omega),\ |\lambda| \leqslant \mathbf{1}\}$ is order bounded in $L^0(\Omega)$ for some $\pi \in \nabla$, $\pi \neq 0$. Then $\pi\delta$ maps any uniformly convergent sequence in $L^\infty(\Omega)$ to an (o)-convergent sequence in $L^0(\Omega)$. The algebra $L^\infty(\Omega)$ coincides with the uniform closure of the linear span of idempotents from ∇. Since $\pi\delta$ is identically zero on ∇ it follows that $\pi\delta \equiv 0$ on $L^\infty(\Omega)$. Since δ commutes with the mixing operation and every element $\lambda \in L^0(\Omega)$ can be represented as $\lambda = \sum_\alpha \pi_\alpha\lambda_\alpha$, where $\{\lambda_\alpha\} \subset L^\infty(\Omega)$, and $\{\pi_\alpha\}$ is a partition of unit in ∇, we have $\delta(\lambda) = \delta(\sum_\alpha \pi_\alpha\lambda_\alpha) = \sum_\alpha \pi_\alpha\delta(\lambda_\alpha) = 0$, i.e. $\pi\delta \equiv 0$ on $L^0(\Omega)$. This is contradicts with $z_\delta = \mathbf{1}$. This contradiction shows that the set $\pi\{\delta(\lambda)\colon \lambda \in L^0(\Omega),\ |\lambda| \leqslant \mathbf{1}\}$ is not order bounded in $L^0(\Omega)$ for all $\pi \in \nabla$, $\pi \neq 0$. Further, since δ commutes with the mixing operations and the set $\{\lambda\colon \lambda \in L^0,\ |\lambda| \leqslant \mathbf{1}\}$ is cyclic, the set $\{\delta(\lambda)\colon \lambda \in L^0(\Omega),\ |\lambda| \leqslant \mathbf{1}\}$ is also cyclic. By Lemma 2.5 there exist a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $L^\infty(\Omega)$ with $|\lambda_n| \leqslant \mathbf{1}$ such that $|\delta(\lambda_n)| \geqslant n\mathbf{1}$, $n \in \mathbb{N}$. The proof is complete. □

Now we are in position to consider derivations on the algebra of locally measurable operators for type I_∞ von Neumann algebras.

Theorem 2.7. *If M is a type I_∞ von Neumann algebra, then any derivation on the algebra $LS(M)$ is inner.*

Proof. Since M is of type I_∞ there exists a sequence of mutually orthogonal and mutually equivalent abelian projections $\{p_n\}_{n=1}^{\infty}$ in M with the central cover $\mathbf{1}$ (i.e. faithful projections).

For any bounded sequence $\Lambda = \{\lambda_k\}$ in Z define an operator x_Λ by

$$x_\Lambda = \sum_{k=1}^{\infty} \lambda_k p_k.$$

Then

$$x_\Lambda p_n = p_n x_\Lambda = \lambda_n p_n. \tag{6}$$

Let D be a derivation on $LS(M)$, and let δ be its restriction onto the center of $LS(M)$, identified with $L^0(\Omega)$.

Take any $\lambda \in L^0(\Omega)$ and $n \in \mathbb{N}$. From the identity

$$D(\lambda p_n) = D(\lambda) p_n + \lambda D(p_n)$$

multiplying it by p_n from both sides we obtain

$$p_n D(\lambda p_n) p_n = p_n D(\lambda) p_n + \lambda p_n D(p_n) p_n.$$

Since p_n is a projection, one has that $p_n D(p_n) p_n = 0$, and since $D(\lambda) = \delta(\lambda) \in L^0(\Omega)$, we have

$$p_n D(\lambda p_n) p_n = \delta(\lambda) p_n. \tag{7}$$

Now from the identity

$$D(x_\Lambda p_n) = D(x_\Lambda) p_n + x_\Lambda D(p_n),$$

in view of (6) one has similarly

$$p_n D(\lambda_n p_n) p_n = p_n D(x_\Lambda) p_n + \lambda_n p_n D(p_n) p_n,$$

i.e.

$$p_n D(\lambda_n p_n) p_n = p_n D(x_\Lambda) p_n. \tag{8}$$

Eqs. (7) and (8) imply

$$p_n D(x_\Lambda) p_n = \delta(\lambda_n) p_n.$$

Further for the center-valued norm $\|\cdot\|$ on $LS(M)$ (see Section 1) we have:

$$\big\| p_n D(x_\Lambda) p_n \big\| \leqslant \| p_n \| \big\| D(x_\Lambda) \big\| \| p_n \| = \big\| D(x_\Lambda) \big\|$$

and

$$\big\| \delta(\lambda_n) p_n \big\| = \big| \delta(\lambda_n) \big|.$$

Therefore

$$\|D(x_\Lambda)\| \geqslant |\delta(\lambda_n)|$$

for any bounded sequence $\Lambda = \{\lambda_n\}$ in Z.

If we suppose that $\delta \neq 0$ then $\pi = z_\delta \neq 0$. By Lemma 2.6 there exists a bounded sequence $\Lambda = \{\lambda_n\}$ in Z such that

$$|\delta(\lambda_n)| \geqslant n\pi$$

for any $n \in \mathbb{N}$. Thus

$$\|D(x_\Lambda)\| \geqslant n\pi$$

for all $n \in \mathbb{N}$, i.e. $\pi = 0$—that is a contradiction. Therefore $\delta \equiv 0$, i.e. D is identically zero on the center of $LS(M)$, and therefore it is Z-linear. By Theorem 2.1 D is inner. The proof is complete. □

We shall now consider derivations on the algebra $LS(M)$ of locally measurable operators with respect to an arbitrary type I von Neumann algebra M.

Let M be a type I von Neumann algebra. There exists a central projection $z_0 \in M$ such that

(a) $z_0 M$ is a finite von Neumann algebra;
(b) $z_0^\perp M$ is a von Neumann algebra of type I_∞.

Consider a derivation D on $LS(M)$ and let δ be its restriction onto its center $Z(S)$. By Theorem 2.7 $z_0^\perp D$ is inner and thus we have $z_0^\perp \delta \equiv 0$, i.e. $\delta = z_0\delta$.

Let D_δ be the derivation on $z_0 LS(M)$ defined as in (3) and consider its extension D_δ on $LS(M) = z_0 LS(M) \oplus z_0^\perp LS(M)$ which is defined as

$$D_\delta(x_1 + x_2) := D_\delta(x_1), \quad x_1 \in z_0 LS(M),\ x_2 \in z_0^\perp LS(M). \tag{9}$$

The following theorem is the main result of this section, and gives the general form of derivations on the algebra $LS(M)$.

Theorem 2.8. *Let M be a type I von Neumann algebra. Each derivation D on $LS(M)$ can be uniquely represented in the form*

$$D = D_a + D_\delta$$

where D_a is an inner derivation implemented by an element $a \in LS(M)$, and D_δ is a derivation of the form (9)*, generated by a derivation δ on the center of $LS(M)$.*

Proof. It immediately follows from Lemma 2.3 and Theorem 2.7. □

3. Derivations on the algebra $S(M)$

In this section we describe derivations on the algebra $S(M)$ of measurable operators affiliated with a type I von Neumann algebra M.

Let M be a type I von Neumann algebra and let $\mathcal{A}$ be an arbitrary subalgebra of $LS(M)$ containing M. Consider a derivation $D : \mathcal{A} \to LS(M)$ and let us show that D can be extended to a derivation $\tilde{D}$ on the whole $LS(M)$.

Since M is a type I, for an arbitrary element $x \in LS(M)$ there exists a sequence $\{z_n\}$ of mutually orthogonal central projections with $\bigvee_{n\in\mathbb{N}} z_n = \mathbf{1}$ and $z_n x \in M$ for all $n \in \mathbb{N}$. Set

$$\tilde{D}(x) = \sum_{n\geqslant 1} z_n D(z_n x). \tag{10}$$

Since every derivation $D : \mathcal{A} \to LS(M)$ is identically zero on central projections of M, the equality (10) gives a well-defined derivation $\tilde{D} : LS(M) \to LS(M)$ which coincides with D on $\mathcal{A}$.

In particular, if D is Z-linear on $\mathcal{A}$, then $\tilde{D}$ is also Z-linear and by Theorem 2.1 the derivation $\tilde{D}$ is inner on $LS(M)$ and therefore D is a spatial derivation on $\mathcal{A}$, i.e. there exists an element $a \in LS(M)$ such that

$$D(x) = ax - xa$$

for all $x \in \mathcal{A}$.

Therefore we obtain the following

Theorem 3.1. *Let M be a type I von Neumann algebra with the center Z, and let $\mathcal{A}$ be an arbitrary subalgebra in $LS(M)$ containing M. Then any Z-linear derivation $D : \mathcal{A} \to LS(M)$ is spatial and implemented by an element of $LS(M)$.*

Corollary 3.2. *Let M be a type I von Neumann algebra with the center Z and let D be a Z-linear derivation on $S(M)$ or $S(M,\tau)$. Then D is spatial and implemented by an element of $LS(M)$.*

We are now in position to improve the last result by showing that in fact such derivations on $S(M)$ and $S(M,\tau)$ are inner.

Let us start by the consideration of the type I_∞ case.

Let M be a type I_∞ von Neumann algebra with the center Z identified with the algebra $L^\infty(\Omega)$ and let ∇ be the Boolean algebra of projection from $L^\infty(\Omega)$.

Denote by $St(\nabla)$ the set of all elements $\lambda \in L^0(\Omega)$ of the form $\lambda = \sum_\alpha \pi_\alpha t_\alpha$, where $\{\pi_\alpha\}$ is a partition of the unit in ∇, and $\{t_\alpha\} \subset \mathbb{R}$ (so-called step-functions).

Suppose that $a \in LS(M)$, $a = a^*$ and consider the spectral family $\{e_\lambda\}_{\lambda\in\mathbb{R}}$ of the operator a. For $\lambda \in St(\nabla)$, $\lambda = \sum_\alpha \pi_\alpha t_\alpha$ put $e_\lambda = \sum_\alpha \pi_\alpha e_{t_\alpha}$.

Denote by $P_\infty(M)$ the family of all faithful projections p from M such that pMp is of type I_∞.

Set

$$\Lambda_- = \{\lambda \in St(\nabla)\colon\ e_\lambda \in P_\infty(M)\}$$

and

$$\Lambda_+ = \{\lambda \in St(\nabla)\colon\ e_\lambda^\perp \in P_\infty(M)\}.$$

Lemma 3.3.

(a) $\Lambda_- \neq \emptyset$ *and* $\Lambda_+ \neq \emptyset$;
(b) *the set* Λ_+ (*resp.* Λ_-) *is bounded from above* (*resp. from below*);
(c) *if* $\lambda_+ = \sup \Lambda_+$ (*resp.* $\lambda_- = \inf \Lambda_-$) *then* $\lambda \in \Lambda_+$ (*resp.* $\lambda \in \Lambda_-$) *for all* $\lambda \in St(\nabla)$ *with* $\lambda + \varepsilon \mathbf{1} \leqslant \lambda_+$ (*resp.* $\lambda - \varepsilon \mathbf{1} \geqslant \lambda_-$) *for some* $\varepsilon > 0$;
(d) *if* $\lambda_+ \in L^\infty(\Omega)$ *and* $\lambda_- \in L^\infty(\Omega)$, *then* $a \in S(M)$.

Proof. (a) Take a sequence of projections $\{z_n\}$ from ∇ such that $z_n a \in M$ for all $n \in \mathbb{N}$. Then for $t_n < -\| z_n a \|_M$ we have $z_n e_{t_n} = 0$ or $z_n e_{t_n}^\perp = z_n$ for all $n \in \mathbb{N}$. Therefore for $\lambda = \sum z_n t_n$ one has $e_\lambda^\perp = \sum z_n e_{t_n}^\perp = \sum z_n = \mathbf{1}$, i.e. $\lambda \in \Lambda_+$ and hence $\Lambda_+ \neq \emptyset$. Similarly $\Lambda_- \neq \emptyset$.

(b) Suppose that the element $\lambda = \sum \pi_\alpha \lambda_\alpha \in St(\nabla)$, satisfies the condition $\pi_0 \lambda \geqslant \pi_0 \|a\| + \varepsilon \pi_0$ for an appropriate non-zero $\pi_0 \in \nabla$, where $\| \cdot \|$ is the center-valued norm on $LS(M)$. Without loss of generality we may assume that $\pi_0 = \pi_\alpha$ for some α, i.e. $\pi_\alpha t_\alpha \geqslant \pi_\alpha \|a\| + \varepsilon \pi_\alpha$. Then $t_\alpha \geqslant \|\pi_\alpha a\|_M + \varepsilon$ and therefore $\pi_\alpha e_{t_\alpha} = \pi_\alpha \mathbf{1}$, i.e. $\pi_\alpha e_{t_\alpha}^\perp = 0$. Since $\pi_\alpha e_{t_\alpha}^\perp = 0$, we have $z(e_{t_\alpha}^\perp) \neq \mathbf{1}$ and so $\lambda \notin \Lambda_+$. Therefore Λ_+ is bounded from above by the element $\|a\|$. Similarly the set Λ_- is bounded from below by the element $-\|a\|$.

(c) Put

$$\lambda_+ = \sup \Lambda_+$$

and

$$\lambda_- = \inf \Lambda_-.$$

Take an element $\lambda \in St(\nabla)$ such that $\lambda + \varepsilon \mathbf{1} \leqslant \lambda_+$, where $\varepsilon > 0$. Suppose that $e_\lambda^\perp \notin P_\infty(M)$. Then $\pi_0 e_\lambda^\perp M e_\lambda^\perp$ is a finite von Neumann algebra for some non-zero $\pi_0 \in \nabla$. Without loss of generality we may assume that $\pi_0 = \pi_\alpha$ for some α, i.e. $\pi_\alpha e_{t_\alpha}^\perp$ is a finite projection. Then $\pi_\alpha e_t^\perp$ is finite for all $t > t_\alpha$. This means that $\pi_\alpha \lambda_+ \leqslant \pi_\alpha t_\alpha$.

On the other hand multiplying by π_α the unequality $\lambda + \varepsilon \mathbf{1} \leqslant \lambda_+$ we obtain that $\pi_\alpha t_\alpha + \pi_\alpha \varepsilon \leqslant \pi_\alpha \lambda_+$. Therefore $\pi_\alpha \varepsilon \leqslant 0$. This contradiction implies that $\lambda \in \Lambda_+$ for all $\lambda \in St(\nabla)$ with $\lambda + \varepsilon \mathbf{1} \leqslant \lambda_+$.

(d) Let $\lambda_+, \lambda_- \in L^\infty(\Omega)$. Take a number $n \in \mathbb{N}$ such that $\lambda_+ \leqslant n\mathbf{1}$ and $\lambda_- \geqslant -n\mathbf{1}$. Take a largest element $\pi \in \nabla$ such that $\pi e_{n+1}^\perp$ is a finite projection and $\pi^\perp e_{n+1}^\perp$ is an infinite projection. For $\lambda' \in \Lambda_+$ put $\lambda'' = \pi \lambda' + \pi^\perp (n+1)$. Then $\lambda'' \in \Lambda_+$ and therefore $\lambda'' \leqslant \lambda_+$. Hence $\pi^\perp \lambda'' \leqslant \pi^\perp \lambda_+$, i.e $\pi^\perp (n+1) \leqslant \pi^\perp \lambda_+$. That contradicts the inequality $\lambda_+ \leqslant n\mathbf{1}$. Therefore $\pi = \mathbf{1}$, i.e. $e_{n+1}^\perp$ is a finite projection. Similarly $e_{-(n+1)}^\perp$ is a finite projection. Therefore $a \in S(M)$. The proof is complete. □

Lemma 3.4. *If* M *is a type* I_∞ *von Neumann algebra then every derivation* $D : M \to S(M)$ *has the form*

$$D(x) = ax - xa, \quad x \in M,$$

for an appropriate $a \in S(M)$.

Proof. By the Remark 1 D maps the center Z of M into the center of $S(M)$ which coincides with Z by Proposition 1.3, i.e. we obtain a derivation D on commutative von Neumann algebra Z. Therefore $D|_Z = 0$. Thus $D(\lambda x) = D(\lambda)x + \lambda D(x) = \lambda D(x)$ for all $\lambda \in Z$, i.e. D is Z-linear.

By Theorem 3.1 there exists an element $a \in LS(M)$ such that $D(x) = ax - xa$ for all $x \in M$.

Let us prove that one can choose the element a from $S(M)$.

For $x \in M$ we have

$$(a + a^*)x - x(a + a^*) = (ax - xa) - (ax^* - x^*a)^* = D(x) - D(x^*)^* \in S(M)$$

and

$$(a - a^*)x - x(a - a^*) = D(x) + D(x^*)^* \in S(M).$$

This means that the elements $a + a^*$ and $a - a^*$ implement derivations from M into $S(M)$. Since $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$, it is sufficient to consider the case where a is a self-adjoint element.

Consider the elements $\lambda_+, \lambda_- \in L^0$ defined in Lemma 3.3(c) and let us prove that $\lambda_+, \lambda_- \in L^\infty(\Omega)$. Lemma 3.3(c) implies that there exists an element $\lambda_1 \in \Lambda_-$ such that $-\frac{1}{4} \leqslant \lambda_- - \lambda_1 \leqslant -\frac{1}{8}$. Since $D(x) = (a - \lambda_1)x - x(a - \lambda_1)$, replacing a by $a - \lambda_1$, we may assume that $-\frac{1}{4} \leqslant \lambda_- \leqslant -\frac{1}{8}$. Then $e_\varepsilon \in P_\infty(M)$ for all $\varepsilon > -\frac{1}{8}$ and e_ε is a finite for all $\varepsilon < -\frac{1}{4}$. In particular $(e_{-\frac{1}{16}} - e_{-\frac{1}{2}})M(e_{-\frac{1}{16}} - e_{-\frac{1}{2}})$ is of type I_∞, and moreover $\lambda_+ \geqslant -\frac{1}{2}$.

Suppose that $\lambda_+ \notin L^\infty(\Omega)$. Since $\lambda_+ \geqslant -\frac{1}{2}$, we have that λ_+ is unbounded from above and thus passing if necessary to the subalgebra zM, where z is a non-zero central projection in M with $z\lambda_+ \geqslant z$, we may assume without loss of generality that $\lambda_+ \geqslant \mathbf{1}$.

First let us consider the particular case where M is of type $\mathrm{I}_{\aleph_0}$, where $\aleph_0$ is the countable cardinal number. Take an element $\lambda_0 \in St(\nabla)$ such that $\lambda_+ - \frac{1}{2} \leqslant \lambda_0 \leqslant \lambda_+ - \frac{1}{4}$. By Lemma 3.3(c) we have $e^\perp_{\lambda_0} \in P_\infty(M)$. Since algebras $e^\perp_{\lambda_0} M e^\perp_{\lambda_0}$ and $(e_{-\frac{1}{16}} - e_{-\frac{1}{2}})M(e_{-\frac{1}{16}} - e_{-\frac{1}{2}})$ are algebras of type $\mathrm{I}_{\aleph_0}$, then there exists a sequences of pairwise equivalent and pairwise orthogonal abelian projections $\{f_k\}_{k\in\mathbb{N}}$ and $\{g_k\}_{k\in\mathbb{N}}$ such that $\bigvee f_k = e^\perp_{\lambda_0}$, $\bigvee g_k = e_{-\frac{1}{16}} - e_{-\frac{1}{2}}$. Since $z(e^\perp_{\lambda_0}) = z(e_{-\frac{1}{16}} - e_{-\frac{1}{2}}) = \mathbf{1}$, then $z(f_k) = z(g_k) = \mathbf{1}$ for all k, and therefore $f_k \sim g_k$ for all k. Thus the projections $p_1 = e^\perp_{\lambda_0}$ and $p_2 = e_{-\frac{1}{16}} - e_{-\frac{1}{2}}$ are equivalent. From $\lambda_0 e^\perp_{\lambda_0} \leqslant a e^\perp_{\lambda_0}$ it follows that $\lambda_0 p_1 \leqslant p_1 a p_1$. Since $p_1 M p_1$ is of type $\mathrm{I}_{\aleph_0}$, the center of the algebra $S(p_1 M p_1)$ coincides with the center of the algebra $p_1 M p_1$ (Proposition 1.3). Due to the fact that $\lambda_0 \notin L^\infty(\Omega)$ and $z(p_1) = \mathbf{1}$, we see that $\lambda_0 p_1$ is an unbounded linear operator from $LS(p_1 M p_1) \setminus S(p_1 M p_1)$. Therefore $a p_1 = p_1 a p_1 \notin S(p_1 M p_1)$.

Let u be a partial isometry in M such that $uu^* = p_1$, $u^*u = p_2$. Put $p = p_1 + p_2$. Consider the derivation D_1 from pMp into $pS(M)p = S(pMp)$ defined as

$$D_1(x) = pD(x)p, \quad x \in pMp.$$

This derivation is implemented by the element $ap = pap$, i.e.

$$D_1(x) = apx - xap, \quad x \in pMp.$$

Since $p_2 = e_{-\frac{1}{16}} - e_{-\frac{1}{2}}$ then $-\frac{1}{2}e_{-\frac{1}{2}} \leqslant (e_{-\frac{1}{16}} - e_{-\frac{1}{2}})a(e_{-\frac{1}{16}} - e_{-\frac{1}{2}}) \leqslant -\frac{1}{16}e_{-\frac{1}{16}}$. Therefore $ap_2 \in pMp$, the element $b = ap_1 = ap - ap_2$ implements a derivation D_2 from pMp into $S(pMp)$.

Since $D_2(u+u^*) = b(u+u^*) - (u+u^*)b$, it follows that $b(u+u^*) - (u+u^*)b \in S(M)$. From $up_1 = p_1u^* = 0$ it follows that $bu - u^*b \in S(M)$. Multiplying this by u from the left side we obtain $ubu - uu^*b \in S(M)$. From $ub = 0$, $uu^* = p_1$, it follows that $p_1 b \in S(M)$, i.e. $ap_1 \in S(M)$. This contradicts the above relation $ap_1 \notin S(M)$. The contradiction shows that $\lambda_+ \in L^\infty(\Omega)$. Now Lemma 3.3(d) implies that $a \in S(M)$.

Let us consider the case of general type I_∞ von Neumann algebra M. Take an element $\lambda_0 \in St(\nabla)$ such that $\lambda_+ - \frac{1}{2} \leqslant \lambda_0 \leqslant \lambda_+ - \frac{1}{4}$. Lemma 3.3(c) implies that $e_{\lambda_0}^\perp \in P_\infty(M)$. Consider projections p_1 and p_2 with the central cover **1** such that $p_1 \leqslant e_{\lambda_0}^\perp$, $p_2 \leqslant e_{\frac{1}{4}}$ and such that $p_i M p_i$ are of type $\mathrm{I}_{\aleph_0}$, $i = 1, 2$. Put $p = p_1 + p_2$. Consider the derivation D_p from pMp into $pS(M)p$ defined as

$$D_p(x) = pD(x)p, \quad x \in pMp.$$

Since pMp is of type $\mathrm{I}_{\aleph_0}$ the above case implies that $pap \in S(M)$ and therefore $p_1ap_1 \in S(M)$. On the other hand $\lambda_0 p_1 \leqslant p_1ap_1$ and $\lambda_0 p_1 \notin S(M)$. From this contradiction it follows that $\lambda_+ \in L^\infty(\Omega)$. By Lemma 3.3(d) we obtain that $a \in S(M)$. The proof is complete. □

From the above results we obtain

Lemma 3.5. *Let M be a type I von Neumann algebra with the center Z. Then every Z-linear derivation D on the algebra $S(M)$ is inner. In particular, if M is a type I_∞ then every derivation on $S(M)$ is inner.*

Now let M be an arbitrary type I von Neumann algebra and let z_0 be the central projection in M such that $z_0 M$ is a finite von Neumann algebra and $z_0^\perp M$ is a von Neumann algebra of type I_∞. Consider a derivation D on $S(M)$ and let δ be its restriction onto its center $Z(S)$. By Lemma 3.5 the derivation $z_0^\perp D$ is inner and thus we have $z_0^\perp \delta \equiv 0$, i.e. $\delta = z_0\delta$.

Since $z_0 M$ is a finite type I von Neumann algebra, we have that $z_0 LS(M) = z_0 S(M)$. Let D_δ be the derivation on $z_0 S(M) = z_0 LS(M)$ defined as in (3).

Finally Lemmas 2.3 and 3.5 imply the following main result the present section.

Theorem 3.6. *Let M be a type I von Neumann algebra. Then every dereivation D on the algebra $S(M)$ can be uniquely represented in the form*

$$D = D_a + D_\delta,$$

where D_a is inner and implemented by an element $a \in S(M)$ and D_δ is the derivation of the form (3) *generated by a derivation δ on the center of $S(M)$.*

4. Derivations on the algebra $S(M, \tau)$

In this section we present a general form of derivations on the algebra $S(M, \tau)$ of τ-measurable operators affiliated with a type I von Neumann algebra M and a faithful normal semi-finite trace τ.

Theorem 4.1. *Let M be a type I von Neumann algebra with the center Z and a faithful normal semi-finite trace τ. Then every Z-linear derivation D on the algebra $S(M, \tau)$ is inner. In particular, if M is a type I_∞ then every derivation on $S(M, \tau)$ is inner.*

Proof. By Theorem 3.1 $D(x) = ax - xa$ for some $a \in LS(M)$ and all $x \in S(M, \tau)$. Let us show that the element a can be chosen from the algebra $S(M, \tau)$. As in Lemma 3.3 we may assume that $a = a^*$.

Case 1. M is a homogeneous type I_n, $n \in \mathbb{N}$ von Neumann algebra. Then $LS(M) = S(M) \cong M_n(L^0(\Omega))$. By [12, Theorem 3.5] a $*$-isomorphism between $S(M)$ and $M_n(L^0(\Omega))$ can be a chosen such that the element a can be represented as $a = \sum_{i=1}^{n} \lambda_i e_{i,i}$, where $\lambda_i = \overline{\lambda_i} \in L^0(\Omega)$, $i = \overline{1, n}$, $\lambda_1 \geqslant \cdots \geqslant \lambda_n$.

Put $u = \sum_{j=1}^{n} e_{j,n-j+1}$. Then

$$D_a(u) = au - ua = \sum_{i=1}^{n} (\lambda_i - \lambda_{n-i+1}) e_{i,n-i+1}$$

and

$$D_a(u)^* = \sum_{i=1}^{n} (\lambda_i - \lambda_{n-i+1}) e_{n-i+1,i}.$$

Therefore $D_a(u)^* D_a(u) = \sum_{i=1}^{n} (\lambda_i - \lambda_{n-i+1})^2 e_{i,i}$, and thus $|D_a(u)| = \sum_{i=1}^{n} |\lambda_i - \lambda_{n-i+1}| e_{i,i}$, Since $\lambda_1 \geqslant \cdots \geqslant \lambda_n$, we have

$$|\lambda_i - \lambda_{n-i+1}| \geqslant |\lambda_i - \lambda_{[\frac{n+1}{2}]}| \tag{11}$$

for all $i \in \overline{1, n}$.

Denote $b = \sum_{i=1}^{n} |\lambda_i - \lambda_{[\frac{n+1}{2}]}| e_{i,i}$. From (11) we obtain that $|D_a(u)| \geqslant b$, and thus $b \in S(M, \tau)$.

Put $v = \sum_{i=1}^{[\frac{n+1}{2}]} e_{i,i} - \sum_{j=[\frac{n+1}{2}]}^{n} e_{j,j}$. Then $vb = a - \lambda_{[\frac{n+1}{2}]} \mathbf{1}$ and $vb \in S(M, \tau)$. Therefore $a - \lambda_{[\frac{n+1}{2}]} \mathbf{1} \in S(M, \tau)$ and this element also implements the derivation D_a.

Case 2. Let M be a finite type I von Neumann algebra. Then

$$LS(M) = S(M) \cong \prod_{n \in F} M_n\big(L^0(\Omega_n)\big),$$

where $F \subseteq \mathbb{N}$. Therefore $a = \{a_n\}$, where $a_n = \sum_{i=1}^{n} \lambda_i^{(n)} e_{i,i}^{(n)}$, $\lambda_1^{(n)} \geqslant \cdots \geqslant \lambda_n^{(n)}$, $\lambda_i^{(n)} \in L^0(\Omega_n)$ and $e_{i,j}^{(n)}$ are the matrix units in $M_n(L^0(\Omega_n))$, $i, j = \overline{1, n}$, $n \in F$.

For each $n \in F$ consider the following elements in $M_n(L^0(\Omega_n)$

$$b_n = \sum_{i=1}^{n} \big|\lambda_i^{(n)} - \lambda_{[\frac{n+1}{2}]}^{(n)}\big| e_{i,i}^{(n)}$$

and

$$v_n = \sum_{i=1}^{[\frac{n+1}{2}]} e_{i,i}^{(n)} - \sum_{j=[\frac{n+1}{2}]}^{n} e_{j,j}^{(n)}.$$

Set $b = \{b_n\}_{n\in F}$ and $v = \{v_n\}_{n\in F}$. Consider the element

$$\lambda = \{\lambda_{[\frac{n+1}{2}]}\}_{n\in F} \in L^0(\Omega) \cong \prod_{n\in F} L^0(\Omega_n).$$

Similar to the case 1 we obtain that $a - \lambda\mathbf{1} = vb \in S(M,\tau)$.

Case 3. M is a type I_∞ von Neumann algebra. Since $S(M,\tau) \subseteq S(M)$ by Lemma 3.4 there exists an element $a \in S(M)$ such that $D(x) = ax - xa$ for all $x \in M$. Let us show that a can be picked from the algebra $S(M,\tau)$. Since $a \in S(M)$, there exists $\lambda \in \mathbb{R}$, $\lambda > 0$ such that $f = e_{-\lambda} \vee e_\lambda^\perp$ is a finite projection.

Suppose that $z_0 \in Z$ is a central projection such that $z_0 gMg$ is a finite von Neumann algebra, where $g = e_{-\lambda}^\perp \wedge e_\lambda = e_\lambda - e_{-\lambda}$. Then $z_0\mathbf{1} = z_0 f + z_0 g$ is a finite projection and thus $z_0 = 0$. Therefore gMg is a type I_∞ von Neumann algebra, in particular $z(g) = \mathbf{1}$. There exists a central projection z in M such that $zf \preceq zg$ and $z^\perp f \succeq z^\perp g$. Since gMg is a type I_∞ von Neumann algebra, we have that $z^\perp g = 0$. From $z(g) = \mathbf{1}$ one has $z^\perp = 0$ and therefore $f \preceq g$. This means that there exists $q \leqslant g$ such that $q \sim f$. Let u be a partial isometry in M such that $uu^* = q$, $u^*u = f$. Similar to Lemma 3.4 we obtain that $uafu - uu^*af \in S(M,\tau)$ and $af = a(e_{-\lambda} \vee e_\lambda^\perp) \in S(M,\tau)$. Therefore $a \in S(M,\tau)$. The proof is complete. □

Let N be a commutative von Neumann algebra, then $N \cong L^\infty(\Omega)$ for an appropriate measure space (Ω, Σ, μ). It has been proved in [3,13] that the algebra $LS(N) = S(N) \cong L^0(\Omega)$ admits nontrivial derivations if and only if the measure space (Ω, Σ, μ) is not atomic.

Let τ be a faithful normal semi-finite trace on the commutative von Neumann algebra N and suppose that the Boolean algebra $P(N)$ of projections is not atomic. This means that there exists a projection $z \in N$ with $\tau(z) < \infty$ such that the Boolean algebra of projection in zN is continuous (i.e. has no atom). Since $zS(N,\tau) = zS(N) = S(zN)$, the algebra $zS(N,\tau)$ admits a nontrivial derivation δ. Putting

$$\delta_0(x) = \delta(zx), \quad x \in S(N,\tau),$$

we obtain a nontrivial derivation δ_0 on the algebra $S(N,\tau)$. Therefore, we have that if a commutative von Neumann algebra N has a nonatomic Boolean algebra of projections then the algebra $S(N,\tau)$ admits a non-zero derivation.

Lemma 4.2. *If N is a commutative von Neumann algebra with a faithful normal semi-finite trace τ and δ is a derivation on $S(N,\tau)$ then $\tau(z_\delta) < \infty$, where z_δ is the support of the derivation δ.*

Proof. Suppose the opposite, i.e. $\tau(z_\delta) = \infty$. Then there exists a sequence of mutually orthogonal projections $z_n \in N$, $n = 1, 2, \ldots$, with $z_n \leqslant z_\delta$, $1 \leqslant \tau(z_n) < \infty$. For $z = \sup_n z_n$ we have $\tau(z) = \infty$. Since $\tau(z_n) < \infty$ for all $n = 1, 2, \ldots$, it follows that $z_n S(N,\tau) = z_n S(N) = S(z_n N)$. Define a derivation $\delta_n : S(z_n N) \to S(z_n N)$ by

$$\delta_n(x) = z_n \delta(x), \quad x \in S(z_n N).$$

Since $z_{\delta_n} = z_n$, Lemma 2.6 implies that for each $n \in \mathbb{N}$ there exists an element $\lambda_n \in z_n N$ such that $|\lambda_n| \leqslant z_n$ and $|\delta_n(\lambda_n)| \geqslant n z_n$.

Put $\lambda = \sum_{n\geqslant 1} \lambda_n$. Then $|\lambda| \leqslant \sum_{n\geqslant 1} z_n \leqslant \mathbf{1}$ and therefore $\lambda \in S(N,\tau)$. On the other hand

$$\left|\delta(\lambda)\right| = \left|\delta\left(\sum_{n\geqslant 1} \lambda_n\right)\right| = \left|\delta\left(\sum_{n\geqslant 1} z_n\lambda_n\right)\right| = \left|\sum_{n\geqslant 1} z_n\delta(\lambda_n)\right| = \sum_{n\geqslant 1}\left|\delta_n(\lambda_n)\right| \geqslant \sum_{n\geqslant 1} nz_n,$$

i.e. $|\delta(\lambda)| \geqslant \sum_{n\geqslant 1} nz_n$. But $\tau(z_n) \geqslant 1$ for all $n \in \mathbb{N}$, i.e. $\sum_{n\geqslant 1} nz_n \notin S(N,\tau)$. Therefore $\delta(\lambda) \notin S(N,\tau)$. The contradiction shows that $\tau(z_\delta) < \infty$. The proof is complete. $\square$

Let M be a homogeneous von Neumann algebra of type $\mathrm{I}_n, n \in \mathbb{N}$, with the center Z and a faithful normal semi-finite trace τ. Then the algebra M is $*$-isomorphic with the algebra $M_n(Z)$ of all $n \times n$- matrices over Z, and the algebra $S(M,\tau)$ is $*$-isomorphic with the algebra $M_n(S(Z,\tau_Z))$ of all $n \times n$ matrices over $S(Z,\tau_Z)$, where τ_Z is the restriction of the trace τ onto the center Z.

Now let M be an arbitrary finite von Neumann algebra of type I with the center Z and let $\{z_n\}_{n\in F}$, $F \subseteq \mathbb{N}$, be a family of central projections from M with $\sup_{n\in F} z_n = \mathbf{1}$ such that the algebra M is $*$-isomorphic with the C^*-product of von Neumann algebras z_nM of type I_n respectively, $n \in F$, i.e.

$$M \cong \bigoplus_{n\in F} z_nM.$$

In this case we have that

$$S(M,\tau) \subseteq \prod_{n\in F} S(z_nM,\tau_n),$$

where τ_n is the restriction of the trace τ onto z_nM, $n \in F$.

Suppose that D is a derivation on $S(M,\tau)$, and let δ be its restriction onto the center $S(Z,\tau_Z)$. Since δ maps each $z_nS(Z,\tau_Z) \cong Z(S(z_nM,\tau_n))$ into itself, δ generates a derivation δ_n on $z_nS(Z,\tau_Z)$ for each $n \in F$.

Let D_{δ_n} be the derivation on the matrix algebra $M_n(z_nZ(S(M,\tau))) \cong S(z_nM,\tau_n)$ defined as in (1). Put

$$D_\delta\left(\{x_n\}_{n\in F}\right) = \left\{D_{\delta_n}(x_n)\right\}, \quad \{x_n\}_{n\in F} \in S(M,\tau). \tag{12}$$

By Lemma 4.2 $\tau(z_\delta) < \infty$, thus

$$z_\delta S(M,\tau) = z_\delta S(M) \cong z_\delta \prod_{n\in F} S(z_nM) = z_\delta \prod_{n\in F} S(z_nM,\tau_n),$$

and therefore $\{D_{\delta_n}(x_n)\} \in z_\delta S(M,\tau)$ for all $\{x_n\}_{n\in F} \in S(M,\tau)$. Hence we obtain that the map D_δ is a derivation on $S(M,\tau)$.

Similar to Lemma 2.3 one can prove the following.

Lemma 4.3. *Let M be a finite von Neumann algebra of type I with a faithful normal semi-finite trace τ. Each derivation D on the algebra $S(M,\tau)$ can be uniquely represented in the form*

$$D = D_a + D_\delta,$$

where D_a is an inner derivation implemented by an element $a \in S(M,\tau)$, and D_δ is a derivation given as (10).

Finally Theorem 4.1 and Lemma 4.3 imply the following main result the present section.

Theorem 4.4. *Let M be a type I von Neumann algebra with a faithful normal semi-finite trace τ. Then every derivation D on the algebra $S(M,\tau)$ can be uniquely represented in the form*

$$D = D_a + D_\delta,$$

where D_a is inner and implemented by an element $a \in S(M,\tau)$ and D_δ is the derivation of the form (12) *generated by a derivation δ on the center of $S(M,\tau)$.*

If we consider the measure topology t_τ on the algebra $S(M,\tau)$ (see Section 1) then it is clear that every non-zero derivation of the form D_δ is discontinuous in t_τ. Therefore the above Theorem 4.4 implies

Corollary 4.5. *Let M be a type I von Neumann algebra with a faithful normal semi-finite trace τ. A derivation D on the algebra $S(M,\tau)$ is inner if and only if it is continuous in the measure topology.*

5. An application to the description of the first cohomology group

Let A be an algebra. Denote by $Der(A)$ the space of all derivations (in fact it is a Lie algebra with respect to the commutator), and denote by $InDer(A)$ the subspace of all inner derivations on A (it is a Lie ideal in $Der(A)$).

The factor-space $H^1(A) = Der(A)/InDer(A)$ is called the first (Hochschild) cohomology group of the algebra A (see [5]). It is clear that $H^1(A)$ measures how much the space of all derivations on A differs from the space on inner derivations.

The following result shows that the first cohomology groups of the algebras $LS(M)$, $S(M)$ and $S(M,\tau)$ are completely determined by the corresponding cohomology groups of their centers (cf. [3, Corollary 3.1]).

Theorem 5.1. *Let M be a type I von Neumann algebra with the center Z and a faithful normal semi-finite trace τ. Suppose that z_0 is a central projection such that $z_0 M$ is a finite von Neumann algebra, and $z_0^\perp M$ is of type* I_∞. *Then*

(a) $H^1(LS(M)) = H^1(S(M)) \cong H^1(S(z_0 Z))$;
(b) $H^1(S(M,\tau)) \cong H^1(S(z_0 Z, \tau_0))$, *where τ_0 is the restriction of τ onto $z_0 Z$.*

Proof. It immediately follows from Theorems 2.8, 3.6 and 4.4. □

Remark 2. In the algebra $S(M,\tau)$ equipped with the measure topology t_τ one can consider another possible cohomology theories. Similar to [8] consider the space $Der_c(A)$ of all continuous derivation on a topological algebra A and define the first cohomology group $H_c^1(A) = Der_c(A)/InDer(A)$.

Under these notations the above results and Corollary 4.5 imply the following result (cf. [8, Theorem 4.4]).

Corollary 5.2. *Let M be a type I von Neumann algebra with the center Z and a faithful normal semi-finite trace τ. Consider the topological algebra $S(M,\tau)$ equipped with the measure topology. Then $H_c^1(S(M,\tau))=\{0\}$.*

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