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Fundamental Study

On the representation of finite deterministic 2-tape automata

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Abstract

This paper presents properties of relations between words that are realized by *deterministic* finite 2-tape automata. It has been made as complete as possible, and is structured by the systematic use of the matrix representation of automata. It is first shown that deterministic 2-tape automata are characterized as those which can be given a *prefix* matrix representation. Schützenberger construct on representations, the one that gives semi-monomial representations for rational functions of words, is then applied to this prefix representation in order to obtain a new proof of the fact that the lexicographic selection of a deterministic rational relation on words is a rational function. © 1999 Published by Elsevier Science B.V. All rights reserved.

Résumé

Cet article donne une présentation des propriétés des relations entre mots réalisées par des automates finis à deux bandes *déterministes*, qu'on a voulu aussi complète que possible. Elle est organisée autour de la notion de représentation matricielle des automates. On montre d'abord que les automates déterministes à deux bandes sont ceux qui admettent une représentation matricielle *préfixe*. La construction de Schützenberger sur les représentations, celle qui donne les représentations semi-monomiales pour les fonctions rationnelles, est alors appliquée à cette représentation préfixe afin d'obtenir une nouvelle preuve du fait que la sélection lexicographique d'une relation rationnelle déterministe est une fonction rationnelle. © 1999 Published by Elsevier Science B.V. All rights reserved.

Keywords: Deterministic k-tape automata; Deterministic rational relations; Uniformization theorem

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0. Introduction

Automata theory is concerned with the study of various models of computational machines. The most basic of these models is probably the finite one-way automaton, with one or many input tapes. As soon as 1959, Rabin and Scott presented in a survey paper [14] – that was the reference for a long time – a number of results and problems on finite one-way automata, the last of which – the decidability of the equivalence of deterministic k-tape automata – has been solved only recently and by means of purely algebraic methods [9].

Relations between sequences of symbols, or *words*, that are computed, or accepted by 2-tape automata, are called here *rational relations* and have proved to be a very powerful concept in formal language theory (cf. [2]) as well as an ubiquitous computation model in Computer Science, from compiler construction to natural language processing. Finite automata with one input tape are easily shown to be equivalent to deterministic ones, whereas this result does not hold anymore for finite automata with two or more input tapes, which means that not every rational relation is deterministic.

The purpose of this paper is to present the properties of relations between words that are realized by *deterministic* finite 2-tape automata. We have tried to make it as complete as possible. It thus contains the description, and the proof, of properties of deterministic rational relations that are often considered as folklore. The presentation is structured by the systematic use of the matrix representation of automata. It aims at a new proof of the Lexicographic Uniformization Theorem, which deserves few words of presentation.

In his treatise on automata, Eilenberg established that every *rational function* can be made *unambiguous*, by means of the so-called *Rational Cross-section Theorem* [4]. As a corollary, he then stated the Rational Uniformization Theorem. [1] Along the line of this Cross-section Theorem, we showed that, under certain conditions, the *lexicographic cross-section* of the mapping equivalence of a morphism is rational [15]. Johnson then raised the problem whether *rational equivalence relations always have rational cross-section* – problem which remains open up to now – and gave a positive answer for deterministic relations, showing that the lexicographic uniformization of a deterministic equivalence relation is rational [10]. On the other hand, Schützenberger gave another method for proving that rational functions are unambiguous, via the construction of *semi-monomial matrix representation* for rational functions [20, 2, 17].

The main contribution of this paper is to show how the Schützenberger construct on representations allows to find again the uniformization result for deterministic relations. After having recalled the definition of deterministic relations (Section 2), we show that deterministic relations are characterized as those that have *prefix* representations (Section 3). We then explain how the Schützenberger construct on representations yields uniformization results when applied to general relations instead of functions. This construction when applied to the prefix representation of a deterministic relation gives then a lexicographic uniformization (Section 5). These two results have been presented in [12].

1. Preliminaries

We first recall the definition of automata as labelled graphs, that makes natural the generalization from automata on a free monoid to automata on direct product of free monoids which is the way we define 2-tape automata. We then present the notions of matrix representation of automata – that yields Kleene–Schützenberger Theorem – and of *covering of automata*.

Notations. The free monoid over a finite alphabet A is denoted by A^*, its identity element, the *empty word*, by 1_{A^*} and the set of words different from 1_{A^*} by A^+. Accordingly, the identity of a monoid M is denoted by 1_M, by 1 if no ambiguity is feared.

The length of a word f in A^* is denoted by $|f|$ and $|f|_a$ is the number of letters a which appear in f. A word f in A^* is a *prefix* (resp. a *strict prefix*) of a word g – denoted by $f \leqslant g$ (resp. $f < g$) – if there exists a word h in A^* (resp. in A^+) such that $g = fh$. If $f \leqslant g$ then $f^{-1}g$ denotes the word h such that $g = fh$.

[1] Cf. Section 5 for a more detailed presentation of this result.

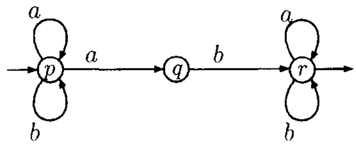

Fig. 1. The automaton $\mathscr{A}_1$.

1.1. The model of labelled graphs for automata

An *automaton* $\mathscr{A}$ *over a finite alphabet* A, $\mathscr{A} = \langle Q, A, E, I, T\rangle$, is a directed graph labelled by elements of A; Q is the set of *states*, $I \subset Q$ is the set of *initial* states, $T \subset Q$ is the set of *terminal* states and $E \subset Q \times A \times Q$ is the set of labelled *edges* called *transitions*. The automaton $\mathscr{A}$ is *finite* if Q is finite; we shall consider only finite automata and thus call them simply *automata* in the sequel. We also note $p \xrightarrow{q}$ for $(p,a,q) \in E$, or even $p \xrightarrow[\mathscr{A}]{a} q$ if there is a possible ambiguity on the automaton. A *computation* c in $\mathscr{A}$ is a finite sequence of transitions:

$$c : p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \cdots p_{n-1} \xrightarrow{a_n} p_n$$

The *label* of c is the element $a_1 a_2 \cdots a_n$ of A^*. The computation c is *successful* if p_0 is in I and p_n in T. The language accepted by $\mathscr{A}$, also called *behaviour* of $\mathscr{A}$, is the subset $|\mathscr{A}|$ of A^* consisting of labels of successful computations of $\mathscr{A}$.

A state q is said to be *accessible* if there exists a path in $\mathscr{A}$ starting in I and ending in q. The *accessible part* of $\mathscr{A}$ is the set of its accessible states together with the corresponding edges. A state q is said to be *co-accessible* if there exists a path in $\mathscr{A}$ starting in q and ending in T. The *co-accessible part* of $\mathscr{A}$ is the set of its co-accessible states together with the corresponding edges. An automaton $\mathscr{A}$ is said to be *trim* if every state q is *accessible and co-accessible*.

The automaton $\mathscr{A}$ is *complete* if for every state p in Q and every letter a in A there exists *at least one* state q such that (p,a,q) is an edge in E; $\mathscr{A}$ is *deterministic* if for every state p in Q and every letter a in A there exists *at most one* state q such that (p,a,q) is an edge in E; $\mathscr{A}$ is *co-deterministic* if for every state q in Q and every letter a in A there exists *at most one* state p such that (p,a,q) is an edge in E. The automaton $\mathscr{A}$ is *unambiguous* if for every *pair* of states (p,q) and every *word* f in A^* there exists *at most one* computation from p to q with label f.

Automata have a natural graphic representation as labelled graphs.

Example 1.1. Fig. 1 shows an automaton $\mathscr{A}_1$ whose behaviour is the set of words with a factor *ab*.

The definition of automata as labelled graphs extends readily to automata *over any monoid* M: an *automaton* $\mathscr{A}$ *over* M, $\mathscr{A} = \langle Q, M, E, I, T\rangle$ is a directed graph the edges of which are labelled by elements of the monoid M. The automaton $\mathscr{A}$ is finite if *the set of edges* $E \subset Q \times M \times Q$ is finite (and thus Q is finite). The label of a computation

$$c : p_0 \xrightarrow{m_1} p_1 \xrightarrow{m_2} p_2 \cdots p_{n-1} \xrightarrow{m_n} p_n$$

is the element $m_1 m_2 \cdots m_n$ of M. The *behaviour* of $\mathscr{A}$ – obviously, we do not say "language accepted" in this case – is the subset $|\mathscr{A}|$ of M consisting of labels of successful computations of $\mathscr{A}$. Two automata are said to be *equivalent* if they have the same behaviour. In this context, an automaton over an alphabet A is to be understood as an automaton over the free monoid A^*.

For a monoid M, the rational closure of finite sets – i.e. the least family of subsets of M containing the finite subsets and closed under *union*, *product* and the "*star*" operation – is denoted by $\operatorname{Rat} M$; its elements are the rational sets of M. The following generalization of Kleene's theorem is due to Elgot and Mezei (cf. [16] for more details).

Theorem 1.1 (Elgot and Mezei [5]). *A subset of M is rational if and only if it is the behaviour of a finite automaton over M, the labels of the edges of the automaton being taken in any set of generators of M.*

The set E of labelled edges of an automaton $\mathscr{A} = \langle Q, M, E, I, T\rangle$ is currently identified with the *incidence matrix* of the graph $\mathscr{A}$. It is a $(Q \times Q)$-matrix the entries of which are finite subsets of M: every $E_{p,q}$ is the set of labels of edges from the state p to the state q. Along the same line, the subsets I and T are identified with a Boolean row-vector, respectively column-vector, of dimension Q. Since Q and M can be recovered by projection from E, one can denote also $\mathscr{A}$ by $\mathscr{A} = \langle I, E, T\rangle$. Obviously, $E^* = \bigcup_{n\in\mathbb{N}} E^n$ and, with these notation,

$$|\mathscr{A}| = I \cdot E^* \cdot T$$

holds where the $\cdot$ indicates the matrix multiplication. The same equation shows that the entries of E, and of I and T, can be taken in $\operatorname{Rat} M$ without changing the generating power of finite automata.

We are concerned here with (finite) 2-tape automata, or 2-*automata* for short, that is, in the above terminology, with automata over a direct product $A^* \times B^*$ of free monoids. (They have also been called in the literature as *generalized sequential machines* or *transducers.*[2]) The behaviour of such an automaton $\mathscr{A} = \langle Q, A^* \times B^*, E, I, T\rangle$ is a subset of $A^* \times B^*$, that is the graph[3] of a relation θ from A^* into B^* :

$$\forall f \in A^* \quad f\theta = \{g \in B^* \mid (f,g) \in |\mathscr{A}|\}.$$

In this case, we also say that $\mathscr{A}$ *realizes* θ.

A relation from A^* into B^* is said to be *rational* if and only if its graph is a rational subset of $A^* \times B^*$, that is, according to Theorem 1.1 if and only if it is the behaviour of an automaton over $A^* \times B^*$. We denote by $\operatorname{Rat} A^* \times B^*$ the set of rational relations over $A^* \times B^*$, by Rat_2 if the alphabet is not specified.

[2] Admittedly, the notation "2-automaton" is not completely satisfactory. In the area of automata reading *numbers*, it comes into collision with the notation "k-automaton" that refers to automata reading numbers written in base k. Moreover, it does not translate easily into French. We stick to it, however, for it supports one of the idea we illustrate here: technics developed for classical automata are relevant to 2-tape automata.

[3] By abuse, we currently identify a relation and its graph and we shall write $(f,g) \in \theta$ for $g \in f\theta$.

A rational relation, or the automaton $\mathscr{A}$ that realizes it, is *unambiguous* if any element of $|\mathscr{A}|$ is the label of a *unique* successful computation in $\mathscr{A}$.

The set $(A \times \{1_{B^*}\}) \cup (\{1_{A^*}\} \times B)$ is a set of generators of $A^* \times B^*$: a 2-automaton the edges of which are labelled by elements of $(A \times \{1_{B^*}\}) \cup (\{1_{A^*}\} \times B)$ is said to be *normalized* and it follows from Theorem 1.1 that any 2-automaton is equivalent to a normalized one. We shall currently denote a normalized 2-automaton over $A^* \times B^*$ as a sextuple $\langle Q, A, B, E, I, T\rangle$.

If $\mathscr{A}$ is a normalized automaton the matrix E can be written as $E = X + Y$ where the entries of X are in $(A \times \{1_{B^*}\})$ and those of Y are in $(\{1_{A^*}\} \times B)$. The straightforward, and classical, following computation,

$$|\mathscr{A}| = I \cdot (X + Y)^* \cdot T = I \cdot (Y^*X)^* Y^* \cdot T = I \cdot (Y^*X)^* \cdot (Y^*T), \tag{1.1}$$

shows that $\mathscr{A}$ is equivalent to the automaton $\mathscr{A}' = \langle I, E', T'\rangle$ with $E' = (Y^*X)$ and $T' = (Y^*T)$. This computation has broken the symmetry between the first and second component of $A^* \times B^*$, between the "two tapes" of $\mathscr{A}$. But as a result, the label of every edge of $\mathscr{A}'$ is in $A \times \operatorname{Rat} B^*$, that is *exactly one letter* is "read" on the first tape *at every move* or transition of $\mathscr{A}'$; such an automaton is often called a *real-time transducer*.

1.2. *Matrix representation of automata*

Automata over a free monoid are classically given matrix representations. The technic extends to automata over a product of free monoids $A^* \times B^*$.

1.2.1. *Boolean matrix representation of automata over A^**

Any finite automaton $\mathscr{A} = \langle Q, A, E, I, T\rangle$ may be given a *matrix representation* (λ, μ, ν) over the Boolean semiring $\mathbb{B}$ where $\mu : A^* \to \mathbb{B}^{Q \times Q}$ is the morphism defined by

$$\forall p, q \in Q, \quad \forall a \in A \;\; a\mu_{p,q} = \begin{cases} 1 & \text{if } (p, a, q) \in E, \\ 0 & \text{otherwise,} \end{cases}$$

and where λ and ν are the row and column vectors, respectively, defined by

$$\forall q \in Q \quad \lambda_q = 1 \Leftrightarrow q \in I, \qquad \forall p \in Q \quad \nu_p = 1 \Leftrightarrow p \in T.$$

The triple (λ, μ, ν) is a *representation* of $\mathscr{A}$ in the sense that

$$|\mathscr{A}| = \{f \in A^* \mid (\lambda \cdot f\mu \cdot \mu) = 1\}.$$

The morphism μ is called *a representation* (of A^* by Boolean matrices) as well.

Example 1.1 (*continued*). The matrix representation of $\mathscr{A}_1$ is

$$\lambda = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad a\mu = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b\mu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \nu = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

1.2.2. Matrix representation of automata over $A^* \times B^*$

Let $\mathscr{A} = \langle I, E, T\rangle$ be a normalized 2-automaton and $\mathscr{A}' = \langle I, E', T'\rangle$ the automaton obtained in (1.1). Every entry of E' is an union of elements of the form

$$(a, K) = (1_{A^*}, K)(a, 1_{B^*}),$$

with a in A and K in $\operatorname{Rat} B^*$. In such writing, $(1_{A^*}, K)$ can be seen as the *coefficient* of $(a, 1_{B^*})$. By slight abuse,[4] and when no ambiguity is feared, we note K instead of $(1_{A^*}, K)$ and a instead of $(a, 1_{B^*})$. With these conventions, the matrix E' can thus be written as

$$E' = \sum_{a\in A} (a\mu)a,$$

where, for every a in A, $a\mu$ is a $(Q\times Q)$-matrix with entries in $\operatorname{Rat} B^*$. Note that, accordingly, I is a Boolean vector and that every entry of T' is in $\operatorname{Rat} B^*$. We put $\lambda = I$ and $\nu = T'$ and we have

$$\forall p, q \in Q, \quad \forall a \in A \quad a\mu_{p,q} = \{v \in B^* \mid \exists s \in Q \quad p \xrightarrow{(1,v)} s \xrightarrow{(a,1)} q\}.$$

Also λ and ν are the row and column vectors, respectively, defined by

$$\forall q \in Q, \quad \lambda_q = 1 \Leftrightarrow q \in I, \quad \forall p \in Q \quad \nu_p = \{v \in B^* \mid \exists t \in T \quad p \xrightarrow{(1,v)} t\}.$$

The mapping μ extends then into a morphism from A^* into $(\operatorname{Rat} B^*)^{Q\times Q}$. We call the triple (λ, μ, ν) *a (matrix) representation* of the automaton $\mathscr{A}$; this definition is justified by the fact that

$$|\mathscr{A}| = \{(u, v) \mid v \in \lambda \cdot u\mu \cdot \nu\}.$$

The decomposition $E = X + Y$ also implies the dual computation

$$|\mathscr{A}| = I \cdot (X + Y)^* \cdot T = I \cdot Y^*(XY^*)^* \cdot T = (IY^*) \cdot (XY^*)^* \cdot T$$

which leads to another representation (λ', μ', ν') of $\mathscr{A}$:

$$\forall p, q \in Q, \quad \forall a \in A \quad a\mu'_{p,q} = \{v \in B^* \mid \exists s \in Q \quad p \xrightarrow{(a,1)} s \xrightarrow{(1,v)} q\},$$

$$\forall q \in Q \quad \lambda'_q = \{v \in B^* \mid \exists i \in I \quad i \xrightarrow{(1,v)} q\}, \quad \forall p \in Q \quad \nu'_p = 1 \Leftrightarrow p \in T.$$

And then again

$$|\mathscr{A}| = \{(u, v) \mid v \in \lambda' \cdot u\mu' \cdot \nu'\}$$

holds. All these constructions are summed up in the following result.

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[4] It is not an abuse indeed. It is the expression of the canonical (semi-ring) isomorphism between $\mathfrak{P}(A^* \times B^*)$ and $(\mathfrak{P}(B^*))\langle\langle A\rangle\rangle$, the semi-ring of formal power series on A^* with coefficient in $\mathfrak{P}(B^*)$. It has not seemed necessary to introduce the heavy formalism of power series to deal with such intuitive evidence.

Theorem 1.2 (Kleene–Schützenberger Theorem). *Let θ be a relation from A^* into B^*. The following assertions are equivalent*:
(i) *θ is a rational relation*;
(ii) *θ is realized by a real-time 2-automaton $\mathcal{A}$ over $A^* \times B^*$*;
(iii) *θ is realized by a matrix representation (λ, μ, ν) with entries in* $\operatorname{Rat} B^*$.

Let $\mathcal{A}$ be a real-time 2-automaton and (λ, μ, ν) its representation. If every non zero entry (in λ, $a\mu$ and ν) is replaced by a 1 (from the Boolean semiring), one gets the representation of a finite 1-automaton by Boolean matrices: we call it the *underlying input automaton* of $\mathcal{A}$ (or of (λ, μ, ν)).

1.2.3. Representations of direct products of automata
The *direct product* of $\mathcal{A} = \langle Q, A, E, I, T\rangle$ and $\mathcal{B} = \langle R, A, F, J, U\rangle$ is by definition the automaton $\mathcal{A} \times \mathcal{B} = \langle Q \times R, A, G, I \times J, T \times U\rangle$ where the set G of labelled edges is defined by

$$G = \{((p,r), a, (q,s)) \mid (p,a,q) \in E,\ (r,a,s) \in F\}.$$

The operation of direct product of automata translates into the *tensor product* of their representations. Let us first recall that the tensor product of two Boolean[5] matrices X and Y of dimension $P \times Q$ and $R \times S$, respectively, is the matrix $X \otimes Y$ of dimension $(P \times R) \times (Q \times S)$ defined by

$$\forall p \in P, \quad \forall q \in Q, \quad \forall r \in R, \quad \forall s \in S \qquad X \otimes Y_{(p,r),(q,s)} = X_{p,q} Y_{r,s}.$$

It is noteworthy that $X \otimes Y$ has a natural *block decomposition* which will be currently used in the sequel: $X \otimes Y$ is a block-matrix of dimension $P \times Q$ of blocks of dimension $R \times S$ (or vice versa). The tensor product of representations makes sense because of the following.

Lemma 1.1. *Let M be any monoid and let $\mu : M \to \mathbb{B}^{Q \times Q}$ and $\kappa : M \to \mathbb{B}^{R \times R}$ be two morphisms. The mapping $\mu \otimes \kappa$ defined for every m in M by*

$$m\mu \otimes \kappa = m\mu \otimes m\kappa$$

is a morphism.

We then have

Proposition 1.2 (Schützenberger [18]). *Let (λ, μ, ν) and (η, κ, ζ) be the Boolean representations of $\mathcal{A} = \langle Q, A, E, I, T\rangle$ and $\mathcal{B} = \langle R, A, F, J, U\rangle$, respectively. Then $(\lambda, \mu, \nu) \otimes (\eta, \kappa, \zeta)$ is the representation of $\mathcal{A} \times \mathcal{B}$.*

[5] This definition and the following two results are valid for matrices with entries in any *commutative* semiring but we do not need such generality here.

By definition $(\lambda,\mu,\nu)\otimes(\eta,\kappa,\zeta)=(\lambda\otimes\eta,\mu\otimes\kappa,\nu\otimes\zeta)$. The key to Proposition 1.2 lies in the fact that, by Lemma 1.1, $\mu\otimes\kappa$ is a morphism from A^* into $\mathbb{B}^{(Q\times R)\times(Q\times R)}$.

1.3. Recognizable sets

A subset R of a monoid M is classically said to be *recognizable* if there exists a morphism φ from M into a *finite* monoid N such that $R=R\varphi\varphi^{-1}$ (cf. [2,4]). The set of recognizable subsets of M is denoted by $\operatorname{Rec} M$.

We shall make use in the sequel of a definition of recognizable sets by a more general construction than morphisms that we present first. A (*right*) *action* of a monoid M over a set S is a mapping from $S\times M$ into S, denoted by a $\cdot$, and satisfying the following conditions:

(i) $\forall s\in S\ s\cdot 1_M=s$;

(ii) $\forall s\in S, \forall m,\ m'\in M\ (s\cdot m)\cdot m'=s\cdot mm'$.

A subset R of M is *recognized* by an action of M over S if there exists an s_0 in S and a subset U of S such that

$$R=\{m\in M \mid s_0\cdot m\in U\},$$

in which case we say, by imitation of automata, that R is *recognized* by the 5-tuple $\langle S,M,\cdot,s_0,U\rangle$. It is straightforward to verify the following.

Lemma 1.3. *A subset of a monoid M is recognizable if and only if it is recognized by an action of M over a finite set.*

The terminology is coherent by virtue of Kleene's Theorem:

Theorem 1.3. *If A is a finite alphabet then* $\operatorname{Rec} A^*=\operatorname{Rat} A^*$.

A recognizable subset of $A^*\times B^*$ is also called a *recognizable relation.* The following are well-known results about recognizable relations.

Proposition 1.4. *The intersection of a rational relation and of a recognizable relation is a rational relation.*

(We reprove it in the next section (cf. Lemma 1.9), in the framework of actions that we shall need later.)

Corollary 1.5. $\operatorname{Rec}(A^*\times B^*)\subset\operatorname{Rat}(A^*\times B^*)$.

Theorem 1.4 (Mezei; cf. Berstel [2]). *A subset of $A^*\times B^*$ is recognizable if and only if it is a finite union of cartesian products of the form $S\times T$ with S in* $\operatorname{Rat} A^*$ *and T in* $\operatorname{Rat} B^*$.

Corollary 1.6. $\operatorname{Rec}(A^*\times B^*)$ *is closed under product.*

1.4. Covering and co-covering of automata

The aim of this section is to adapt and extend to automata the notion of *covering* as defined by Stallings [22] for graphs. This has been already published, partly in [8], more completely in [17], and is included here for sake of completeness.

1.4.1. Morphism of automata

Given an automaton $\mathscr{A} = \langle Q, M, E, I, T \rangle$, the set E of labelled edges is canonically equipped with three mappings (the three projections):

$$\iota : E \to Q, \ \tau : E \to Q \quad \text{and} \quad \varepsilon : E \to M.$$

The vertices $e\iota$ and $e\tau$ are respectively the *origin* and the *end* of the edge e; $e\varepsilon$ is the *label* of the edge e.

A morphism φ from *an automaton* $\mathscr{B} = \langle R, M, F, J, U \rangle$ *into an automaton* $\mathscr{A} = \langle Q, M, E, I, T \rangle$ is indeed a pair of mappings (both denoted by φ) $\varphi : R \to Q$ and $\varphi : F \to E$, which satisfy the three properties [6] :

$$\varphi \circ \iota = \iota \circ \varphi \quad \text{and} \quad \varphi \circ \tau = \tau \circ \varphi, \tag{1.2}$$

$$\varphi \circ \varepsilon = \varepsilon, \tag{1.3}$$

$$J\varphi \subseteq I \quad \text{and} \quad U\varphi \subseteq T. \tag{1.4}$$

Conditions (1.2) imply that *the image of a path in* $\mathscr{B}$ *is a path in* $\mathscr{A}$. Condition (1.3) implies that *the label of a path in* $\mathscr{B}$ *is the same as the label of the image of that path in* $\mathscr{A}$. Conditions (1.4) imply that *the image of a successful path in* $\mathscr{B}$ *is a successful path in* $\mathscr{A}$ *– and with the same label.* In particular $|\mathscr{B}| \subseteq |\mathscr{A}|$.

Example 1.2. The classical construction of *direct product* of automata gives rise to an important instance of morphism of automata. Let $\mathscr{A} \times \mathscr{B} = \langle Q \times R, A, G, I \times J, T \times U \rangle$ be the direct product of $\mathscr{A} = \langle Q, A, E, I, T \rangle$ and $\mathscr{B} = \langle R, A, F, J, U \rangle$. The projections $\pi_{\mathscr{A}}$ and $\pi_{\mathscr{B}}$ from the set $Q \times R$ on the first and on the second components respectively, together with the corresponding mappings from G into E and F are clearly morphisms from $\mathscr{A} \times \mathscr{B}$ onto $\mathscr{A}$ and $\mathscr{B}$, respectively.

1.4.2. Covering and co-covering

For every state q of an automaton $\mathscr{A} = \langle Q, M, E, I, T \rangle$, let us denote by $\mathrm{Out}_{\mathscr{A}}(q)$ the set of edges of $\mathscr{A}$ the origin of which is q, that is, edges that are "going out" of q:

$$\mathrm{Out}_{\mathscr{A}}(q) = \{ e \in E \mid e\iota = q \},$$

and let us denote by $\mathrm{In}_{\mathscr{A}}(q)$ the set of edges of $\mathscr{A}$ the end of which is q, that is, edges that are "arriving at" q:

$$\mathrm{In}_{\mathscr{A}}(q) = \{ e \in E \mid e\tau = q \}.$$

[6] Though we use the postfixed notation for functions (e.g. $e\iota$) we find it clearer to indicate composition of functions explicitely by a symbol ($\circ$) than with the mere concatenation.

If φ is a morphism from $\mathscr{B} = \langle R, M, F, J, U \rangle$ into $\mathscr{A} = \langle Q, M, E, I, T \rangle$ then for every r in R, φ maps $\mathrm{Out}_{\mathscr{B}}(r)$ into $\mathrm{Out}_{\mathscr{A}}(r\varphi)$, and $\mathrm{In}_{\mathscr{B}}(r)$ into $\mathrm{In}_{\mathscr{A}}(r\varphi)$.

We say that φ is *Out-surjective* (resp. *Out-bijective, Out-injective*) if for every r in R the restriction of φ to $\mathrm{Out}_{\mathscr{B}}(r)$ is surjective onto $\mathrm{Out}_{\mathscr{A}}(r\varphi)$ (resp. bijective between $\mathrm{Out}_{\mathscr{B}}(r)$ and $\mathrm{Out}_{\mathscr{A}}(r\varphi)$, injective). Accordingly, we say that φ is *In-surjective* (resp. *In-bijective, In-injective*) if for every r in R the restriction of φ to $\mathrm{In}_{\mathscr{B}}(r)$ is surjective onto $\mathrm{In}_{\mathscr{A}}(r\varphi)$ (resp. bijective between $\mathrm{In}_{\mathscr{B}}(r)$ and $\mathrm{In}_{\mathscr{A}}(r\varphi)$, injective).

What we call *Out-bijective morphism* is exactly what Stallings calls a *covering (of graphs)*. The definition of *covering of automata* we are now coining is consistent with the one of covering of graphs and puts also in relation the initial states and the terminal states respectively.

Definition 1.1. A morphism φ from an automaton $\mathscr{B} = \langle R, M, F, J, U \rangle$ into an automaton $\mathscr{A} = \langle Q, M, E, I, T \rangle$ is a *covering* if the following conditions hold:
(i) φ is *Out-bijective*;
(ii) for every i in I, there exists a *unique* j in J such that $j\varphi = i$;
(iii) for every t in T, $t\varphi^{-1} \subset U$ (i.e. by (1.4) $T\varphi^{-1} = U$).

We also need the dual definition:

Definition 1.2. A morphism φ from $\mathscr{B}$ into $\mathscr{A}$ is a *co-covering* if the following conditions hold:
(i) φ is *In-bijective*;
(ii) for every i in I, $i\varphi^{-1} \subset J$ (i.e. by (1.4) $I\varphi^{-1} = J$);
(iii) for every t in T, there exists a *unique* s in S such that $s\varphi = t$.

The immediate consequence of these definitions is the following (cf. [17]).

Proposition 1.7. *If $\varphi : \mathscr{B} \to \mathscr{A}$ is a covering, or a co-covering, then for every successful path c in $\mathscr{A}$ there exists a unique successful path d in $\mathscr{B}$ such that $d\varphi = c$ (and thus $\mathscr{B}$ is equivalent to $\mathscr{A}$).*

Corollary 1.8. *A trim covering (resp. co-covering) of an unambiguous automaton is an unambiguous automaton.*

The last definitions we need are those of *immersion* and *co-immersion.*

Definition 1.3. A morphism φ from $\mathscr{B}$ into $\mathscr{A}$ is an *immersion* if the following conditions hold:
(i) φ is *Out-injective*;
(ii) for every i in I there exists *at most one* j in J such that $j\varphi = i$.

Definition 1.4. A morphism φ from $\mathscr{B}$ into $\mathscr{A}$ is a *co-immersion* if the following conditions hold:
(i) φ is *In-injective*;
(ii) for every t in T there exists *at most one* u in U such that $u\varphi = t$.

Roughly speaking an immersion (resp. a co-immersion) is a covering (resp. a co-covering) from which some edges have been removed and where some states have lost the property of being initial or terminal.

If $\varphi : \mathscr{B} \to \mathscr{A}$ is an immersion (resp. a co-immersion), it is not only true that $|\mathscr{B}| \subseteq |\mathscr{A}|$ – which holds as soon as there exists a morphism from $\mathscr{B}$ into $\mathscr{A}$ – but φ is moreover *an injection* from the set of successful pathes of $\mathscr{B}$ into the set of successful pathes of $\mathscr{A}$.

Example 1.3. A *subautomaton* $\mathscr{B}$ of $\mathscr{A}$, that is an automaton obtained from $\mathscr{A}$ by deleting edges and/or by *suppressing the quality of being initial or terminal* to certain states is an immersion (the morphism being the identity mapping on the set of states).

It will be convenient to say that $\mathscr{B}$ *covers* $\mathscr{A}$ or is a *covering of* $\mathscr{A}$ (resp. is an *immersion in* $\mathscr{A}$) if there exists a morphism $\varphi : \mathscr{B} \to \mathscr{A}$ that is a covering (resp. an immersion).

1.4.3. An example

As an illustration of the above definition let us state a refinement of the classical proposition asserting that, *in any monoid*, the intersection of a rational set with a recognizable set is rational.

Lemma 1.9. *Let $\mathscr{A}$ be an automaton over a monoid M and R a recognizable subset of M. There exists then an immersion $\mathscr{B}$ in $\mathscr{A}$ such that $|\mathscr{B}| = |\mathscr{A}| \cap R$.*

Proof. Let $\mathscr{A} = \langle Q, M, E, I, T \rangle$. The classical construction is indeed what is needed to establish the lemma and can be performed on *any action* $\langle P, M, \cdot, p_0, U \rangle$ that recognizes R.

Let $\mathscr{B}' = \langle Q \times P, M, F, J, S' \rangle$ be defined as follows:

$$F = \{((q, p), m, (q', p \cdot m)) \mid p \in P, \quad (q, m, q') \in E\}, \quad J = I \times \{p_0\} \quad \text{and}$$
$$S' = T \times P.$$

The projection of $Q \times P$ onto Q induces then, by construction, a covering of $\mathscr{A}$ by $\mathscr{B}'$. By induction on the length of the computations,

$$(q, p_0) \xrightarrow[\mathscr{B}']{m} (q', p) \quad \text{implies that} \quad p = p_0 \cdot m.$$

Let $\mathscr{B} = \langle Q \times P, M, F, J, S \rangle$ with $S = T \times U$. Then $|\mathscr{B}| = |\mathscr{B}'| \cap R = |\mathscr{A}| \cap R$ holds. $\square$

1.5. *Miscellaneous*

We end this preliminary section with two properties that are easy exercises, but not classical enough to be simply called by reference.

1.5.1. *Factorizing the elements of $A^* \times B^*$*

The monoid $A^* \times B^*$ is not a free monoid, i.e. every element *has not a unique factorization* into elements of a base. In a sense however, "it is not far" from being free. This is what we try to state formally here and that will be used in the next section.

The set $X = (A \times \{1_{B^*}\}) \cup (\{1_{A^*}\} \times B)$ is the minimal generating system of $A^* \times B^*$. That is, every element (u,v) of $A^* \times B^*$ is factorized as a product of elements of X: $(u,v) = x_1x_2 \ldots x_n$, $x_i \in X$. As said before, this factorization is not unique, e.g. $(a,b) = (a,1)(1,b) = (1,b)(a,1)$. We define the *length* of (u,v), denoted by $|(u,v)|$, to be the quantity $|u| + |v|$. It is the common length of *all* factorizations of (u,v) over X. This terminology is legitimate since $|(u,v)(u',v')|, = |(u,v)| + |(u',v')|$.

An element (u,v) of $A^* \times B^*$ is a *prefix* of (f,g) if there exists (h,k) such that $(f,g) = (u,v)(h,k)$; in such case, u is a prefix of f, $f = uh$, and v is a prefix of g, $g = vk$. The relation "being a prefix" is a partial ordering of $A^* \times B^*$ called the *prefix ordering*. The set of all prefixes of any element (f,g) is a lattice for the prefix ordering. A maximal chain of prefixes of (f,g) uniquely determines a factorization of (f,g) over X and conversely.

Lemma 1.10. *Let (f,g) in $A^* \times B^*$ and let (u,v) and (u',v') be any two prefixes of (f,g) which may be equal but such that none is a strict prefix of the other. Let $(u,v) = x_1x_2 \ldots x_n$ and $(u',v') = y_1y_2 \ldots y_m$ be any two distinct*[7] *factorizations of these elements and let $x_1x_2 \ldots x_{i-1} = y_1y_2 \ldots y_{i-1}$ be the longest common prefix of these two factorizations (as sequences of X^* i.e. $x_j = y_j$ for every j, $0 \leqslant j \leqslant i-1$, and $x_i \neq y_i$). Then $x_i \in (A \times \{1_{B^*}\})$ and $y_i \in (\{1_{A^*}\} \times B)$, or vice versa.*

Proof. Let $(h,k) = x_1x_2 \ldots x_{i-1}$. Suppose that both x_i and y_i belong to $(A \times \{1_{B^*}\}) : x_i = (a',1)$ and $y_i = (a'',1)$. Then $(h,k)x_i = (ha',k)$ and $(h,k)y_i = (ha'',k)$, ha' and ha'' are both *the* prefix of f of length $|h| + 1$. Thus $a' = a''$, a contradiction with $x_i \neq y_i$. The same contradiction arises if x_i and y_i are both in $(\{1_{A^*}\} \times B)$. □

1.5.2. *Prefix families of languages*

A subset K of A^* is said to be *prefix* if no element of K is a prefix of another element of K, i.e.

$$\forall f,h \in A^* \quad f \in K \text{ and } fh \in K \;\Rightarrow\; h = 1_{A^*}.$$

[7] This has to be specified since $(u,v) = (u',v')$ is possible.

A family $\{K_j\}_{j\in J}$ of subsets of A^* is called a *prefix family* if the following two conditions are met: (i) the K_j are pairwise disjoint, (ii) the union $K = \bigcup_{j\in J} K_j$ is prefix, i.e.

$$\forall i, j \in J, \quad \forall f, h \in A^* \; f \in K_i \text{ and } fh \in K_j \;\Rightarrow\; h = 1_{A^*} \text{ and } i = j.$$

We want to characterize families of prefix rational sets by means of automata. First, we call *exit automaton*, or *e-automaton* for short, an automaton where there is no edge "going out" from a terminal state (i.e. as soon as a terminal state is reached, the computation halts).

Lemma 1.11. *A rational language K is prefix if and only if it is the behaviour of a trim deterministic e-automaton.*

Proof. Let K be a rational prefix language and let $\mathscr{A}$ be a trim deterministic automaton which recognizes K. Assume that there exists an edge $t \xrightarrow{a} q$, where t is a terminal state. Since $\mathscr{A}$ is trim, t is accesssible and q is co-accesssible. There are thus two words f and g such that f and fag are in K, a contradiction with K prefix.

Conversely, let K be the behaviour of a trim deterministic e-automaton $\mathscr{A} = \langle Q, A, E, q_0, T\rangle$. If f and fh are elements of K, then the two states t and s uniquely defined by $q_0 \xrightarrow{f} t \xrightarrow{h} s$ are terminal. By hypothesis, $s = t$ and $h = 1_{A^*}$, hence K is prefix. $\square$

By a slight abuse, we say that an automaton $\mathscr{A} = \langle Q, A, E, I, T\rangle$ *recognizes a finite family of languages* $\{K_j\}_{j\in J}$ if there exists a *partition* $\{T_j\}_{j\in J}$ of T such that every K_j is the behaviour of $\mathscr{A}_j = \langle Q, A, E, I, T_j\rangle$.

Any finite family of rational languages is recognized by a finite automaton. This can be derived from the first part of proof of the following statement.

Lemma 1.12. *A finite family of languages $\{K_j\}_{j\in J}$ is a prefix family of rational languages if and only if there exists a trim deterministic e-automaton which recognizes the family $\{K_j\}_{j\in J}$.*

Proof. By Lemma 1.11, a trim deterministic automaton $\mathscr{A}$ that recognizes $K = \bigcup_{j\in J} K_j$ is an e-automaton. But $\mathscr{A}$ does not necessarily answers the question for it might be the case that a word in K_i and a word in K_j, $i \neq j$, both reach the same terminal state in $\mathscr{A}$. The construction requires slightly more care.

For every j in J, let $\mathscr{A}_j = \langle Q_j, A, E_j, q_{0,j}, U_j\rangle$ be a trim deterministic e-automaton that recognizes K_j (as given by Lemma 1.11) and let $\mathscr{A}$ be the product of the $\mathscr{A}_j$:

$$\mathscr{A} = \prod_{j\in J} \mathscr{A}_j = \left\langle \prod_{j\in J} Q_j, A, G, (q_{0,1}, q_{0,2}, \ldots, q_{0,n}), U \right\rangle,$$

where G is defined as in Section 1.2.3 and where

$$\begin{aligned} U &= \{(q_1, q_2, \ldots, q_n) \mid \exists j \in J \; q_j \in U_j\} \\ &= (U_1 \times Q_2 \times \cdots \times Q_n) \cup (Q_1 \times U_2 \times \cdots \times Q_n) \cup \cdots \cup (Q_1 \times Q_2 \times \cdots \times U_n) \end{aligned}$$

The automaton $\mathscr{A}$ is deterministic and recognizes $\bigcup_{j\in J} K_j$. Let $\mathscr{A}'$ be the *accessible part* of $\mathscr{A}$, R its set of states and, for every j, $T_j = (Q_1 \times \cdots \times U_j \times \cdots \times Q_n) \cap R$. Any two distinct T_j are disjoint for any two distinct K_j are disjoint and the $\mathscr{A}_j$ are deterministic. Hence $\mathscr{A}'$ is a trim e-automaton and *recognizes the family* $\{K_j\}_{j\in J}$.

Conversely, it should be clear that any family of languages recognized by a trim deterministic e-automaton is a prefix family. □

2. Deterministic relations

The original model of k-tape automaton of Rabin and Scott that we quoted in the introduction had two features that we have not yet mentioned. First, the word written on every tape ends with a special symbol that does not appear anywhere else, called an *endmarker* and almost universally denoted by \$. Second, the automaton behaves *deterministically*.

The model of 2-automaton as labelled graph does not feature normally any endmarker for it would not change the generating power. It is possible to translate the property of *determinism* in this model but it is necessary then to reintroduce an endmarker in order to keep the same generating power as the (deterministic) 2-tape automata. This is the object of the next two subsections.

2.1. Deterministic 2-automata

A normalized 2-automaton will be said deterministic if, roughly speaking, every state determines unambiguously on which tape the input is read and if the letter read on the adequate tape defines a unique move of the automaton. Let us state it formally.

Definition 2.1. A normalized 2-automaton $\mathscr{A} = \langle Q, A, B, E, I, T\rangle$ is *deterministic* if the following conditions hold:

(i) there exists a *partition* of the set of states $Q = Q_A \cup Q_B$, such that the label of an edge whose origin is in Q_A is in $(A \times \{1_{B^*}\})$, respectively the label of an edge whose origin is in Q_B is in $(\{1_{A^*}\} \times B)$;

(ii) for every p in Q and every label x in $(A \times \{1_{B^*}\}) \cup (\{1_{A^*}\} \times B)$, there exists at most one q in Q such that (p,x,q) is in E;

(iii) $\operatorname{Card} I = 1$ (i.e. there is a unique initial state, denoted by q_0 in the sequel).

This definition is the transcription, in the model of labelled graphs, of the conditions of determinism for 2-tape automata. It can probably be considered as folklore; the only place where we have seen it in the literature is Johnson's papers [10, 11]. Note that some authors call "deterministic" 2-automata that are in general (and more wisely) called *(left) sequential*, that is, real-time 2-automata whose underlying input automaton is deterministic (cf. [2] for instance).

It directly follows from the definition:

Proposition 2.1. *Any covering or immersion of a deterministic 2-automaton is a deterministic 2-automaton.*

The determinism of a 2-automaton $\mathscr{A}$ implies that its computations share the most important properties with the computations of a deterministic automaton, though $\mathscr{A}$ works over a non-free monoid. This is what we describe in the remaining of the section; let us first begin with a definition.

Let $\mathscr{A}$ be an automaton over a monoid M and let c be a computation of $\mathscr{A}$:

$$c : p_0 \xrightarrow{m_1} p_1 \xrightarrow{m_2} p_2 \cdots p_{n-1} \xrightarrow{m_n} p_n.$$

A prefix (resp. *the* prefix of length k) of c is a (resp. the) computation d,

$$d : p_0 \xrightarrow{m_1} p_1 \xrightarrow{m_2}, p_2 \cdots p_{k-1} \xrightarrow{m_k} p_k$$

with k smaller than, or equal to, n.

By definition of a free monoid, any word f of A^* has a unique factorization $f = a_1a_2 \ldots a_n$ over A. Let $\mathscr{A}$ be a deterministic automaton over A^*. By definition of determinism, a computation $c : p \xrightarrow[\mathscr{A}]{f} q$ starting in p and with label f, *if it exists*, is unique, uniquely determined by p and f. The determinism also implies that if g is *the* prefix of f of length k ($k \leqslant n$), then the computation starting from p with the label g is *the* prefix of length k of the computation c. On the other hand, every f of A^* is represented by a (partial) function from Q into itself; this representation defines *an action* of A^* over Q and may be denoted as such: $p \cdot f = q$.

This situation somewhat extends to deterministic 2-automata with the even stronger feature that the factorization of the label of a computation does not exist *a priori* but *is determined by the automaton.*

Let us first state that computations are unique *once the factorizations are given* – which is a mere extension of the case of (1-)automata, with the same (easy) proof by induction on the length of the computations.

Lemma 2.2. *Let $\mathscr{A}$ be a deterministic 2-automaton over $A^* \times B^*$ and let*

$$c : p = p_0 \xrightarrow{x_1} p_1 \xrightarrow{x_2} p_2 \cdots p_{n-1} \xrightarrow{x_n} p_n$$

and

$$c' : p = p_0 \xrightarrow{x_1} p'_1 \xrightarrow{x_2} p'_2 \cdots p'_{n-1} \xrightarrow{x_n} p'_n$$

be two computations starting in a same state p, with the same label (f, g), *and that correspond to the* same factorization *of $(f, g) = x_1x_2 \ldots x_n$. Then $c = c'$, i.e. for every i, $p_i = p'_i$.*

It is consistent, and convenient for induction proofs, to consider that for every state p of a 2-automaton $\mathscr{A}$ there exists a computation of length 0, starting in p and with label $(1_{A^*}, 1_{B^*})$.

The fact that the computations of $\mathscr{A}$ determine a factorization for the elements of $A^* \times B^*$ is expressed by the following.

Lemma 2.3 (Corrugated Cardboard Lemma). *Let $\mathscr{A}$ be a deterministic 2-automaton over $A^* \times B^*$. Let p be any state of $\mathscr{A}$ and (f,g) any element of $A^* \times B^*$. The set of prefixes (u,v) of (f,g) such that there exists a computation starting in p with label (u,v) is a chain for the prefix ordering, maximal between $(1_{A^*}, 1_{B^*})$ and its largest element.*

Proof. By virtue of the convention we have just taken, the set of considered prefixes of (f,g) is not empty. Let

$$c : p = p_0 \xrightarrow{x_1} p_1 \xrightarrow{x_2} p_2 \cdots \xrightarrow{x_n} p_n$$

and

$$d : p = p_0 \xrightarrow{y_1} p'_1 \xrightarrow{y_2} p'_2 \cdots \xrightarrow{y_m} p'_m$$

be two computations of $\mathscr{A}$ with labels (u,v) and (u',v'), respectively, such that both (u,v) and (u',v') are prefixes of (f,g).

Suppose that none is a prefix of the other and let

$$x_1 x_2 \ldots x_{i-1} = y_1 y_2 \ldots y_{i-1}$$

be the longest common prefix of the corresponding factorizations (as sequences of X^*, i.e. $x_j = y_j$ for all j such that $0 \leqslant j \leqslant i-1$ and $x_i \neq y_i$). As noted in Lemma 2.2, the prefixes of length $i-1$ of c and d thus coincide

$$c : p = p_0 \xrightarrow{x_1} p_1 \xrightarrow{x_2} p_2 \cdots \xrightarrow{x_{i-1}} p_{i-1} \xrightarrow{x_i} p_i \cdots \xrightarrow{x_n} p_n$$

and

$$d : p = p_0 \xrightarrow{x_1} p_1 \xrightarrow{x_2} p_2 \cdots \xrightarrow{x_{i-1}} p_{i-1} \xrightarrow{y_i} p'_i \cdots \xrightarrow{y_m} p'_m.$$

From Lemma 1.10, it follows that $x_i = (a, 1_{B^*})$ and $y_i = (1_{A^*}, b)$ (or vice versa) a contradiction: p_{i-1} cannot be the origin of two edges, one with label in $A \times 1_{B^*}$ and one with label in $1_{A^*} \times B$. Thus, of (u,v) and (u',v'), one is a prefix of the other and the prefixes of (f,g) that are labels of computations starting in p form a chain. Let (h,k) be its largest element; the computation

$$d = p \xrightarrow{(h,k)} q$$

determines a factorization of (h,k) over $(A \times \{1_{B^*}\}) \cup (\{1_{A^*}\} \times B)$, that in turn defines a *maximal* chain between $(1_{A^*}, 1_{B^*})$ and (h,k). $\square$

An immediate consequence of both Lemmas 2.2 and 2.3 is that if there exists a computation starting in p with label (f,g), *it is unique.*

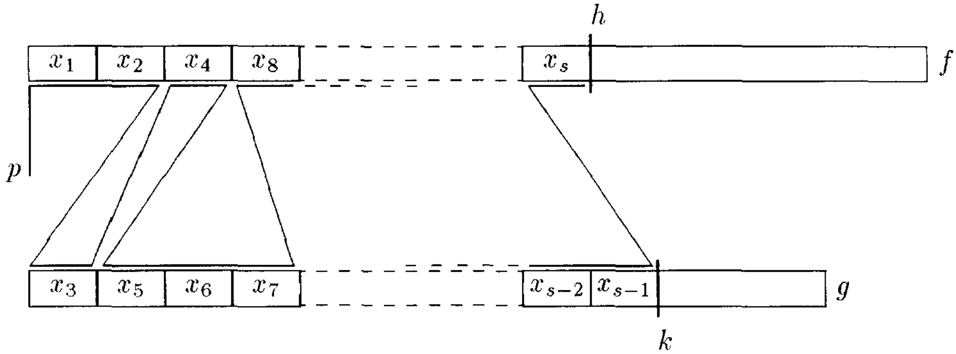


Fig. 2. A path oscillating between f and g.

As in the case of automata, if a deterministic 2-automaton $\mathscr{A}$ over $A^* \times B^*$ is given, every (f,g) of $A^* \times B^*$ is thus represented by a *partial* function from the state set Q of $\mathscr{A}$ into itself. Remark that, in contrast to the case of 1-automata, this *does not define an action* of $A^* \times B^*$ over Q. We note

$$p \circ (f,g) = q \quad \text{if } c : p \xrightarrow{(f,g)} q.$$

A 2-automaton $\mathscr{A} = \langle Q, A, B, E, I, T \rangle$ thus computes, for every (f,g) in $A^* \times B^*$ for which $q_0 \circ (f,g)$ is defined, *a unique factorization*; we refer to it as *the* factorization of (f,g) if no ambiguity is to be feared.

Remark 2.1. Given p and (f,g), Lemma 2.3 ensures existence and uniqueness of a computation starting at p and having a prefix of (f,g) as label. A natural way for representing that computation will be a path oscillating forth and back between f and g as in Fig. 2; hence the name we gave to the lemma.

The consequences of Lemma 2.3 are conveniently summarized in the following.

Corollary 2.4. *Let $\mathscr{A}$ be a deterministic 2-automaton over $A^* \times B^*$. Let p be a state of $\mathscr{A}$ and let (u,v) and (h,k) be two elements of $A^* \times B^*$.*
(i) *If $p \circ (u,v) = q$ and $q \circ (h,k) = r$, then $p \circ (uh, vk) = r$.*
(ii) *If $p \circ (u,v) = q$ and $p \circ (uh, vk) = r$, then $q \circ (h,k) = r$.*
(iii) *If both $p \circ (uh, v)$ and $p \circ (u, vk)$ exist, then $h = 1_{A^*}$ or $k = 1_{B^*}$.*

Proof. (i) follows directly from the definition: (uh, vk) is the label of the computation obtained by the concatenation of the two computations:

$$p \xrightarrow{(u,v)} q \quad \text{and} \quad q \xrightarrow{(h,k)} r.$$

(ii) Since (u,v) is the unique prefix of length $|(u,v)|$ of (uh, vk) such that $p \circ (u,v)$ is defined, the computation $p \xrightarrow{(u,v)} q$ is necessarily a prefix of the computation $p \xrightarrow{(uh,vk)} r$ which thus may be written as $p \xrightarrow{(u,v)} q \xrightarrow{(h,k)} r$.

(iii) (uh, v) and (u, vk) are both prefixes of (uh, vk); Lemma 2.3 implies that one should be prefix of the other, which makes the condition $h = 1_{A^*}$ or $k = 1_{B^*}$ necessary. □

2.2. Deterministic relations

Deterministic 2-automata allow to define deterministic relations. In order to give these 2-automata their full generative power, we have first to define their behaviour when they are endowed with an *endmarker*.

2.2.1. 2-automata with endmarker

Let \$ be a symbol that, by convention, does not belong to any alphabet and, for every alphabet A, let us denote by $A_\$$ the set $A \cup \{\$\}$.

A 2-automaton $\mathscr{A}$ over a monoid $A_\$^* \times B_\$^*$ will be called a *2-automaton with endmarker* over $A^* \times B^*$. The *\$-behaviour* of such an automaton $\mathscr{A}$ is, by definition, the set $|\mathscr{A}|_\$$:

$$|\mathscr{A}|_\$ = \{(u, v) \in A^* \times B^* \mid (u\$, v\$) \in |\mathscr{A}|\}.$$

Since $|\mathscr{A}|_\$ = (|\mathscr{A}| \cap A^*\$ \times B^*\$)\pi$ – where π is the morphism of $A_\$^* \times B_\$^*$ onto $A^* \times B^*$ that erases the \$ – the \$-behaviour of an automaton with endmarker is a rational relation and converserly any rational relation is the \$-behaviour of an automaton with endmarker since for any 2-automaton $\mathscr{A}$, $|\mathscr{A}|(\$, \$)$ is a rational set.

2.2.2. Deterministic and pure deterministic relations

Definition 2.2. A rational relation from A^* into B^* is *deterministic* if it is the \$-behaviour of a deterministic 2-automaton with endmarker over $A^* \times B^*$. A rational relation from A^* into B^* is *pure deterministic* if it is the behaviour of a deterministic 2-automaton over $A^* \times B^*$.

It does not directly follow from the definition that a pure deterministic relation is a deterministic one. However, an easy construction restores consistency.

We denote by DRat $A^* \times B^*$ (resp. D′Rat $A^* \times B^*$) the set of deterministic (rational) relations (resp. of pure deterministic (rational) relations) over $A^* \times B^*$. We denote them also by DRat_2 and $\mathrm{D'Rat}_2$, respectively, if the alphabet is not specified.

It is well-known that DRat_2 is a proper subclass of Rat_2 which contains Rec_2, that it is closed under complementation and that it is not decidable within Rat_2 [7]. [8] Example 2.1 is the classical paradigm for rational relations that are *not* deterministic.

Example 2.1. The function $\alpha_1 : \{x\}^* \to \{x\}^*$ defined by

$$x^{2n}\alpha_1 = x^{2n} \quad \text{and} \quad x^{2n+1}\alpha_1 = x^n$$

for every n in $\mathbb{N}$ is a rational function, but not a deterministic one.

[8] This means that, given a 2-automaton, it is not decidable whether there exists or not an equivalent deterministic one.

It is "intuitively" clear that α_1 is not deterministic. But if one wants *to prove* that fact, one cannot help from establishing an iteration lemma. An iteration lemma for rational relation is given for instance in [2, Lemma III.3.3]. Such a lemma can be made more precise for *deterministic* (and *pure deterministic*) relations.

Lemma 2.5 (Iteration lemma). *For every deterministic relation θ of $A^* \times B^*$ there exists a positive integer N with the following property. Every pair (u,v) in θ, with both u and v of length greater than N, admits a factorization*

$$(u,v)=(u_1,v_1)(f,g)(u_2,v_2)$$

which meets the following conditions:

(i) $|(f,g)| \geqslant 0$;

(ii) $|(u_1 f, v_1 g)| < N$;

(iii) *for any h and k such that $(u_1 fh, v_1 gk)$ is in θ, then $(u_1 f^n h, v_1 g^n k)$ is in θ for any integer n.*

If θ is pure deterministic, *the condition "$|u|>N$ and $|v|>N$" may be replaced by "$|(u,v)|>N$" while the same conclusion holds.*

Proof. Let $\mathscr{A} = \langle Q, A_\$, B_\$, E, q_0, T\rangle$ be a deterministic 2-automaton which realizes θ and let $N=|Q|$. Let $(u,v)\in\theta$ such that $|u|>N$ and $|v|>N$, then $q_0 \circ (u\$, v\$) = t \in T$. Let (u',v') be *the* prefix of length N of *the* factorization of $(u\$, v\$)$ determined by $\mathscr{A}$. Since the length of both u and v is greater than N, the final \$ neither belongs to u' nor to v'. The computation $q_0 \circ (u',v')$ may thus be written as

$$q_0 \xrightarrow{c_1} q_1 \xrightarrow{c_2} q_2 \cdots q_{N-1} \xrightarrow{c_N} q_N \tag{2.1}$$

with every c_i in $(A \times \{1_{B^*}\}) \cup (\{1_{A^*}\} \times B)$. The rest of the proof mimics the classical one for the iteration lemma for (1-)automata. Since $N=|Q|$, there exist *distinct* i and j such that $q_i = q_j = q$. Let us note

$$(u_1,v_1)=c_1\cdots c_i \quad \text{and} \quad (f,g)=c_{i+1}\cdots c_j.$$

Then $q_0 \circ (u_1,v_1)=q$ and $q\circ(f,g)=q$. Thus, for any (h,k) such that $q\circ(h,k)=t'\in T$, and for any integer n,

$$q_0 \circ [(u_1,v_1)(f,g)^n(h,k)] = t'$$

holds. The three conditions of the lemma are thus met.

If θ is pure deterministic, it is realized by a deterministic 2-automaton $\mathscr{A} = \langle Q, A, B, E, I, T\rangle$ and the computation (2.1) exists as soon as $|(u,v)|>N$. $\square$

The necessity of considering 2-automata with endmarker is expressed by the following.

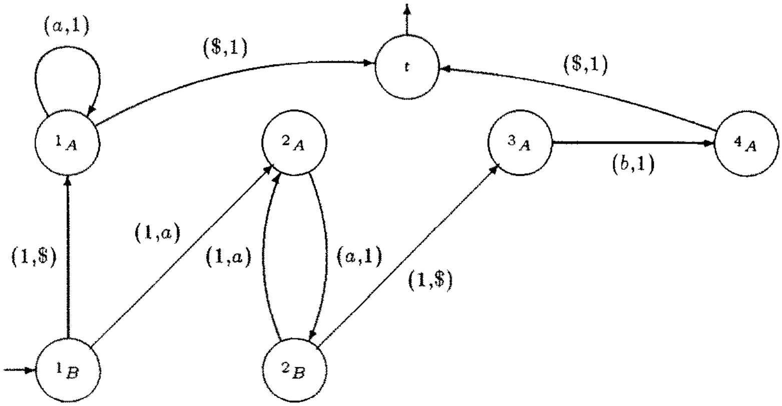


Fig. 3. A deterministic automaton with endmarker for θ_2.

Proposition 2.6. $\mathrm{D'Rat}_2$ *is a proper subclass of* DRat_2.

Proof. Condition (iii) of Corollary 2.4 takes indeed a special form for pure deterministic relations: *If θ is a pure deterministic relation from A^* into B^*, then*

$$\forall u,f\in A^*,\ \forall v,g\in B^* \quad (uf,v)\in\theta \text{ and } (u,vg)\in\theta \Rightarrow f=1_{A^*} \text{ or } g=1_{B^*}. \tag{2.2}$$

The universal relation, i.e. the relation the graph of which is the whole set $A^*\times B^*$, which is recognizable, and thus deterministic, does not meet (2.2). □

The following example shows that, conversely, (2.2) is not a sufficient condition for a deterministic relation to be a pure deterministic relation.

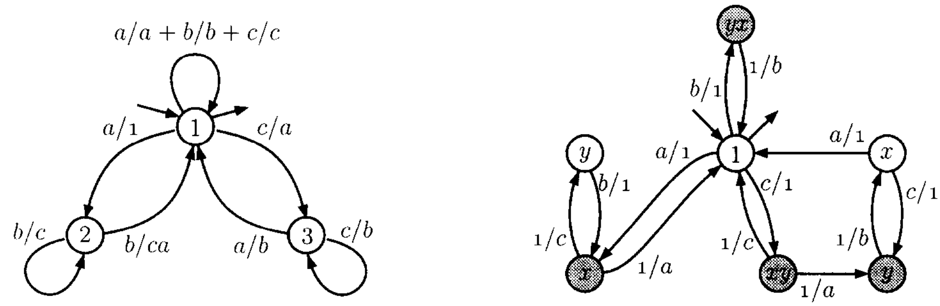
Example 2.2. The relation $\theta_2=\{(a^nb,a^n)\mid n>0\}\cup\{(a^n,1)\mid n\in\mathbb{N}\}$ is deterministic since it is the \$-behaviour of the deterministic automaton with endmarker drawn in Fig. 3. Moreover, θ_2 satisfies (2.2): let u, v, f and g be such that $(uf,v)\in\theta_2$ and $(u,vg)\in\theta_2$. Assume that $g\neq 1_{B^*}$; then $(u,vg)=(a^nb,a^n)$ and $(uf,v)=(a^mb,a^m)$ (since a^nb is not a prefix of a^m). Since a^nb has to be a prefix of a^mb, we obtain $m=n$ and $g=1_{B^*}$.

On the other hand, θ_2 does not satisfy the specialization of Lemma 2.5 for pure deterministic relations.

Let us end this section by the fact that we do not know whether $\mathrm{D'Rat}_2$ is decidable within DRat_2 or not.

2.2.3. An example

Morphisms, and thus inverse morphisms, as well as intersection with rational sets are (pure) deterministic relations. This implies, by the way, that (pure) deterministic relations are not closed by composition. The next result gives an interesting example of non-trivial deterministic relations (resp. pure deterministic) as well as a lemma for



(a) A 2-automaton for $\varphi\varphi^{-1}$

(b) A deterministic one

Fig. 4. Two 2-automata that realize $\varphi\varphi^{-1}$.

later use. Recall that a morphism φ from A^* into B^* is *continuous* if $A\varphi \subset B^+$, i.e. if no letter of A is mapped by φ on 1_{B^*}.

Proposition 2.7. *Let $\varphi : A^* \to B^*$ be a morphism. Then the mapping equivalence $\varphi\varphi^{-1}$: $A^* \to A^*$ is a deterministic relation. Moreover, if φ is continuous then $\varphi\varphi^{-1}$ is pure deterministic.*

Example 2.3. Let $A = \{a, b, c\}$ and $B = \{x, y\}$. Also let $\varphi : A^* \to B^*$ be the morphism defined by

$$a\varphi = x, \qquad b\varphi = yx \quad \text{and} \quad c\varphi = xy.$$

Then straightforward computations lead to the 2-automaton shown at Fig. 4(a) for $\varphi\varphi^{-1}$. Also normalization and rearrangement yield the deterministic 2-automaton of Fig. 4(b); states in "Q_B" are shown in grey and the names of states match the names used in the construction given below for the proof in the general case.

Proof of Proposition 2.7. Let us first prove the statement for continuous morphisms. Thus, let $\varphi : A^* \to B^*$ be a continuous morphism. Let S be the set of prefixes of $A\varphi$:

$$S = \{s \in B^* \mid \exists a \in A \;\; s \leqslant a\varphi\}.$$

Then the automaton $\mathscr{A}$ we are buiding will have the set $Q = S \times \{1, 2\}$ as set of states; the unique initial state is $(1_{B^*}, 1)$, which is the unique terminal state as well. The transitions of $\mathscr{A}$, described in the functional setting, are the following:

$$(s,1) \circ (a, 1_{A^*}) = \begin{cases} ((a\varphi)^{-1}s, 1) & \text{if } a\varphi \leqslant s, \\ (s^{-1}(a\varphi), 2) & \text{if } s < a\varphi, \\ \text{undefined} & \text{otherwise,} \end{cases} \tag{2.3}$$

$$(s,2)\circ(1_{A^*},a)=\begin{cases}((a\varphi)^{-1}s,1) & \text{if } a\varphi\leqslant s,\\ (s^{-1}(a\varphi),2) & \text{if } s<a\varphi,\\ \text{undefined} & \text{otherwise.}\end{cases} \tag{2.4}$$

Let us prove, by induction on $|f|+|g|$, that, for every s in S, the two equivalences hold:

$$s(g\varphi)=f\varphi \ \Leftrightarrow\ (s,1)\circ(f,g)=(1_{B^*},1), \tag{2.5}$$

$$s(f\varphi)=g\varphi \quad\text{and}\quad s\neq 1_{B^*} \ \Leftrightarrow\ (s,2)\circ(f,g)=(1_{B^*},1). \tag{2.6}$$

Transition (2.3) gives (2.5) and transition (2.4) gives (2.6) for $|f|+|g|=1$.

Suppose $s(g\varphi)=f\varphi$; then $|f|\geqslant 1$, $f=af'$ with a in A and either $a\varphi\leqslant s$, in which case

$$(s,1)\circ(a,1_{A^*})=(s',1) \quad\text{and}\quad s'(g\varphi)=f'\varphi$$

or $s<a\varphi$, in which case

$$(s,1)\circ(a,1_{A^*})=(s',2) \quad\text{and}\quad s'(f'\varphi)=g\varphi.$$

In both the cases, the induction hypothesis yields

$$(s,1)\circ(f,g)=(1_{B^*},1).$$

Suppose conversely that $(s,1)\circ(f,g)=(1_{B^*},1)$; from the definition of the transitions of $\mathscr{A}$ then necessarily $f=af'$ with a in A and either $a\varphi\leqslant s$, in which case

$$(s,1)\circ(a,1_{A^*})=(s',1) \quad\text{and}\quad (s',1)\circ(f',g)=(1_{B^*},1)$$

or $s<a\varphi$, in which case

$$(s,1)\circ(a,1_{A^*})=(s',2) \quad\text{and}\quad (s',2)\circ(f',g)=(1_{B^*},1).$$

In both the cases, the induction hypothesis yields

$$s(g\varphi)=f\varphi$$

Equivalence (2.6) is treated exactly the same way. Now, noting that $(1_{B^*},1)$ is the unique initial state as well as the unique terminal state of $\mathscr{A}$, the equivalence (2.5), taken for $s=1_{B^*}$, expresses exactly that $\varphi\varphi^{-1}=|\mathscr{A}|$.

Suppose now that $\varphi: C^*\to B^*$ is not a continuous morphism. There exist then a partition $C=A+D$ such that the restriction of φ to A^* is a continuous morphism and for every d in D $d\varphi=1_{B^*}$. It is clear then that $(u,v)\in\varphi\varphi^{-1}$ if and only if u and v may be written as

$$u=x_1f_1x_2f_2\cdots x_pf_px_{p+1} \quad\text{and}\quad v=y_1g_1y_2g_2\cdots y_qg_qy_{q+1} \tag{2.7}$$

with the f_i's and g_j's in A, the x_l's and y_k's in D^*, and such that $(f_1f_2\cdots f_p, g_1g_2\cdots g_q)\in\varphi\varphi^{-1}$.

Let us build an automaton as above for the restriction of φ to A^* and let us add, for every s in S and every d in D, the following transitions:

$$(s,1)\circ(d,1_{C^*})=(s,1) \quad \text{and} \quad (s,2)\circ(1_{C^*},d)=(s,2).$$

It is not difficult to check that the behaviour of such a transformed automaton will be exactly the set of pairs (u,v) such that $y_{q+1}=1_{D^*}$ in the factorization (2.7). In order to recover the whole graph of $\varphi\varphi^{-1}$ we must use an endmarker and add another two states t and t' with the following transitions:

$$(1_{B^*},1)\circ(\$,1_{C^*})=t \quad \forall d\in D, \quad t\circ(1_{C^*},d)=t \quad t\circ(1_{C^*},\$)=t',$$

and the terminal state is not $(1_{B^*},1)$ any more but t'. □

2.3. Serialization of deterministic automata

The aim of this section is to show that any deterministic automaton with endmarker may be put into a kind of normal form in which the endmarker is read once and only once on each "tape". We first begin with the specialization of Proposition 1.4 for the case of deterministic relations.

Corollary 2.8. *The intersection of a deterministic relation and a recognizable relation is a deterministic relation.*

Proof. Let θ be the \$-behaviour of a deterministic 2-automaton with endmarker $\mathscr{A}=\langle Q,A_\$,B_\$,E,q_0,T\rangle$, then $|\mathscr{A}|\cap A^*\$\times B^*\$=\theta(\$,\$)$. Let $R\in\mathrm{Rec}(A^*\times B^*)$, then $R(\$,\$)\in\mathrm{Rec}(A_\$^*\times B_\$^*)$ since recognizable relations are closed under product (Corollary 1.6). By Lemma 1.9 and Proposition 2.1, there exists a deterministic 2-automaton $\mathscr{B}$ over $A_\$^*\times B_\$^*$ such that $|\mathscr{A}|\cap R(\$,\$)=|\mathscr{B}|$. Hence

$$\begin{aligned}|\mathscr{B}|\cap A^*\$\times B^*\$ &= |\mathscr{A}|\cap A^*\$\times B^*\$\cap R(\$,\$)\\ &=\theta(\$,\$)\cap R(\$,\$)\\ &=(\theta\cap R)(\$,\$)\end{aligned}$$

holds, and $\theta\cap R$ is the \$-behaviour of the deterministic 2-automaton $\mathscr{B}$. □

Remark that Lemma 1.9 also implies that the intersection of a pure deterministic relation from A^* into B^* and of a recognizable relation from A^* into B^* is a pure deterministic relation from A^* into B^*.

Proposition 2.9. *Any deterministic relation from A^* into B^* is the \$-behaviour of a deterministic 2-automaton $\mathscr{D}$ such that $|\mathscr{D}|$ is a subset of $A^*\$\times B^*\$$. Moreover, we may assume that $\mathscr{D}$ has a unique terminal state t, which belongs to Q_A.*

Proof. Let θ be the \$-behaviour of a deterministic 2-automaton with endmarker $\mathscr{A}=\langle Q,A_\$,B_\$,E,I,T\rangle$. As we did in the preliminaries, we identify the set E with a

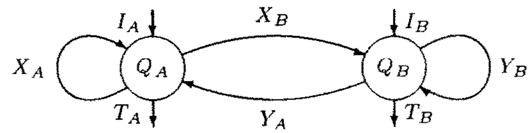


Fig. 5. The block decomposition of $\mathscr{A}$.

$(Q \times Q)$-matrix which can be written $X+Y$ (where the entries of X are in $(A \times \{1_{B^*}\})$ and those of Y in $(\{1_{A^*}\} \times B))$ and we identify the set I with a row vector and the set T with a column vector. The partition $Q = Q_A \cup Q_B$ yields the following block decomposition of X and Y:

$$X = \begin{pmatrix} X_A & X_B \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ Y_A & Y_B \end{pmatrix},$$

where X_A (resp. X_B) is a $(Q_A \times Q_A)$-matrix (resp. $(Q_A \times Q_B)$-matrix) over $(A_\$ \times \{1_{B^*}\})$ and Y_A (resp. Y_B) is a $Q_B \times Q_A$-matrix (resp. $Q_B \times Q_B$-matrix) over $(\{1_{A^*}\} \times B_\$)$. The Boolean vectors I and T can be written as well as

$$I = \begin{pmatrix} I_A & I_B \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} T_A \\ T_B \end{pmatrix},$$

where I_A and T_A, and I_B and T_B, are respectively of dimension Q_A and Q_B. This is illustrated in Fig. 5.

The aim of Proposition 2.9 is to separate the endmarkers \$ from the letters of A and B so it will be convenient to write the matrix X as the disjoint union of two matrices: one, X', with entries in $(A \times \{1_{B^*}\})$ and the other, X'', with entries in $\{(\$, 1_{B^*})\}$. In the same way, the matrix Y is written as the disjoint union of a matrix Y' with entries in $(\{1_{A^*}\} \times B)$ and a matrix Y'', with entries in $\{(1_{A^*}, \$)\}$. It induces the following block decomposition:

$$X = X' + X'' = \begin{pmatrix} X'_A & X'_B \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} X''_A & X''_B \\ 0 & 0 \end{pmatrix},$$

$$Y = Y' + Y'' = \begin{pmatrix} 0 & 0 \\ Y'_A & Y'_B \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ Y''_A & Y''_B \end{pmatrix}.$$

The set $A^*\$ \times B^*\$$ is recognized by the action $\mathscr{B} = (R, A_\$^* \times B_\$^*, \cdot, \{1\}, \{4\})$ as drawn in Fig. 6.

By Lemma 1.9, the set $|\mathscr{A}| \cap |\mathscr{B}|$ (which is equal to $|\mathscr{A}| \cap (A^*\$ \times B^*\$)$) is the behaviour of the deterministic 2-automaton $\mathscr{C} = \langle Q \times R, A_\$, B_\$, G, I \times \{1\}, T \times \{4\} \rangle$, where the set G of labelled edges is defined by

$$G = \{((p,r), c, (q,s)) \mid (p,c,q) \in E,\ r \cdot c = s\}.$$

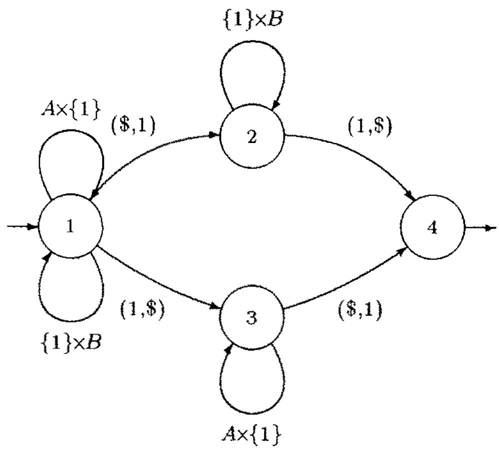


Fig. 6. An action $\mathscr{B}$ recognizing $A^*\$ \times B^*\$$.

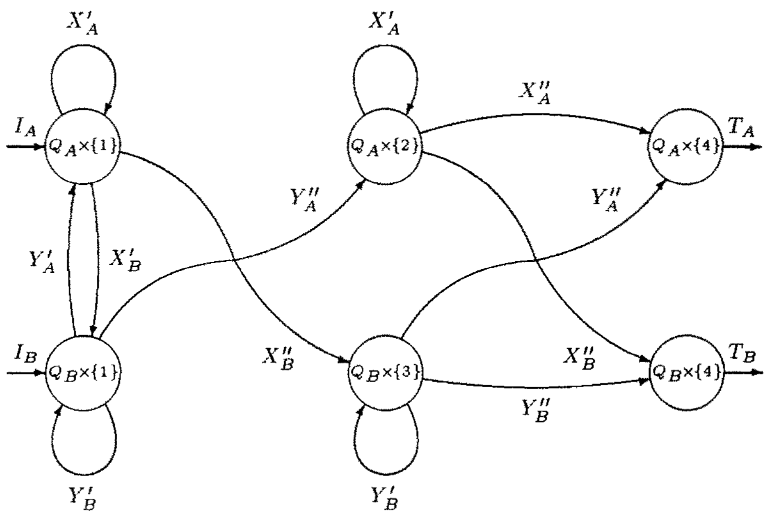


Fig. 7. The automaton $\mathscr{C}$ with behaviour $|\mathscr{A}| \cap (A^*\$ \times B^*\$)$.

The definition of G implies that

$$((p,r),c,(q,s)) \in G \;\Rightarrow\; \begin{cases} p \in Q_A \quad c \in A_\$ \times \{1_{B_\$^*}\} \quad r \in \{1,2\} \\ \qquad\qquad\qquad \text{or} \\ p \in Q_B \quad c \in \{1_{A_\$^*}\} \times B_\$ \quad r \in \{1,3\}. \end{cases}$$

Remark that the states of $(Q_A \times \{3\}) \cup (Q_B \times \{2\})$ are not co-accessible since they are not terminal and there is no edge which starts from these states. So we may delete them and we obtain the 2-automaton as drawn in Fig. 7.

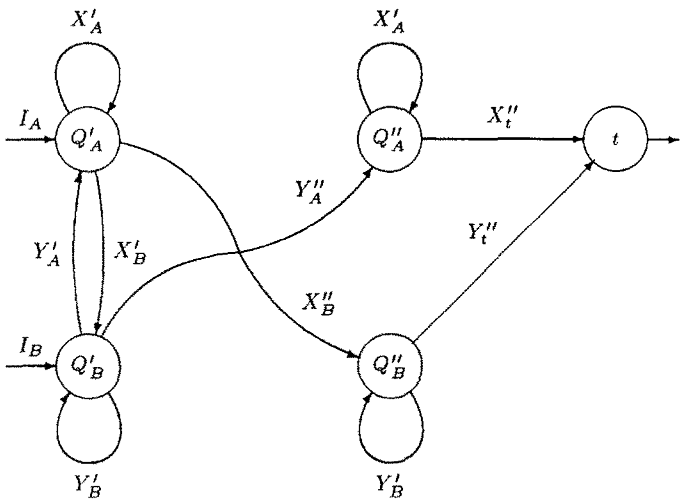


Fig. 8. The serialized 2-automaton $\mathscr{D}$.

There is no edge which starts from the states of $(Q_A \times \{4\}) \cup (Q_B \times \{4\})$ so we may identify all these states with a unique terminal state t. Finaly, let us consider the following notation,

$$X''_t = X''_A T_A + X''_B T_B \quad \text{and} \quad Y''_t = Y''_A T_A + Y''_B T_B,$$

$$Q'_A = Q_A \times \{1\}, \qquad Q''_A = Q_A \times \{2\}, \qquad Q'_B = Q_B \times \{1\} \quad \text{and} \quad Q''_B = Q_B \times \{3\},$$

which allows to define the deterministic 2-automaton $\mathscr{D}$ as drawn in Fig. 8. The set $|\mathscr{A}| \cap (A^*\$ \times B^*\$)$ is the behaviour of $\mathscr{D}$ and θ is its \$-behaviour. This automaton $\mathscr{D}$ is called *serialized.* □

2.4. Synchronized rational relations

The only families of deterministic relations we have seen so far are the recognizable relations and the mapping equivalences of morphisms (between free monoids). Other works on rational relations have shown the usefulness of another subfamily of deterministic relations: the *synchronized rational relations* (cf. [8]). For they will be considered in the last section of this paper, we briefly review here their definition, some of their properties, and we give some examples of such relations.

Roughly speaking, synchronized rational relations are those relations realized by 2-automata where the two reading heads move simultaneously on the two input tapes (if one is considering the Rabin–Scott model of automata). More precisely, and in the labelled graph model of automata, one defines first two classes of 2-automata: the letter-to-letter 2-automata and the 2-automata with terminal function. A *letter-to-letter* 2-automaton is a 2-automaton with edges labelled in $A \times B$. A *2-automaton with terminal function* is a 2-automaton in which the set of terminal states – which can be considered as a function from Q into $\mathbb{B}$ – is replaced with a function ω

from Q into $\mathfrak{P}(A^* \times B^*)$. The *behaviour* of such an automaton is the set of pairs of words $(f'f'', g'g'')$ for which there exist an initial state p and a state q such that $p \xrightarrow{(f',g')} q$ and $(f'', g'') \in q\omega$.

Definition 2.3 (*Frougny and Sakarovitch* [8]). A rational relation θ is *synchronized* if it is realized by a letter-to-letter 2-automaton with a terminal function taking its value in the set

$$\mathrm{Diff}_{\mathrm{Rat}} = \{(S \times 1_{B^*}) \cup (1_{A^*} \times T) \mid S \in \mathrm{Rat}\, A^*, T \in \mathrm{Rat}\, B^*\}.$$

Let us mention at this point that, like determinism and unlike recognizability or being letter-to-letter, being synchronized is an *oriented notion* as the automata read words *from left to right*.

The family of synchronized relations strictly contains Rec_2 and is an effective Boolean algebra. [9]

It will be convenient to have a more general characterization of synchronized relations:

Proposition 2.10. *A rational relation θ is synchronized if and only it is realized by a letter-to-letter 2-automaton with a terminal function taking its value in* $\mathrm{Rec}(A^* \times B^*)$.

Proof. Since $\mathrm{Diff}_{\mathrm{Rat}}$ is a subset of $\mathrm{Rec}(A^* \times B^*)$, the condition is obviously necessary.

Let θ be a relation from A^* into B^* realized by a letter-to-letter 2-automaton $\mathscr{A} = \langle Q, A^* \times B^*, E, I, \omega\rangle$ with a terminal function ω taking its value in $\mathrm{Rec}(A^* \times B^*)$. Since the union of two synchronized relations is synchronized, we may assume that there is only one state t such that $t\omega \neq 0$ and that t is not an initial state. Since every recognizable relation is a synchronized relation, there exists a letter-to-letter 2-automaton with terminal function $\mathscr{B} = \langle P, A^* \times B^*, F, J, \varepsilon\rangle$ which realizes $t\omega$. Let $\mathscr{C} = \langle Q \cup P, A^* \times B^*, G, I, \varepsilon\rangle$ with

$$G = E \cup F \cup \{(p, a, b, j) \mid j \in J \text{ and } (p, a, b, t) \in E\};$$

then $\mathscr{C}$ is a letter-to-letter 2-automaton with terminal function taking its value in $\mathrm{Diff}_{\mathrm{Rat}}$ and $\mathscr{C}$ realizes θ. $\square$

Proposition 2.11 (Frougny and Sakarovitch [8]). *Synchronized rational relations are deterministic.*

We shall prove in Section 5.4 that the family of synchronized rational relations is closed under composition (Proposition 5.7), unlike the family of deterministic rational relations.

[9] This means that if two elements of this family are (effectively) given, by a finite 2-automata say, one can compute a 2-automaton that recognizes the intersection, the complement, and one can also decide if the elements are empty or not.

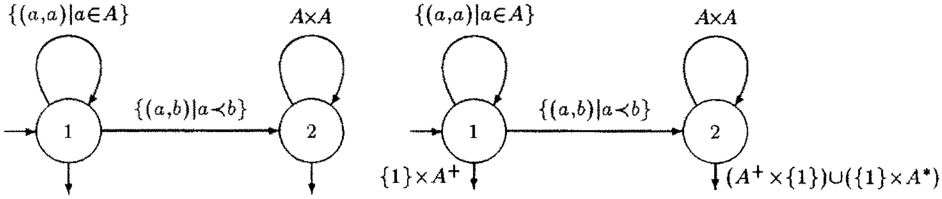


Fig. 9. Lexicographic ordering: induced relations.

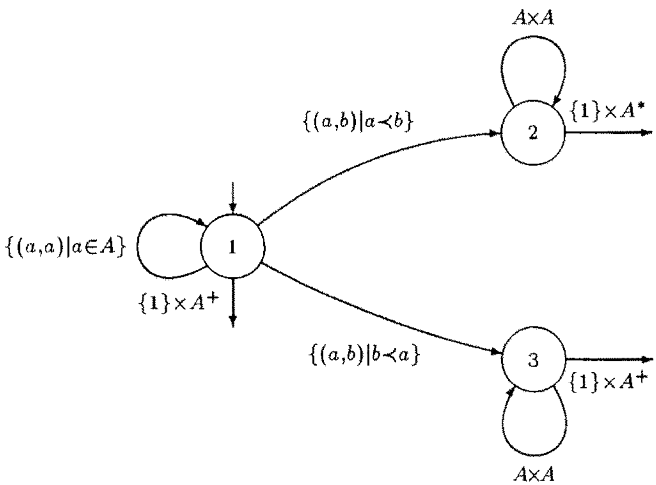


Fig. 10. The relation θ_5.

Example 2.4. The classical orderings on words provide useful examples of synchronized rational relations (and they will be used in Section 5).

Let $\preccurlyeq$ be an ordering on the alphabet A and let $\prec$ be the strict ordering associated to it. The *lexicographic ordering* on A^*, denoted by $\preccurlyeq$, is defined by

$$f \preccurlyeq g \Leftrightarrow \begin{cases} g = fh & \text{with } h \in A^+ \text{ or} \\ f = uav, \quad g = ubw & \text{with } a \prec b \end{cases}$$

The *military ordering* on A^*, denoted by $\sqsubseteq$, is defined by

$$f \sqsubseteq g \Leftrightarrow \begin{cases} |f| < |g| & \text{or} \\ |f| = |g| & \text{and} \quad f \preccurlyeq g \end{cases}$$

(i) The relation θ_3 which associates to every word u the set of words v of same length as u and greater than u in the lexicographic (or military) ordering is a synchronized relation. It is realized by the letter-to-letter 2-automaton as drawn in Fig. 9(a).
(ii) The relation θ_4 which associates to every word u the set of words v greater than u in the lexicographic ordering is a synchronized relation. It is realized by the letter-to-letter 2-automaton with output function, as drawn in Fig. 9(b).

(iii) The relation θ_5 which associates to every word u the set of words v greater than u in the military ordering is a synchronized relation. It is realized by the synchronized 2-automaton drawn in Fig. 10.

3. Representation of deterministic relations

In this section we prove that a deterministic rational relation can always be given a representation of a special form (Proposition 3.2). We have first to define what is a matrix representation for a 2-automaton with endmarker, and what is this "special form".

3.1. Prefix matrices

We extend here the definition of prefix subsets to matrices of subsets.

Let M be a $(P \times Q)$-matrix with entries in $\mathfrak{P}(A^*)$. For every p in P we denote by $M_{p,\bullet}$ the union of the entries of the line p of M, i.e. $M_{p,\bullet} = \bigcup_{q \in Q} M_{p,q}$.

Definition 3.1. A $(P \times Q)$-matrix M with entries in $\mathfrak{P}(A^*)$ is *prefix* if every row of M forms a prefix family of languages, i.e. for every p in P: (i) the entries $M_{p,q}$ are pairwise disjoint; (ii) $M_{p,\bullet}$ is a prefix subset of A^*.

The product of two prefix subsets is a prefix subset. A slightly stronger property indeed holds.

Property 3.1. *Let X be a prefix subset of A^*. Then*

$$\forall x, x' \in X, \quad \forall y, y' \in A^* \qquad xy \leqslant x'y' \Rightarrow x = x' \quad \textit{and} \quad y \leqslant y'.$$

Proof. If xy is a prefix of $x'y'$ then x is a prefix of x' or x' is a prefix of x. Since X is prefix, it follows that $x = x'$. Now $xy \leqslant xy'$ implies that $y \leqslant y'$. □

Proposition 3.1. *The product of two prefix matrices is a prefix matrix.*

Proof. Let M be a prefix $(P \times Q)$-matrix and M' be a prefix $(Q \times R)$-matrix, both with entries in $\mathfrak{P}(A^*)$. Let p in R, let r, s in R and let u, v in A^* such that

$$u \in (MM')_{p,r} \quad v \in (MM')_{p,s} \quad u \leqslant v.$$

By definition of the product of matrices, there exist j and k in Q such that

$$\exists f \in M_{p,j}, \quad \exists f' \in M'_{j,r} \quad u = ff' \quad \text{and} \quad \exists g \in M_{p,k}, \quad \exists g' \in M'_{k,s} \quad v = gg'.$$

Since both f and g belong to the prefix subset $M_{p,\bullet}$, we have (Property 3.1)

$$ff' \leqslant gg' \Rightarrow f = g \quad \text{and} \quad f' \leqslant g'.$$

Since the $(M_{p,q})_{q\in Q}$ are pairwise disjoint, $f=g$, with f in $M_{p,j}$ and g in $M_{p,k}$, implies that $j=k$. Hence, f' and g' both belong to the same row j of M':

$$f'\in M'_{j,r} \quad \text{and} \quad g'\in M'_{j,s}.$$

Now $f'\leqslant g'$ implies $f'=g'$ and $r=s$. Thus $u=v$ and $r=s$. Hence the pth row of MM' forms a prefix family. $\square$

3.2. Representation of 2-automata with endmarker

We have recalled in Section 1 that every 2-automaton over $A^*\times B^*$ can be given a so-called representation (λ,μ,ν) where μ is a morphism from A^* into a monoid of square matrices with entries in $\operatorname{Rat} B^*$. Since we are going to discuss representations of *deterministic* relations and since such relations are defined via 2-automata with endmarker, we have first to describe what is a representation of such an automaton.

Let $\mathscr{A}$ be a 2-automaton over a monoid $A_\$^*\times B_\$^*$. Then there exists $\mu: A_\$^*\to(\operatorname{Rat} B_\$^*)^{Q\times Q}$ and λ and ν vectors with entries in $\operatorname{Rat} B_\$^*$ such that

$$|\mathscr{A}|=\{(u,v)\mid v\in\lambda\cdot u\mu\cdot\nu\}.$$

The \$-behaviour of $\mathscr{A}$ is thus represented in the following way:

$$(u,v)\in|\mathscr{A}|_\$ \;\Leftrightarrow\; v\$\in\lambda\cdot u\mu\cdot(\$\mu\cdot\nu).$$

Note that, since it is always possible to assume that $|\mathscr{A}|\subset A^*\$\times B^*\$$, this implies, if $\mathscr{A}$ is also chosen to be trim, that the entries of λ, of $\nu'=\$\mu\cdot\nu$, and of $\lambda\cdot u\mu\cdot\nu'$, $\lambda\cdot u\mu$, $u\mu\cdot\nu'$ and $u\mu$, for every u in A^*, all belong to $\operatorname{Rat} B^*$ or to $(\operatorname{Rat} B^*)\$$.

Accordingly, a *representation with endmarker* of a relation θ from A^* into B^* is a triple (λ,μ,ν) where μ is a morphism $\mu: A^*\to(\operatorname{Rat} B_\$^*)^{Q\times Q}$, $\lambda\in(\operatorname{Rat} B_\$^*)^{1\times Q}$ and $\nu\in(\operatorname{Rat} B_\$^*)^{Q\times 1}$ such that

$$\forall(u,v)\in A^*\times B^* \quad (u,v)\in\theta \;\Leftrightarrow\; v\$\in\lambda\cdot u\mu\cdot\nu.$$

3.3. Representation of deterministic relations

A representation (λ,μ,ν) from A^* into B^* will be said to be *prefix*, if λ, ν and $a\mu$, for every a in A, are prefix matrices. In such a case, it follows from Proposition 3.1 that $\lambda\cdot u\mu\cdot\nu$, $\lambda\cdot u\mu$, $u\mu\cdot\nu$ and $u\mu$, for every u in A^*, are prefix matrices.

We are now in a position to state the characterization of deterministic relations we are aiming at.

Proposition 3.2. *A rational relation is deterministic if and only if it has a representation with endmarker that is prefix.*

Proof. (A) Let θ be the \$-behaviour of a deterministic 2-automaton with endmarker $\mathscr{A}=\langle Q,A_\$,B_\$,E,I,T\rangle$ – with the understated partition $Q=Q_A\cup Q_B$. With the same notation as in Section 2.3, the $Q\times Q$-matrix E is written as $E=X+Y$, and the partition

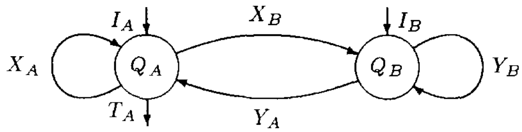


Fig. 11. The block decomposition of $\mathscr{A}$.

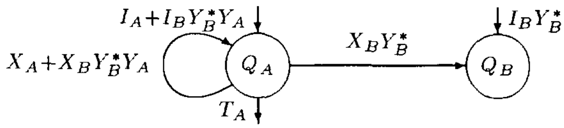


Fig. 12. An automaton $\mathscr{B}$ equivalent to $\mathscr{A}$.

$Q = Q_A \cup Q_B$ yields a block decomposition of I, X, Y and T. By Proposition 2.3, we may assume that $\mathscr{A}$ is serialized and we thus get the representation of (the block decomposition of) $\mathscr{A}$ as shown in Fig. 11.

As the equality $E = X + Y$ gives

$$|\mathscr{A}| = (IY^*) \cdot (XY^*)^* \cdot T,$$

the block decomposition of $\mathscr{A}$ induces a block decomposition of IY^* and XY^*:

$$IY^* = (I_A + I_B Y_B^* Y_A \quad I_B Y_B^*), \tag{3.1}$$

$$XY^* = \begin{pmatrix} X_A + X_B Y_B^* Y_A & X_B Y_B^* \\ 0 & 0 \end{pmatrix}, \tag{3.2}$$

and then gives the following expression for $|\mathscr{A}|$:

$$|\mathscr{A}| = (I_A + I_B Y_B^* Y_A) \cdot (X_A + X_B Y_B^* Y_A)^* \cdot T_A, \tag{3.3}$$

which corresponds to an automaton $\mathscr{B}$ equivalent to $\mathscr{A}$ and represented as in Fig. 12.

Starting from (3.3), we define a representation of $\mathscr{B}$, (λ, μ, γ), of dimension Q_A of A^* by matrices over $\operatorname{Rat} B_\$^*$ such that

$$\forall (u,v) \in A_\$^* \times B_\$^* \quad (u,v) \in |\mathscr{A}| \Leftrightarrow v \in \lambda \cdot u\mu \cdot \gamma. \tag{3.4}$$

(The representation (λ, μ, v) is built in the same way as the representation (λ', μ', v') in Section 1.2.2.) The vector $I_A + I_B Y_B^* Y_A$ is the block of dimension Q_A of the vector IY^* (by (3.1)) and thus

$$\forall p \in Q_A \quad \lambda_p = \{v \in B_\$^* \mid q_0 \circ (1, v) = p\}.$$

The matrix $X_A + X_B Y_B^* Y_A$ is the block of dimension $Q_A \times Q_A$ of XY^* (by (3.2)) and thus

$$\forall p, q \in Q_A \quad \forall x \in A_\$ \quad x\mu_{pq} = \{v \in B_\$^* \mid p \circ (x, 1) \circ (1, v) = q\}$$

and γ is the Boolean vector of dimension Q_A defined by

$$\forall q \in Q_A \quad \gamma_q = 1 \Leftrightarrow q \in T_A.$$

Claim 3.1. *The representation* (λ, μ, γ) *is prefix.*

Proof. (i) The vector γ is a prefix matrix since it is a (Boolean) column vector.

(ii) Let p in Q_A. Assume that there exist q, q' in Q_A and v, h in $B_\$^*$ such that

$$v \in x\mu_{p,q} \quad \text{and} \quad vh \in x\mu_{p,q'}.$$

Then we would have

$$p \circ (x,1) \circ (1,v) = q \quad \text{and} \quad p \circ (x,1) \circ (1,vh) = q'$$

which implies, by Corollary 2.4, that $q \circ (1,h) = q'$. Since q is in Q_A and $(1,h)$ in $1 \times B_\$^*$, we necessarily have

$$h = 1 \quad \text{and} \quad q = q'.$$

Therefore, the pth row of $x\mu$ is a prefix family and $x\mu$ is a prefix matrix.

(iii) The row vector λ is shown to be prefix in the same way. □

Moreover, the definition of \$-behaviour and (3.4) imply that

$$\forall (f,g) \in A^* \times B^* \qquad (f,g) \in \theta \Leftrightarrow g\$ \in \lambda \cdot (f\$)\mu \cdot \gamma.$$

Let $v = (\$\mu) \cdot \gamma$. Then v is prefix (as the product of two prefix matrices) and

$$(f,g) \in \theta \Leftrightarrow g\$ \in \lambda \cdot f\mu \cdot v.$$

Hence (λ, μ, v) is a prefix representation with endmarker of the relation θ (we identify μ with its restriction to A^*).

(B) Let (λ, μ, v) be a prefix representation with endmarker of the relation θ and let R be the dimension of (λ, μ, v).

Since $(\lambda_p)_{p \in R}$ is a prefix family of rational languages of $B_\$^*$, there exist a deterministic e-automaton $\mathscr{L} = \langle L, B_\$, D, i, R\rangle$ which recognizes this family. By definition, we have

$$v \in \lambda_p \Leftrightarrow i \xrightarrow[\mathscr{L}]{v} p. \tag{3.5}$$

For every state p in R and for every letter a in A, the pth row of $a\mu$ is a prefix family of rational languages of $B_\$^*$. Let $\mathscr{M}_{a,p} = \langle M_{a,p}, B_\$, E_{a,p}, j_{a,p}, R\rangle$ be a deterministic e-automaton which recognizes this family. By definition, we have

$$v \in (a\mu)_{p,q} \Leftrightarrow j_{a,p} \xrightarrow[\mathscr{M}_{a,p}]{v} q. \tag{3.6}$$

Remark that $\mathscr{L}$ and all the $\mathscr{M}_{a,p}$ share the same set R of terminal states. This is possible since there is no edge starting from terminal states.

For every p in R, let $\mathscr{N}_p = \langle N_p, B_\$, F_p, k_p, \{t\}\rangle$ be a deterministic e-automaton which recognizes the prefix language v_p. By definition, we have

$$v \in v_p \Leftrightarrow k_p \xrightarrow[\mathscr{N}_p]{v} t. \tag{3.7}$$

Remark once again that all the $\mathcal{N}_p$ may share the same terminal state t since there is no edge starting from terminal states.

Let us now build a deterministic 2-automaton with endmarker $\mathcal{B} = \langle Q, A_\$, B_\$, E, q_0, T\rangle$, the \$-behaviour of which is equal to θ. Let

$$Q_A = R \cup \{t\} \quad \text{and} \quad Q_B = \left[\left(\bigcup_{a\in A,\, p\in R} M_{a,p}\right) \cup L \cup \left(\bigcup_{p\in R} N_p\right)\right] \Big\backslash\, Q_A.$$

By identification of the elements of $B_\$^*$ with the elements of $\{1\} \times B_\$^*$, the edges of the automata $\mathcal{L}$, $(\mathcal{M}_{a,p})_{a\in A,\, p\in R}$ and $(\mathcal{N}_p)_{p\in R}$ become the edges of the automaton $\mathcal{B}$, the labels of which are in $\{1\} \times B_\$$. It is clear that every such edge starts from a state of Q_B. The edges of the automaton $\mathcal{B}$, the labels of which are in $A_\$ \times \{1\}$, connect the automata $\mathcal{L}$, $(\mathcal{M}_{a,p})_{a\in A,\, p\in R}$, $(\mathcal{N}_p)_{p\in R}$ together, by linking terminal states to initial states. More precisely, for every p in R, we define an edge with label $(a, 1)$ from p to $j_{a,p}$ and an edge with label $(\$, 1)$ from p to k_p. The initial state of $\mathcal{B}$ is equal to i and the set of terminal states of $\mathcal{B}$ is equal to $\{t\}$.

It is clear that $\mathcal{B}$ is deterministic on the second tape (since $\mathcal{B}$ coincides with one of the deterministic automata $\mathcal{L}$, $\mathcal{M}_{a,p}$ or $\mathcal{N}_p$). If there exist a transition $p \xrightarrow{(a,1)} q$ (resp. $p \xrightarrow{(\$,1)} q$), then $q = j_{a,p}$ (resp. $q = k_p$), and thus $\mathcal{B}$ is also deterministic on the first tape.

The equivalences (3.5)–(3.7) and the construction of $\mathcal{B}$ yield the following equivalences:

$$u \in \lambda_p \Leftrightarrow i \xrightarrow[\mathcal{B}]{(1,u)} p, \quad w \in \nu_q \Leftrightarrow q \xrightarrow[\mathcal{B}]{(\$,w)} t, \quad \text{and} \quad v \in (a\mu)_{p,q} \Leftrightarrow p \xrightarrow[\mathcal{B}]{(a,v)} q. \tag{3.8}$$

By induction on the length of f, it follows from (3.8)

$$v \in (f\mu)_{p,q} \;\Leftrightarrow\; p \xrightarrow[\mathcal{B}]{(f,v)} q.$$

We then have the following series of equivalence:

$$g\$ \in \lambda \cdot (f\mu) \cdot \nu \quad\Leftrightarrow\quad \begin{cases} \exists p, q \in R, & \exists u, v, w \in B_\$^* \\ g\$ = uvw, & u \in \lambda_p, \quad v \in (f\mu)_{p,q} \quad \text{and} \quad w \in \nu_q \end{cases}$$

$$\Leftrightarrow\quad \begin{cases} \exists p, q \in R, & \exists u, v, w \in B_\$^* \\ g\$ = uvw, & i \xrightarrow[\mathcal{B}]{(1,u)} p, \quad p \xrightarrow[\mathcal{B}]{(f,v)} q \quad \text{and} \quad q \xrightarrow[\mathcal{B}]{(\$,w)} t \end{cases}$$

$$\Leftrightarrow\; i \xrightarrow[\mathcal{B}]{(f\$,g\$)} t \;\Leftrightarrow\; (f\$, g\$) \in |\mathcal{A}|,$$

which is exactly what is to be established. □

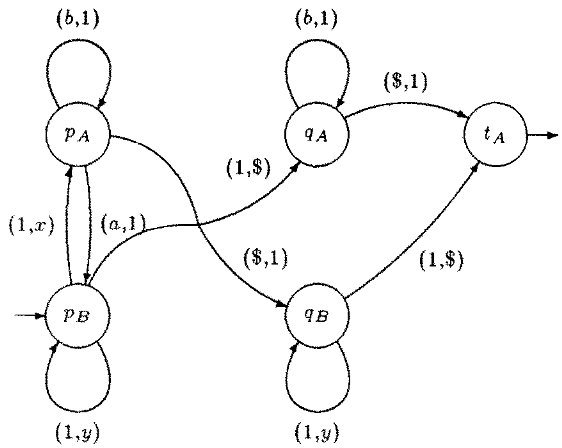


Fig. 13. A deterministic 2-automaton $\mathcal{A}$ for θ_6.

Example 3.1. Let $A=\{a,b\}$ and $B=\{x,y\}$ and let θ_6 be the relation from A^* into B^* defined by $(f,g)\in\theta_6 \Leftrightarrow |f|_a \leqslant |g|_x \leqslant |f|_a+1$.

The relation θ_6 is the \$-behaviour of the deterministic 2-automaton with endmarker $\mathcal{A}$ as drawn in Fig. 13.

The partition of Q is $Q_A=\{p_A,q_A,t_A\}$ and $Q_B=\{p_B,q_B\}$; it yields the block decomposition of I, X, Y and T:

$$I_A=(0\quad 0\quad 0),\quad I_B=(1\quad 0),\quad T_A=\begin{pmatrix}0\\0\\1\end{pmatrix}\quad\text{and}\quad T_B=\begin{pmatrix}0\\0\end{pmatrix},$$

$$X_A=\begin{pmatrix}(b,1) & 0 & 0\\ 0 & (b,1) & (\$,1)\\ 0 & 0 & 0\end{pmatrix},\qquad X_B=\begin{pmatrix}(a,1) & (\$,1)\\ 0 & 0\\ 0 & 0\end{pmatrix},$$

$$Y_A=\begin{pmatrix}(1,y) & (1,\$) & 0\\ 0 & 0 & (1,\$)\end{pmatrix}\quad\text{and}\quad Y_B=\begin{pmatrix}(1,x) & 0\\ 0 & (1,y)\end{pmatrix}.$$

We then get

$$I_A+I_BY_B^*Y_A=((1,y^*x)\quad (1,y^*\$)\quad 0),$$

$$X_A+X_BY_B^*Y_A=\begin{pmatrix}(b,1)+(a,y^*x) & (a,y^*\$) & (\$,y^*\$)\\ 0 & (b,1) & (\$,1)\\ 0 & 0 & 0\end{pmatrix},$$

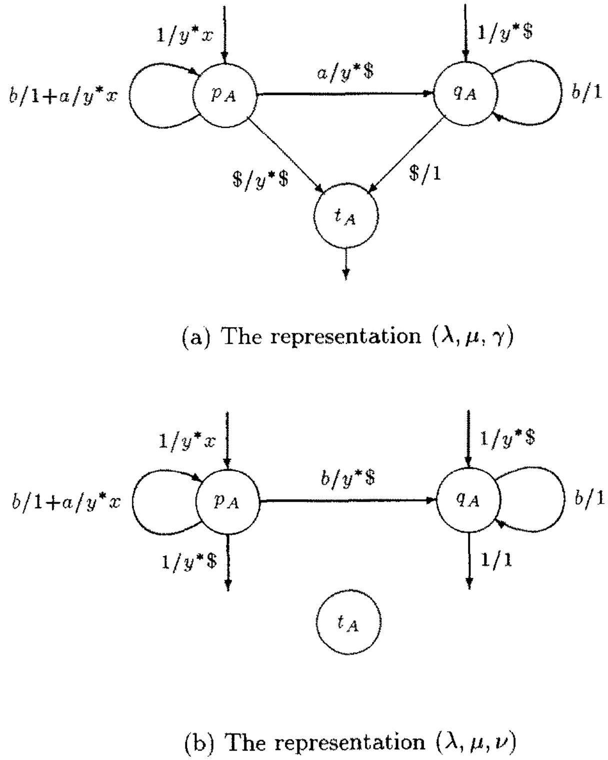


Fig. 14. The computation of a prefix representation for θ_6.

which defines the prefix representation (λ, μ, γ):

$$\lambda = (y^*x \quad y^*\$ \quad 0), \quad \gamma = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$a\mu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad b\mu = \begin{pmatrix} y^*x & y^*\$ & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \$\mu = \begin{pmatrix} 0 & 0 & y^*\$ \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and, finally, the prefix representation with endmarker (λ, μ, v) : λ and m as above and v defined by

$$v = (\$\mu) \cdot \gamma = \begin{pmatrix} y^*\$ \\ 1 \\ 0 \end{pmatrix}.$$

The representations (λ, μ, γ) and (λ, μ, v) correspond to the 2-automata drawn in Fig. 14(a) and (b).

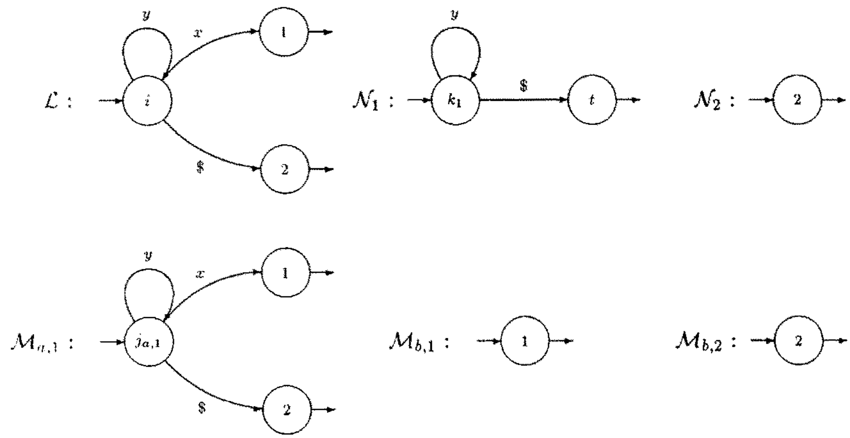


Fig. 15. The automata $\mathscr{L}$, $\mathscr{M}_{a,1}$, $\mathscr{M}_{b,1}$, $\mathscr{M}_{b,2}$, $\mathscr{N}_1$ and $\mathscr{N}_2$.

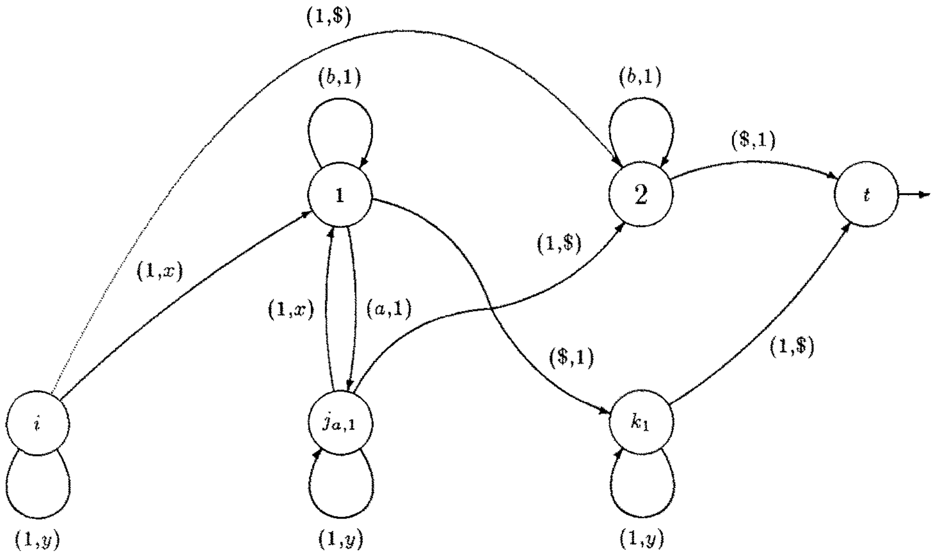


Fig. 16. The automaton $\mathscr{B}$.

Conversely, θ_6 has the prefix representation with endmarker (λ, μ, ν) of dimension $R = \{1, 2\}$ defined by

$$\lambda = (y^*x \quad y^*\$), \quad a\mu = \begin{pmatrix} y^*x & y^*\$ \\ 0 & 0 \end{pmatrix}, \quad b\mu = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \nu = \begin{pmatrix} y^*\$ \\ 1 \end{pmatrix}.$$

The corresponding automata $\mathscr{L}$, $\mathscr{M}_{a,1}$, $\mathscr{M}_{b,1}$, $\mathscr{M}_{b,2}$, $\mathscr{N}_1$ and $\mathscr{N}_2$ constructed in part B of the proof of Proposition 3.2 are drawn in Fig. 15 and the resulting deterministic 2-automaton with endmarker $\mathscr{B}$ is drawn in Fig. 16.

We have not yet taken full advantage of the possible assumption that $\mathscr{A}$ is serialized. This assumption allows indeed to give more information on the prefix representation that has been constructed in the proof of the preceding proposition. In order to present the refined version of the statement, we have to define a notation which allows to describe certain families of matrices.

Notation. Let E_1, E_2, E_3, E_4 be four sets. We denote by

$$\mathsf{Block}\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}$$

the family of matrices M which have a block decomposition

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

where the entries of every M_i are in E_i. We shall also use the notation

$$\mathsf{Block}\ [E_1 \quad E_2], \quad \mathsf{Block}\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \quad \text{and, more generally,} \quad \mathsf{Block}\begin{bmatrix} E_{1,1} & \dots & E_{1,n} \\ \vdots & & \vdots \\ E_{m,1} & \dots & E_{m,n} \end{bmatrix}$$

with the obvious meaning.

Proposition 3.3. *For any deterministic relation from A^* into B^*, there exist a prefix representation (λ, μ, ν) such that, for all $a \in A$,*

$$\lambda \in \mathsf{Block}\ [B^* \quad B^*\$] \quad a\mu \in \mathsf{Block}\begin{bmatrix} B^* & B^*\$ \\ 0 & \mathbb{B} \end{bmatrix} \qquad \nu \in \mathsf{Block}\begin{bmatrix} B^*\$ \\ \mathbb{B} \end{bmatrix}.$$

Proof. Let θ be the \$-behaviour of a deterministic 2-automaton with endmarker $\mathscr{A} = \langle Q, A_\$, B_\$, E, I, T\rangle$. By Proposition 2.9, we may assume that $\mathscr{A}$ is a serialized automaton. We then prove that the representation (λ, μ, ν) built in Proposition 3.2 has the required form. As seen before, $|\mathscr{A}|$ may be written as

$$|\mathscr{A}| = (I_A + I_B Y_B^* Y_A) \cdot (X_A + X_B Y_B^* Y_A)^* \cdot T_A.$$

Since $\mathscr{A}$ is serialized, Q_A and Q_B admit the following partitions:

$$Q_A = Q'_A \cup Q''_A \cup \{t\} \quad \text{and} \quad Q_B = Q'_B \cup Q''_B$$

and the matrices I_A, I_B, X_A, X_B, Y_A, Y_B, T_A and T_B have the following block decompositions:

$$I_A = \begin{pmatrix} I'_A & 0 & 0 \end{pmatrix}, \quad I_B = \begin{pmatrix} I'_B & 0 \end{pmatrix}, \quad T_A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad T_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$X_A = \begin{pmatrix} X'_A & 0 & 0 \\ 0 & X'_A & X''_t \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_B = \begin{pmatrix} X'_B & X''_B \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.9}$$

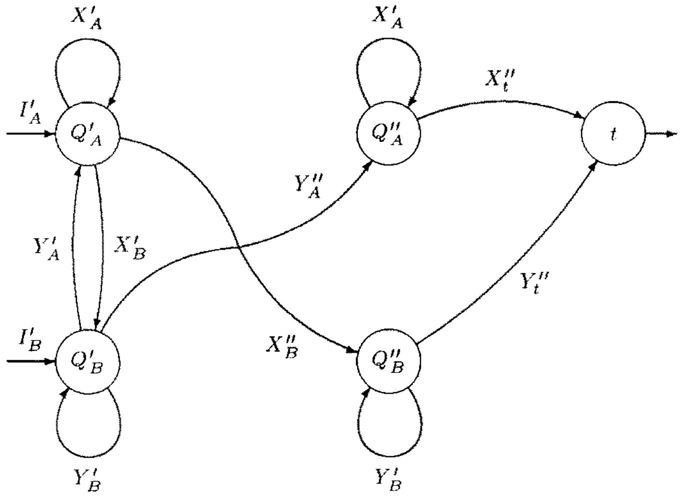


Fig. 17. The block decomposition of the serialized 2-automaton $\mathscr{A}$.

$$Y_A = \begin{pmatrix} Y'_A & Y'_A & 0 \\ 0 & 0 & X''_t \end{pmatrix} \quad \text{and} \quad Y_B = \begin{pmatrix} Y'_B & 0 \\ 0 & X''_B \end{pmatrix}, \tag{3.10}$$

where the entries of X'_A and X'_B (resp. Y'_A and Y'_B) are in $A^* \times \{1\}$ (resp. $\{1\} \times B^*$) and the entries of X''_B and X''_t (resp. Y''_A and Y''_t) are in $\{(\$, 1)\}$ (resp. $\{(1\ ,\$)\}$). These block decompositions are illustrated in Fig. 17. With the previous decompositions of I_A, I_B, X_A, X_B, Y_A, Y_B, we obtain

$$I_A + I_B Y_B^* Y_A = \begin{pmatrix} I'_A + I'_B {Y'_B}^* Y'_A & I'_B {Y'_B}^* Y''_A & 0 \end{pmatrix}, \tag{3.11}$$

$$X_A + X_B Y_B^* Y_A = \begin{pmatrix} X'_A + X'_B {Y'_B}^* Y'_A & X'_B {Y'_B}^* Y''_A & X''_B {Y'_B}^* Y''_t \\ X'_A & X''_t & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.12}$$

These computations are illustrated in Fig. 18. Let us recall the construction made in Proposition 3.2. We first built a prefix representation (λ, μ, γ) of dimension $Q'_A \cup Q''_A \cup \{t\}$ from $A^*_\$$ into $B^*_\$$ such that

$$(f, g) \in \theta \Leftrightarrow g\$ \in \lambda \cdot (f\$)\mu \cdot \gamma.$$

For every states p and q of $Q'_A \cup Q''_A \cup \{t\}$ and for every letter x of $A_\$$,

$$\lambda_p = \{v \in B^*_\$ \mid (1, v) \in (I_A + I_B Y_B^* Y_A)_p\}, \quad \gamma_q = \begin{cases} 1 & \text{if } q = t, \\ 0 & \text{otherwise.} \end{cases}$$

$$x\mu_{p,q} = \{v \in B^*_\$ \mid (x, v) \in (X_A + X_B Y_B^* Y_A)_{p,q}\}.$$

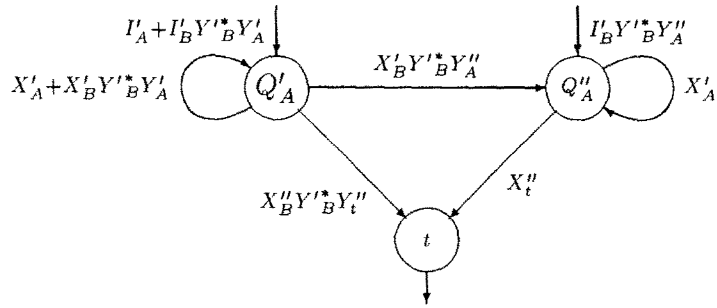


Fig. 18. The computations (3.11) and (3.12).

From the block decomposition of $I_A+I_BY_B^*Y_A$ and $X_A+X_BY_B^*Y_A$ (in (3.11) and (3.12)) and by definition of X'_A, X'_B, Y'_A, Y'_B, X''_B, X''_t, Y''_B, Y''_t, it should be clear that

$$\lambda \in \mathsf{Block}\,[B^* \quad B^*\$ \quad 0], \qquad \gamma \in \mathsf{Block}\begin{bmatrix}0\\0\\1\end{bmatrix},$$

$$a\mu \in \mathsf{Block}\begin{bmatrix}B^* & B^*\$ & 0\\ 0 & \mathbb{B} & 0\\ 0 & 0 & 0\end{bmatrix} \quad \text{for } a \text{ in } A, \text{ and} \quad \$\mu \in \mathsf{Block}\begin{bmatrix}0 & 0 & B^*\$\\ 0 & 0 & \mathbb{B}\\ 0 & 0 & 0\end{bmatrix}.$$

We then defined a prefix representation (λ,μ,ν) from A^* into $B^*_\$$: μ is identified with its restriction to A^* and $\nu=(\$\mu).\gamma$, thus

$$\nu \in \mathsf{Block}\begin{bmatrix}B^*\$\\ \mathbb{B}\\ 0\end{bmatrix}.$$

Since all entries in row and column t are 0, the dimension of (λ,μ,ν) is in fact $Q'_A \cup Q''_A$. Therefore, for every a in A, we have

$$\lambda \in \mathsf{Block}\,[B^* \quad B^*\$], \quad a\mu \in \mathsf{Block}\begin{bmatrix}B^* & B^*\$\\ 0 & \mathbb{B}\end{bmatrix} \quad \text{and} \quad \nu \in \mathsf{Block}\begin{bmatrix}B^*\$\\ \mathbb{B}\end{bmatrix}. \qquad \square$$

4. Complement of a deterministic relation

If the complement of a rational relation is not, in general, a rational relation, it is not difficult to figure out, starting from the definition of a deterministic 2-automaton and the description of its behaviour, as done in Section 2.1, that the complement of a deterministic rational relation is again a deterministic rational relation. Such a statement is given, for instance, in [14] with no more explanation than that.

The prefix representation of deterministic rational relations yields a canonical construction of the complement relation. We give it here as an illustration of the potential of such a representation. It goes through the definition of the prefix complement of a prefix subset.

For any subset L of A^*, let $\mathrm{Pre}(L)$ (resp. $\mathrm{Pre}_+(L)$) be the set of prefixes (resp. the set of proper prefixes) of the elements of L. If L is prefix, then $\mathrm{Pre}_+(L) = \mathrm{Pre}(L)\backslash L$. Obviously, $L \subset \mathrm{Pre}_+(L) \cdot A$ and $L \cap \mathrm{Pre}_+(L) = \emptyset$ if L is prefix. Let us denote by $\widehat{L}$ the set

$$\widehat{L} = [\mathrm{Pre}_+(L) \cdot A]\backslash \mathrm{Pre}_+(L).$$

Then $\widehat{L}$ is prefix and $L \subset \widehat{L}$ if L is prefix.

Definition 4.1. Let L be a prefix subset of A^* that we suppose to be non-empty nor equal to 1_{A^*}. We call *prefix complement* of L, and we denote by $\widetilde{L}$, the complement of L with respect to $\widehat{L}$:

$$\widetilde{L} = \widehat{L}\backslash L.$$

It will be consistent to take $\widehat{L} = 1_{A^*}$ in the cases where L is empty or equal to 1_{A^*}; then we have $\widetilde{\emptyset} = 1_{A^*}$ and $\widetilde{1_{A^*}} = \emptyset$. It follows from the definition that the prefix complement of a rational prefix subset is rational.

Proposition 4.1. *The complement of a deterministic relation is a deterministic relation.*

Proof. Let θ be a deterministic relation from A^* into B^* and let (λ, μ, ν) be a prefix representation of θ of dimension Q. We may assume that there exists a partition $Q_1 \cup Q_2$ of Q such that, for every a in A,

$$\lambda \in \mathsf{Block}\ [B^* \quad B^*\$], \qquad a\mu \in \mathsf{Block}\begin{bmatrix} B^* & B^*\$ \\ 0 & \mathbb{B} \end{bmatrix}, \qquad \nu \in \mathsf{Block}\begin{bmatrix} B^*\$ \\ B^* \end{bmatrix}.$$

Let t_1 and t_2 be two new states and let

$$Q'_1 = Q_1 \cup \{t_1\}, \qquad Q'_2 = Q_2 \cup \{t_2\} \quad \text{and} \quad Q' = Q'_1 \cup Q'_2.$$

We define the representation β of dimension Q' together with the four vectors α, γ, δ and δ' by the following:

$$\begin{array}{cccccc} & Q_1 & t_1 & Q_2 & t_2 & \\ \alpha = \Big(& \boxed{\cdots \ \lambda_p \ \cdots} & \boxed{\widetilde{\lambda_\bullet} \cap B^*} & \boxed{\cdots \ \lambda_p \ \cdots} & \boxed{\widetilde{\lambda_\bullet} \cap B^*\$} & \Big), \end{array}$$

$$a\beta = \begin{array}{c} \begin{array}{cccc} \quad Q_1 \quad & \quad t_1 \quad & \quad Q_2 \quad & \quad t_2 \quad \end{array} \\ \begin{pmatrix} \begin{matrix} \vdots \\ \cdots \ a\mu_{p,q} \ \cdots \\ \vdots \end{matrix} & \begin{matrix} \vdots \\ \widetilde{a\mu_{p,\bullet}} \cap B^* \\ \vdots \end{matrix} & \begin{matrix} \vdots \\ \cdots \ a\mu_{p,q} \ \cdots \\ \vdots \end{matrix} & \begin{matrix} \vdots \\ \widetilde{a\mu_{p,\bullet}} \cap B^*\$ \\ \vdots \end{matrix} \\ \\ \cdots \ 0 \ \cdots & 1 & \cdots \ 0 \ \cdots & 0 \\ \\ \begin{matrix} \vdots \\ \cdots \ 0 \ \cdots \\ \vdots \end{matrix} & \begin{matrix} \vdots \\ 0 \\ \vdots \end{matrix} & \begin{matrix} \vdots \\ \cdots \ a\mu_{p,q} \ \cdots \\ \vdots \end{matrix} & \begin{matrix} \vdots \\ \widetilde{a\mu_{p,\bullet}} \\ \vdots \end{matrix} \\ \\ \cdots \ 0 \ \cdots & 0 & \cdots \ 0 \ \cdots & 1 \end{pmatrix} \end{array},$$

$$\delta = \begin{pmatrix} \begin{matrix} \vdots \\ B^*\$ \\ \vdots \end{matrix} \\ \\ B^*\$ \\ \\ \begin{matrix} \vdots \\ 1 \\ \vdots \end{matrix} \\ \\ 1 \end{pmatrix} \quad \gamma = \begin{pmatrix} \begin{matrix} \vdots \\ v_q \\ \vdots \end{matrix} \\ \\ 0 \\ \\ \begin{matrix} \vdots \\ v_q \\ \vdots \end{matrix} \\ \\ 0 \end{pmatrix} \quad \gamma' = \begin{pmatrix} \begin{matrix} \vdots \\ B^*\$ \backslash v_q \\ \vdots \end{matrix} \\ \\ B^*\$ \\ \\ \begin{matrix} \vdots \\ 1 \backslash v_q \\ \vdots \end{matrix} \\ \\ 1 \end{pmatrix}.$$

By construction, (α,β,δ), (α,β,γ) and (α,β,γ') are prefix representations.

Let u be in A^*; the definition of (α,β,γ) gives the following three implications:

$$\begin{cases} \qquad\qquad p\in Q \;\Rightarrow\; \alpha_p=\lambda_p \\ (q\in Q \quad \text{and} \quad (u\beta)_{p,q}\neq 0) \;\Rightarrow\; (p\in Q \text{ and } (u\beta)_{p,q}=(u\mu)_{p,q}) \\ \qquad\qquad \gamma_q\neq 0 \;\Rightarrow\; (q\in Q \text{ and } \gamma_q=\nu_q) \end{cases}$$

Hence, we have

$$\alpha\cdot u\beta\cdot\gamma=\sum_{p,q\in Q'}\alpha_p(u\beta)_{p,q}\gamma_q=\sum_{p,q\in Q}\lambda_p(u\mu)_{p,q}\nu_q=\lambda\cdot u\mu\cdot\nu.$$

Therefore (α,β,γ) is a prefix representation (with endmarker) of θ.

Claim. *(α,β,δ) is a prefix representation (with endmarker) of the whole set $A^*\times B^*$.*

First remark that, for every a in A,

$$\forall p\in Q_2',\ \exists q\in R_2'\quad a\beta_{p,q}=1. \tag{4.1}$$

Let u in A^* and v in B^*. If $\alpha_\bullet$ does not contain any prefix of v, it contains $v\$$ since it is the smallest prefix complete subset which contains $\lambda_\bullet$. Moreover, $v\$$ belongs to α_{t_2} and since $\delta_{t_2}=1=(u\beta)_{t_2,t_2}$ for every u, $v\$$ belongs to $\alpha\cdot u\beta\cdot\delta$.

Assume now that $\alpha_\bullet$ contains a prefix of v and let v_1 be the longest prefix of v such that there exists a prefix u_1 of u such that

$$v_1\in(\alpha\cdot u_1\beta)_\bullet.$$

Such u_1 is unique; let $v_2=(v_1)^{-1}v$ and $u_2=(u_1)^{-1}u$. Since v_1 is in B^*, the state p in Q' such that $v_1\in(\alpha\cdot u_1\beta)_p$ is in Q_1' and thus $\delta_p=B^*\$$.

If $u=u_1$, then $v\$=v_1v_2\$$ is in $(\alpha\cdot u\beta)_p\delta_p$, contained in $\alpha\cdot u\beta\cdot\delta$.

If $u\neq u_1$, then $u=u_1au_2$. By construction, $(a\beta)_{p,\bullet}$ does not contain any prefix of v_2. It thus contains $v_2\$$ since it is the smallest prefix complete subset which contains $(a\mu)_{p,\bullet}$. Moreover, $v_2\$$ belongs to $(a\beta)_{p,t_2}$ and since $\delta_{t_2}=1=(u_2\beta)_{t_2,t_2}$ for every u_2, $v_1v_2\$$ belongs to $\alpha\cdot u_1au_2\beta\cdot\delta$. And the claim is proved.

Let now θ' be the relation from A^* into B^* realized the representation (α,β,γ'); since this representation is prefix, θ' is deterministic. Since $\delta=\gamma\cup\gamma'$, (α,β,δ) is a representation of $\theta\cup\theta'$ and thus

$$\theta\cup\theta'=A^*\times B^*.$$

Assume that $\theta\cap\theta'$ is not empty, i.e. there exists a pair (u,v) such that

$$v\$\in(\alpha\cdot u\beta\cdot\gamma)\cap(\alpha\cdot u\beta\cdot\gamma').$$

It follows then

$$\exists q\in Q',\ \exists v_1\in(\alpha\cdot u\beta)_q,\ \exists v_2\in\gamma_q\qquad v\$=v_1v_2$$

and

$$\exists q' \in Q',\ \exists v_1' \in (\alpha \cdot u\beta)_{q'},\ \exists v_2' \in \gamma_{q'} \qquad v\$ = v_1' v_2'.$$

Obviously, either v_1 is a prefix of v_1' or v_1' is a prefix of v_1, and since the family $\{(\alpha \cdot u\beta)_p\}_{p \in Q}$ is prefix, then

$$v_1 = v_1' \quad \text{(and thus } v_2 = v_2'\text{)} \quad \text{and} \quad q = q'.$$

But $v_2 = v_2'$ contradicts $\gamma_q \cap \gamma_q' = \emptyset$ and thus $\theta \cap \theta' = \emptyset$: θ' is the complement of θ and we have already seen that this complement is a deterministic relation. □

Example 4.1. Let $A = \{a, b, c\}$ and let θ_7 be the mapping equivalence of the morphism that erases the letter a. Then θ_7 has the following prefix representation (λ, μ, ν) of dimension $Q = Q_1 = \{1\}$:

$$\lambda = (1) \quad \nu = (a^*\$) \quad a\mu = (1) \quad b\mu = (a^*b) \quad c\mu = (a^*c) \quad \nu = (a^*\$).$$

Let $Q' = Q \cup \{t, t_\$\}$. Let (α, β, δ) be the representation of dimension Q' defined by

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \qquad \delta = \begin{pmatrix} A^*\$ \\ A^*\$ \\ 1 \end{pmatrix},$$

$$a\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad b\beta = \begin{pmatrix} a^*b & a^*c & a^*\$ \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad c\beta = \begin{pmatrix} a^*c & a^*b & a^*\$ \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(for every letter x of A, $x\beta_{1,1} = x\mu_{1,1}$ and $x\beta_{1,t_1}$ (resp. $x\beta_{1,t_2}$) is the prefix complement of $x\mu_{1,1}$ in A^* (resp. in $A^*\$$) and α is obtained in an analoguous way). The column-vectors γ and γ' are defined by

$$\gamma = \begin{pmatrix} a^*\$ \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \gamma' = \begin{pmatrix} A^*\$ \backslash a^*\$ \\ A^*\$ \\ 1 \end{pmatrix}.$$

5. Uniformization of deterministic rational relations

Let $\theta : A^* \to B^*$ be any relation; a function $\alpha : A^* \to B^*$ is said to *uniformize* θ, or to be a *uniformization* [10] of θ, if it selects one element in $f\theta$ for every f in $\operatorname{Dom}\theta$. In other words, α is a *function* such that

$$\operatorname{Dom}\alpha = \operatorname{Dom}\theta \quad \text{and} \quad \forall f \in \operatorname{Dom}\theta \quad f\alpha \in f\theta.$$

It is easy to derive the following result from Eilenberg's Rational Cross-Section Theorem [4].

[10] This terminology comes from logic; cf. for instance [1, p. 368].

Theorem 5.1 (Rational Uniformization Theorem). *Any rational relation is uniformized by an unambiguous rational function.*

As a corollary, one gets that every rational function is unambiguous.

We have explained in [17] how a construction on automata due to Schützenberger yields an enlightening proof of Theorem 5.1. This construction naturally takes place in the framework of *covering* of automata. We have also shown in the same paper that this result, instead of being a corollary of some other proposition, can be given a central position in the theory of rational relations. In spite of its restored preeminence, Theorem 5.1 remains unsatisfactory by one aspect: it says that any rational relation θ can be uniformized by a rational function α, the proof guarantees that such uniformizing α is effectively computable from a 2-automaton $\mathscr{A}$ that realizes θ, but it does not say anything on the "nature" of α. In particular, α *will not be intrinsic* to θ and, in fact, the computed α will depends not only on the automaton $\mathscr{A}$ one starts from, but also on a number of choices that have to be made in the course of the construction. Let us explain what could be an intrinsic uniformization of a relation.

Suppose first that a (total) ordering $\preccurlyeq$ is chosen on the alphabet B. This ordering extends, as in Section 2.4, Example 2.4, to the *lexicographic ordering* on B^*. The lexicographic ordering is total, but is *not a well ordering*: a subset may well have no smallest element (e.g. a^*b, with $a \prec b$).

Given any relation $\theta : A^* \to B^*$, we define $\theta_{\text{lex}} : A^* \to B^*$ as the function that associates to every element f of A^* the smallest element of $f\theta$ *when this smallest element exists*. We call θ_{lex} the *lexicographic selection* of θ. Obviously, $\operatorname{Dom}\theta_{\text{lex}} \subseteq \operatorname{Dom}\theta$; if $\operatorname{Dom}\theta_{\text{lex}} = \operatorname{Dom}\theta$, i.e. if $f\theta$ contains a smallest element for every f in $\operatorname{Dom}\theta$, then θ_{lex} is a uniformization, called *lexicographic uniformization* of θ. Clearly, the lexicographic uniformization of θ is *intrinsic* to θ.

In general, the lexicographic selection of a rational relation is *not* a rational function. Though it is difficult to state it formally, it is very likely that any reasonable attempt to define *intrinsic* uniformizations would fail in giving *rational* uniformizations. The following result, proved by Johnson [10], settles the case for *deterministic* relations.

Theorem 5.2 (Lexicographic Uniformization Theorem). *The lexicographic selection of a deterministic rational relation is an unambiguous rational function.*

The purpose of this section is to show how Schützenberger construct yields a proof for this result as well, though we have to proceed in way dual to the one we naturally used for Theorem 5.1. For the sake of completeness, and because of the "duality", we recall the Schützenberger construct as it is presented in [17]. Let us first add few comments to Theorem 5.2.

In [15], we showed that the lexicographic selection of the equivalence mapping of a morphism between free monoids is a rational relation. As such equivalence mappings are deterministic relations (cf. Section 2.2.3), this appears now as a particular case of Johnson's theorem. [11]

In the same paper, we also considered the "military" selection of a relation, using the military ordering instead of the lexicographic ordering for the choice of an element in a subset of B^*; since the military ordering is a well ordering, we define thus a "military uniformization" in any case. But it turns out that, even in the very special case of the equivalence mapping of a morphism, the military uniformization is *not* a rational relation. Which shows that Theorem 5.2 does not state only a property of deterministic relation, but also a property of the lexicographic ordering, or, to be more accurate, a property of the relationship between deterministic relation and lexicographic ordering.

To tell the truth, Johnson proved in [10] that the lexicographic selection (of a deterministic relation) is a (functional) *deterministic* relation, which is stronger statement than Theorem 5.2. We are interested here in giving the Lexicographic Uniformization Theorem, that applies to *deterministic* relations, the *same proof* as the one we gave in [17] for Rational Uniformization Theorem for *general rational* relations, in order to point up *where* the hypothesis of determinism really plays a rôle and *how* it allows to construct the lexicographic selection. It is an open question whether this could be achieved while retaining the full strength of Johnson's result.

5.1. The Schützenberger construct on automata

The Schützenberger construct operates on "classical" automata (not on 2-automata) and yields Theorem 5.3 below. We first describe the construct in terms of labelled graphs and then we rephrase it by means of matrix representations. We shall stress on the dual construct for it is the one we shall use later.

Theorem 5.3. *Let $\mathcal{A}$ be an automaton on A^*. There exists an unambiguous automaton that is equivalent to $\mathcal{A}$ and that is an* immersion *in $\mathcal{A}$. There exists an unambiguous automaton that is equivalent to $\mathcal{A}$ and that is a* co-immersion *in $\mathcal{A}$.*

The essence of this statement – which is the way the construction of [21] is presented in [17] – lies in the fact that the quoted equivalent automaton is an *immersion* (or a *co-immersion*) in $\mathcal{A}$. For otherwise, the deterministic automaton $\mathcal{A}_{\mathcal{D}}$, and the co-deterministic automaton $\mathcal{A}_{\mathrm{co}\mathcal{D}}$, associated to $\mathcal{A}$ by the so-called "subset method" are obviously unambiguous and equivalent to $\mathcal{A}$. But they are not an immersion, nor a co-immersion, in $\mathcal{A}$: for instance, there is no relationships between the pathes in $\mathcal{A}$ and those in $\mathcal{A}_{\mathrm{co}\mathcal{D}}$, as it can be observed in Fig. 19.

The immersion we shall get is a subautomaton of a special covering, that we shall call *S-covering* of $\mathcal{A}$ and that is *the accessible part of the direct product of $\mathcal{A}_{\mathcal{D}}$*

[11] The relations that are considered in [15] are indeed slightly more general, since they are the composition of an equivalence mapping of a morphism with the intersection with a rational set. Such relations can be shown to be deterministic as well.

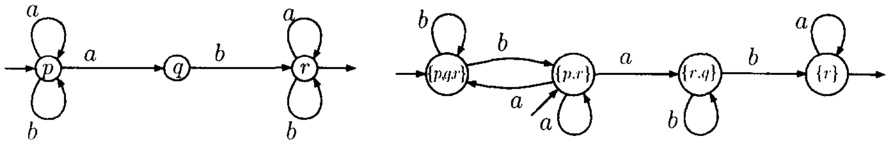


Fig. 19. The automaton $\mathscr{A}_1$ and $\mathscr{A}_{1\mathrm{co}\mathscr{D}}$, its co-determinized by the subset method.

with $\mathscr{A}$; accordingly, the co-immersion we shall get is a subautomaton of a special co-covering, the *S-co-covering* of $\mathscr{A}$, which is *the co-accessible part of the direct product of* $\mathscr{A}_{\mathrm{co}\mathscr{D}}$ *with* $\mathscr{A}$.

The S-covering and the S-immersion have been built in [17]. We recall here these constructions and we adapt them to the S-co-covering and S-co-immersion.

5.1.1. The construct on labelled graphs

As we just did, we note $\mathscr{A}_{\mathscr{D}}$ the deterministic automaton obtained from an automaton $\mathscr{A}$ over a free monoid by the subset method (and we call it the *determinized* of $\mathscr{A}$).

Theorem & Definition 5.4. *Let* $\mathscr{A}$ *be an automaton and* $\mathscr{A}_{\mathscr{D}}$ *its determinized. Let* $\mathscr{S}$ *be the accessible part of* $\mathscr{A}_{\mathscr{D}} \times \mathscr{A}$. *It then holds*:
(i) $\pi_{\mathscr{A}}$ *is a covering of* $\mathscr{S}$ *onto* $\mathscr{A}$;
(ii) $\pi_{\mathscr{A}_{\mathscr{D}}}$ *is an In-surjective morphism from* $\mathscr{S}$ *onto* $\mathscr{A}_{\mathscr{D}}$.
We call $\mathscr{S}$ *the* S-covering *of* $\mathscr{A}$.

Example 1.1 (*Continued*). The S-covering $\mathscr{S}_1$ of $\mathscr{A}_1$, together with $\mathscr{A}_1$ and $\mathscr{A}_{1\mathscr{D}}$ are shown on Fig. 20. The transitions of $\mathscr{S}_1$ on which $\pi_{\mathscr{A}_{1\mathscr{D}}}$ is *not In-injective* are marked up as bold arrows.

As already mentioned, we denote by $\mathscr{A}_{\mathrm{co}\mathscr{D}}$ the co-deterministic automaton obtained from an automaton $\mathscr{A}$ by the subset method (and we call it the *co-determinized* of $\mathscr{A}$).

Theorem & Definition 5.5. *Let* $\mathscr{A}$ *be an automaton and* $\mathscr{A}_{\mathrm{co}\mathscr{D}}$ *its co-determinized. Let* $\mathscr{S}'$ *be the co-accessible part of* $\mathscr{A}_{\mathrm{co}\mathscr{D}} \times \mathscr{A}$. *Then the following holds*:
(i) $\pi_{\mathscr{A}}$ *is a co-covering of* $\mathscr{S}'$ *onto* $\mathscr{A}$;
(ii) $\pi_{\mathscr{A}_{\mathrm{co}\mathscr{D}}}$ *is an Out-surjective morphism from* $\mathscr{S}'$ *onto* $\mathscr{A}_{\mathrm{co}\mathscr{D}}$.
We call $\mathscr{S}'$ *the* S-co-covering *of* $\mathscr{A}$.

Example 1.1 (*Continued*). The S-co-covering $\mathscr{S}'_1$ of $\mathscr{A}_1$, together with $\mathscr{A}_1$ and $\mathscr{A}_{1\mathrm{co}\mathscr{D}}$ are shown on Fig. 21. The transitions of $\mathscr{S}_1$ on which $\pi_{\mathscr{A}_{1\mathrm{co}\mathscr{D}}}$ is not *Out-injective* are marked up as bold arrows.

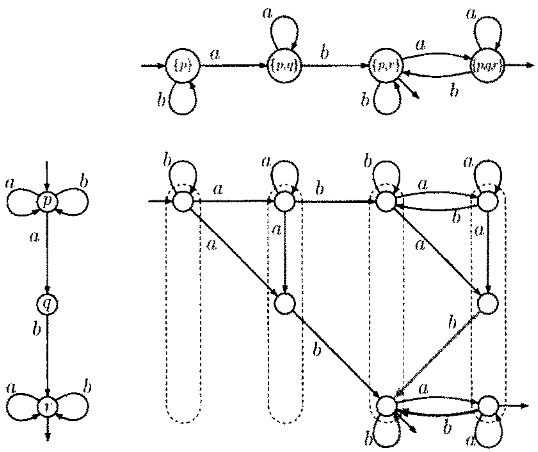


Fig. 20. The S-covering of $\mathscr{A}_1$.

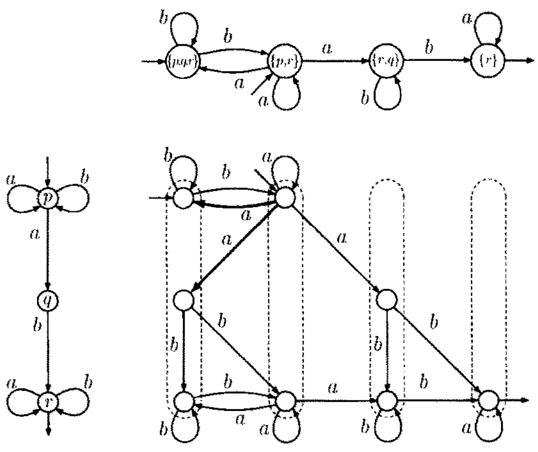


Fig. 21. The S-co-covering of $\mathscr{A}_1$.

In order to prove Theorem 5.4, we first state two easy properties of morphisms of automata (cf. [17]). Let $\mathscr{A} = \langle Q, A, E, I, T \rangle$ and $\mathscr{B} = \langle R, A, F, J, U \rangle$ be two automata on A^*.

Property 5.1. *Let $\mathscr{B}$ be a* deterministic *and* complete *automaton on A. For any automaton $\mathscr{A}$ on A, $\pi_{\mathscr{A}}$ is an* Out-bijective *morphism from $\mathscr{B} \times \mathscr{A}$ onto $\mathscr{A}$.*

Property 5.2. *Let $\varphi : \mathscr{B} \to \mathscr{A}$ be an* Out-bijective *morphism and let $\mathscr{C}$ be the* accessible part *of $\mathscr{B}$. Then the restriction of φ to $\mathscr{C}$ is an* Out-bijective *morphism from $\mathscr{C}$ onto $\mathscr{A}$.*

Condition (i) of Theorem 5.4 is then seen as a particular case of a more general statement.

Proposition 5.1. *Let $\mathscr{A}$ be an automaton, $\mathscr{B}$ a deterministic automaton* equivalent to $\mathscr{A}$, *and $\mathscr{S}$ the accessible part of $\mathscr{B}\times\mathscr{A}$. Then $\pi_{\mathscr{A}}$ is a covering from $\mathscr{S}$ onto $\mathscr{A}$.*

Proof. By Properties 5.1 and 5.2, $\pi_{\mathscr{A}}$ is an *Out-bijective* morphism from $\mathscr{S}$ onto $\mathscr{A}$.

Since $\mathscr{B}$ (as any deterministic automaton) has only one initial state $J=\{r_0\}$, then for every initial state i of $\mathscr{A}$ there exists one and only one initial state in $i\pi_{\mathscr{A}}^{-1}$: (r_0,i).

Let now (r,p) in $R\times Q$ be an accessible state of $\mathscr{B}\times\mathscr{A}$, i.e. there exist f in A^* and i in I such that

$$(r_0,i)\xrightarrow[\mathscr{B}\times\mathscr{A}]{f}(r,p)\quad\text{thus } r_0\xrightarrow[\mathscr{B}]{f}r\quad\text{and}\quad i\xrightarrow[\mathscr{A}]{f}p.$$

If p is in T, then f is in $|\mathscr{A}|$ and r is in U since $\mathscr{B}$ is equivalent to $\mathscr{A}$: every state of $\mathscr{S}$ that is mapped onto p by $\pi_{\mathscr{A}}$ is terminal.

The three conditions for being a covering have thus been checked for $\pi_{\mathscr{A}}:\mathscr{S}\to\mathscr{A}$. □

Proof of Theorem 5.4. It remains to prove condition (ii).

Let $\mathscr{A}_{\mathscr{D}}=\langle 2^Q,A,F,J,U\rangle$.[12] By definition,

$$F=\{(P,a,S)\in 2^Q\times A\times 2^Q\mid S=\{s\mid \exists p\in P(p,a,s)\in E\}\},$$
$$J=\{I\}\quad\text{and}\quad U=\{S\in 2^Q\mid S\cap T\neq\emptyset\}.$$

From this definition, it follows that

$$P\xrightarrow[\mathscr{A}_{\mathscr{D}}]{a}S\;\Leftrightarrow\;S=\left\{q\mid\exists p\in P\quad p\xrightarrow[\mathscr{A}]{a}q\right\}$$

and then

$$\begin{aligned}\forall P,S\subset Q\quad\forall q\in S\quad P\xrightarrow[\mathscr{A}_{\mathscr{D}}]{a}S\;&\Rightarrow\;\exists p\in P\quad p\xrightarrow[\mathscr{A}]{a}q\\&\Rightarrow\;\exists p\in P\quad (P,p)\xrightarrow[\mathscr{A}_{\mathscr{D}}\times\mathscr{A}]{a}(S,q)\end{aligned}$$

which expresses that $\pi_{A_{\mathscr{D}}}:\mathscr{S}\to\mathscr{A}_{\mathscr{D}}$ is In-surjective. □

Proof of Theorem 5.5 goes in a way dual to the one of Theorem 5.4. Statement of intermediary properties is useful for it helps understanding, but formal proofs are not really needed since the dual of what is true for $\mathscr{A}$ holds for $\mathscr{A}^t$.

[12] This should not be confusing, for $\mathscr{A}_{\mathscr{D}}$ and $\mathscr{B}$ never appear in the same statement; on the contrary, $\mathscr{A}_{\mathscr{D}}$ happens to be a special case of an automaton $\mathscr{B}$.

Property 5.3. *Let* $\mathscr{B}$ *be a* co-deterministic *and* co-complete *automaton on* A*. For any automaton* $\mathscr{A}$ *on* A*,* $\pi_{\mathscr{A}}$ *is an* In-bijective *morphism from* $\mathscr{B} \times \mathscr{A}$ *onto* $\mathscr{A}$*.*

Property 5.4. *Let* $\varphi : \mathscr{B} \to \mathscr{A}$ *be an* In-bijective *morphism and let* $\mathscr{C}$ *be the* co-accessible part *of* $\mathscr{B}$*. Then the restriction of* φ *to* $\mathscr{C}$ *is an* In-bijective *morphism from* $\mathscr{C}$ *onto* $\mathscr{A}$*.*

Condition (i) of Theorem 5.5 is then seen as a particular case of a more general statement.

Proposition 5.2. *Let* $\mathscr{A}$ *be an automaton,* $\mathscr{B}$ *a co-deterministic automaton* equivalent to $\mathscr{A}$*, and* $\mathscr{S}'$ *the co-accessible part of* $\mathscr{B} \times \mathscr{A}$*. Then* $\pi_{\mathscr{A}}$ *is a co-covering from* $\mathscr{S}'$ *onto* $\mathscr{A}$*.*

Proof of Theorem 5.5. It remains to prove condition (ii).
Let $\mathscr{A}_{\mathrm{co}\mathscr{D}} = \langle 2^Q, A, F', J', U' \rangle$. By definition,

$$F' = \{(P,a,S) \in 2^Q \times A \times 2^Q \mid P = \{p \mid \exists s \in S \quad (p,a,s) \in E\}\},$$
$$J' = \{P \in 2^Q \mid P \cap I \neq \emptyset\} \quad \text{and} \quad U' = \{T\}.$$

From this definition, it follows:

$$P \xrightarrow[\mathscr{A}_{\mathrm{co}\mathscr{D}}]{a} S \;\Leftrightarrow\; P = \{p \mid \exists q \in S \quad p \xrightarrow[\mathscr{A}]{a} q\}$$

and then

$$\forall P,S \subset Q, \quad \forall p \in P \quad P \xrightarrow[\mathscr{A}_{\mathrm{co}\mathscr{D}}]{a} S \;\Rightarrow\; \exists q \in S \quad p \xrightarrow[\mathscr{A}]{a} q$$
$$\Rightarrow\; \exists q \in S \quad (P,p) \xrightarrow[\mathscr{A}_{\mathrm{co}\mathscr{D}} \times \mathscr{A}]{a} (S,q)$$

which expresses that $\pi_{\mathscr{A}_{\mathrm{co}\mathscr{D}}} : \mathscr{S}' \to \mathscr{A}_{\mathrm{co}\mathscr{D}}$ is Out-surjective. □

Proof of Theorem 5.3. Let $\mathscr{S}$ be the S-covering of an automaton $\mathscr{A}$. Since $\pi_{\mathscr{A}_{\mathscr{D}}}$ is In-surjective from $\mathscr{S}$ onto $\mathscr{A}_{\mathscr{D}}$, it is possible, by deleting some edges in $\mathscr{S}$ if $\pi_{\mathscr{A}_{\mathscr{D}}}$ is not In-injective, and by suppressing if necessary their quality of being terminal to certain states, to construct a sub-automaton $\mathscr{T}$ of $\mathscr{S}$ that is a *co-covering* of $\mathscr{A}_{\mathscr{D}}$. Such a $\mathscr{T}$ is thus unambiguous (Corollary 1.8), and equivalent to $\mathscr{A}_{\mathscr{D}}$ and thus to $\mathscr{A}$. Since $\mathscr{S}$ is a covering of $\mathscr{A}$, $\mathscr{T}$ is an immersion in $\mathscr{A}$.

Let $\mathscr{S}'$ be the S-co-covering of $\mathscr{A}$. Since $\pi_{\mathscr{A}_{\mathrm{co}\mathscr{D}}}$ is Out-surjective from $\mathscr{S}'$ onto $\mathscr{A}_{\mathrm{co}\mathscr{D}}$, it is possible, by deleting some edges in $\mathscr{S}'$ if $\pi_{\mathscr{A}_{\mathrm{co}\mathscr{D}}}$ is not Out-injective, and by suppressing if necessary their quality of being initial to certain states, to construct a sub-automaton $\mathscr{T}'$ of $\mathscr{S}'$ that is a *covering* of $\mathscr{A}_{\mathrm{co}\mathscr{D}}$. Such a $\mathscr{T}'$ is thus *unambiguous*

(Corollary 1.8), and equivalent to $\mathscr{A}_{\mathrm{co}\mathscr{D}}$ and thus to $\mathscr{A}$. Since $\mathscr{S}'$ is a co-covering of $\mathscr{A}$, $\mathscr{T}'$ is a co-immersion in $\mathscr{A}$. □

5.1.2. The construct on matrix representations

The above construct may be rephrased in terms of matrix representations. In itself, this does not bring in anything new. But the framework of representations proves to be better suited for the application we are aiming at: the uniformization of deterministic relations. In [17], we have presented the computation of the matrix representation of both the S-covering and a S-immersion of an automaton $\mathscr{A}$. Here, we give the computation of the matrix representation of the S-co-covering and of a S-co-immersion of $\mathscr{A}$ since it is this matrix representation that we shall use next. As we have already observed, the two computations are indeed the same and a sheer transposition of matrices yields one from the other.

Let (λ,μ,ν) be the representation of $\mathscr{A}=\langle Q,A,E,I,T\rangle$ and (η',κ',ξ') the representation of $\mathscr{A}_{\mathrm{co}\mathscr{D}}=\langle 2^Q,A,F',J',U'\rangle$, i.e.

$$\forall P,S\in Q,\quad \forall a\in A \quad a\kappa'_{P,S}=1 \Leftrightarrow P=\{p\mid \exists q\in S \quad a\mu_{p,q}=1\},$$
$$\eta'_S=1 \Leftrightarrow \exists q\in S\ \lambda_q=1 \quad\text{and}\quad \xi'_P=1 \Leftrightarrow P=\{p\mid \nu_p=1\}.$$

By definition, $a\kappa'$ is *column-monomial*, i.e. every column has at most one non zero entry (this is clearly equivalent to the fact that $\mathscr{A}_{\mathrm{co}\mathscr{D}}$ is co-deterministic).

By Proposition 1.2 the representation of $\mathscr{A}_{\mathrm{co}\mathscr{D}}\times\mathscr{A}$ is $(\eta',\kappa',\xi')\otimes(\lambda,\mu,\nu)$. Any matrix $(f)\kappa'\otimes\mu$ is a $2^Q\times 2^Q$ block matrix made of blocks of size $Q\times Q$.

In order to describe the representation of the S-co-covering, the co-accessible part of $\mathscr{A}_{\mathrm{co}\mathscr{D}}\times\mathscr{A}$, we need another notation. Let α be any $(Q\times R)$-matrix and let S be any subset of R. We denote by $\alpha^{|S|}$ the matrix whose columns are those of α if their index is in S and 0 otherwise, [13] i.e.

$$(\alpha^{|S|})_{p,q}=\begin{cases}\alpha_{p,q} & \text{if } q\in S,\\ 0 & \text{otherwise.}\end{cases}$$

The dimension of the representation (ζ',σ',ω') of the S-co-covering is $2^Q\times Q$, the same as the one of $(\eta',\kappa',\xi')\otimes(\lambda,\mu,\nu)$. For every a in A, the matrix $a\sigma'$ is a $2^Q\times 2^Q$-matrix of blocks obtained by replacing *the* non zero entry in *the column* S of $a\kappa'$ by the $Q\times Q$-matrix $a\mu^{|S|}$, i.e.

$$a\sigma'_{(P,Q),(S,Q)}=\begin{cases}a\mu^{|S|} & \text{if } a\kappa'_{P,S}=1,\\ 0 & \text{otherwise.}\end{cases}$$

[13] Here is the only subtle difference with (the notation of) [17]. For a $Q\times R$-matrix α, we denoted there by $\alpha^{[P]}$ the matrix the lines of which are equal to those of α if their index is in P and to 0 otherwise. What we need here, by transposition, is to retain *columns* of α instead of *lines*; but the notation $\alpha^{[S]}$ would have been ambiguous when confronted to $\alpha^{[P]}$. If we were to use also $\alpha^{[P]}$ in the present paper – which we purposely avoided – or if we had to rewrite [17], we would denote it by $\alpha^{\underline{\overline{P}}}$.

Accordingly,

$$\forall S \subset Q \quad \zeta'_{(S,Q)} = \lambda^{|S|} \quad \text{and} \quad \omega'_{(P,Q)} = \begin{cases} \nu & \text{if } \xi'_P = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.1 (*Continued*). The matrix representation of the automaton $\mathscr{A}_1$ is

$$\lambda = \begin{pmatrix} 1\,0\,0 \end{pmatrix}, \qquad a\mu = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b\mu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \nu = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The matrix representation of $\mathscr{A}_{1\,\mathrm{co}\mathscr{D}}$ is

$$\eta' = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}, \qquad a\kappa' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad b\kappa' = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\xi' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and the representation $(\zeta', \sigma', \omega')$ of the S-co-covering $\mathscr{S}'$ of $\mathscr{A}_1$ is then

$$\zeta' = \begin{pmatrix} 1\,0\,0 & 1\,0\,0 & 0\,0\,0 & 0\,0\,0 \end{pmatrix}, \qquad \omega' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$a\sigma' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} & \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} & \begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \end{pmatrix}$$

and

$$
b\sigma' = \begin{pmatrix}
\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{smallmatrix} & \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{smallmatrix} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{smallmatrix} & \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{smallmatrix} \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

where the “big zeroes” represent (3×3)-blocks of zeroes.

The reader will easily check that this is the representation of the automaton shown in Fig. 21, with the difference that, in order to have all blocks of the same size (3×3), every dashed box in the figure is supposed to contain 3 states, the missing ones being the initial or terminal state of no edge whatsoever.

The definition of an S-co-immersion is then straightforward from $(\zeta', \sigma', \omega')$. For every letter a in A^*, every non-zero block $a\sigma'_{(P,Q),(S,Q)}$ is replaced by a *row monomial* block which has the same non zero rows as the original block. In other words, every non monomial row of any $(Q \times Q)$-block of $a\sigma'$ is made row monomial, *but not zero*, by the deletion of arbitrary entries. The same operation is performed on the $(1 \times Q)$-block vectors of ζ'.

Example 1.1 (*Continued*). There are two S-co-immersions $\mathscr{T}'_1$ and $\mathscr{T}'_2$ in $\mathscr{A}_1$ with representations $(\chi'_1, \tau'_1, \omega')$ and $(\chi'_2, \tau'_2, \omega')$, respectively. Obviously, $b\tau'_1 = b\tau'_2 = b\sigma'$ and $\chi'_1 = \chi'_2 = \zeta'$. The construction has a real impact on $a\tau'_1$ and $a\tau'_2$ only:

$$
a\tau'_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\begin{smallmatrix} 1 & \boxed{0} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} & \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} & \begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}
\end{pmatrix}
$$

and

$$
a\tau'_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\begin{smallmatrix} \boxed{0} & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} & \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} & \begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}
\end{pmatrix},
$$

where the "deleted" entries have been marked by a box for easier location. They correspond to the transitions that have been marked up in Fig. 21.

The example shows clearly that it is *the blocks* of $a\sigma'$ (or $b\sigma'$) that are transformed into row monomial matrices, and not the whole matrix $a\sigma'$ (or $b\sigma'$).

It is probably under this matrix representation that the "idea" that makes the S-co-covering so powerful appears more clearly. For instance, the product $\mathscr{A}_{\mathrm{co}\mathscr{G}} \times \mathscr{A}$, is a co-covering of $\mathscr{A}$, as is $\mathscr{S}'$, but it does not allow the construction of S-co-immersions by (arbitrary) deletions of entries with a simple criterion (as "transforming any block into a column-monomial one").

We now extend the construct from automata over A^* to automata over $A^* \times B^*$.

5.2. S-uniformization of rational relations

The above construction of S-co-immersions yields a proof of Rational Uniformization Theorem.

Proof of Theorem 5.1. Let $\theta : A^* \to B^*$ be a rational relation and let (λ, μ, ν) be a matrix representation – of dimension Q – of θ. Let $\mathscr{C}$ be the *real-time* 2-automaton defined by (λ, μ, ν) and $\mathscr{A}$ its underlying input automaton. Let (η', κ', ξ') be the (Boolean) representation – of dimension 2^Q – of $\mathscr{A}_{\mathrm{co}\mathscr{G}}$. It is the virtue of the notion of co-covering (and of the notation) that the definition of the matrix representation $(\zeta', \sigma', \omega')$ of the S-co-covering $\mathscr{S}'$ of $\mathscr{C}$ is identical as the one in the previous section. The only difference is that the entries of the matrices (and vectors) are in $\mathfrak{P}(B^*)$ instead of $\mathbb{B}$:

$$a\sigma'_{(P,Q),(S,Q)} = \begin{cases} a\mu^{|S|} & \text{if } a\kappa'_{P,S} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Accordingly,

$$\forall S \subset Q \quad \zeta'_{(S,Q)} = \lambda^{|S|} \quad \text{and} \quad \omega'_{(P,Q)} = \begin{cases} \nu & \text{if } \xi'_P = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, since $\mathscr{S}'$ is equivalent to $\mathscr{C}$, we have $\zeta' \cdot f\sigma' \cdot \omega' = \lambda \cdot f\mu \cdot \nu$ for every f in A^*.

The making of the representation (χ', τ', ψ') of a S-co-uniformization goes also as above. For every letter a in A, every non-zero block $a\sigma'_{(P,Q),(S,Q)}$ is replaced by a *row monomial* block which has the same non-zero rows as the original one. More precisely, for each non-zero row p of the block (P,S) of $a\sigma'$: $a\sigma'_{(P,p),(S,Q)}$, *we choose a single* s in Q (such that $a\sigma'_{(P,p),(S,s)}$ is not empty) and *we then choose a single word* $g_{p,s}$ in $a\sigma'_{(P,p),(S,s)}$ (i.e. in $(a\mu)_{p,s}$) and these choices define τ':

$$\forall P, S \subset Q, \quad \forall p \in P \quad a\tau'_{(P,p),(S,q)} = \begin{cases} g_{p,s} \in a\mu_{p,s} & \text{if } q = s, \\ 0 & \text{otherwise.} \end{cases}$$

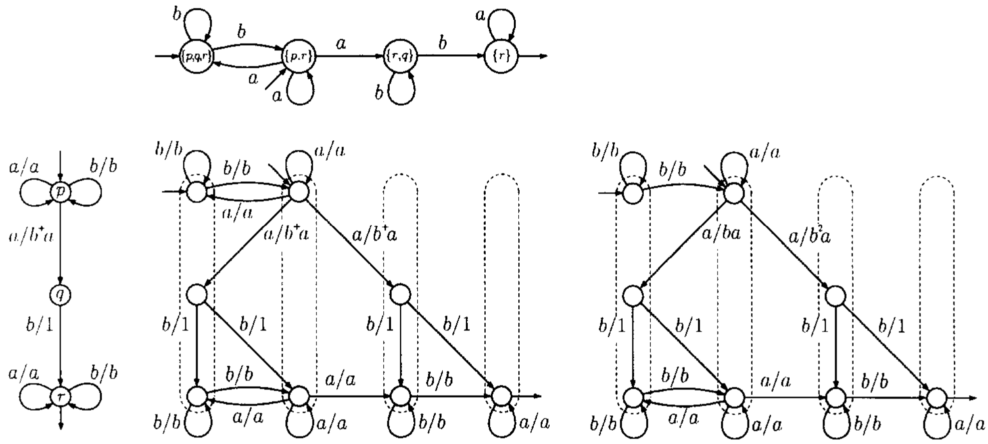


Fig. 22. The 2-automaton $\mathscr{C}_1$, its S-co-covering, and a S-uniformization of θ_1, built from the S-co-covering.

The same operation is performed on the $(1 \times Q)$-block vectors of ζ' and on the $(Q \times 1)$-block vectors of ω' in order to define χ' and ψ', respectively.

Let α be the relation realized by (χ', τ', ψ'). Remark that the underlying input automaton of (χ', τ', ψ') is a S-co-immersion in $\mathscr{A}$, the underlying input automaton of $\mathscr{C}$ and this implies that $\mathrm{Dom}\,\alpha$ is equal to the behaviour of $\mathscr{A}$, i.e. to $\mathrm{Dom}\,\theta$. This underlying input automaton is thus unambiguous, and as every entry of χ', ψ', and of every $a\tau'$, is a single word (in B^*), $f\alpha = \chi' \cdot f\tau' \cdot \psi'$ is a single word as well: α is a function. Finally, as χ', ψ', and every $a\tau'$, are obtained from ζ', ω', and every $a\sigma'$ by taking "subsets" one has

$$f\alpha = \chi' \cdot f\tau' \cdot \psi' \in \zeta' \cdot f\sigma' \cdot \omega' = \lambda \cdot f\mu \cdot \nu = f\theta. \qquad \square$$

Example 1.1 (*Continued*). Let θ_1 be the relation from $\{a,b\}^*$ into $\{a,b\}^*$ that replaces in any word one of its factor ab by a word in b^+a . The Fig. 22 shows an automaton $\mathscr{C}_1$ that realises θ_1, the underlying input automaton of which is $\mathscr{A}_1$ (on the left, vertically), the S-co-covering of $\mathscr{C}_1$ and a S-co-immersion computed as in the above proof.

The matrix representation of $\mathscr{C}_1$ is

$$\lambda_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \qquad a\mu_1 = \begin{pmatrix} a & b^+a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}, \qquad b\mu_1 = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & b \end{pmatrix},$$

$$\nu_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

And the representation $(\zeta', \sigma', \omega')$ of the S-co-covering of $\mathscr{C}_1$ is then:

$$\zeta' = (\,1\,0\,0 \quad 1\,0\,0 \quad 0\,0\,0 \quad 0\,0\,0\,), \qquad \omega' = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix},$$

$$a\sigma' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \begin{smallmatrix} a & b^+a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{smallmatrix} & \begin{smallmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{smallmatrix} & \begin{smallmatrix} 0 & b^+a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{smallmatrix} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{smallmatrix} \end{pmatrix}$$

and

$$b\sigma' = \begin{pmatrix} \begin{smallmatrix} b & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & b \end{smallmatrix} & \begin{smallmatrix} b & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & b \end{smallmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & b \end{smallmatrix} & \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & b \end{smallmatrix} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The representation (χ', τ', ψ') of the S-uniformization of θ_1 shown in Fig. 22 is given by

$$a\tau' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \begin{smallmatrix} \boxed{0} & \boxed{ba} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{smallmatrix} & \begin{smallmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{smallmatrix} & \begin{smallmatrix} 0 & \boxed{b^2a} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{smallmatrix} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{smallmatrix} \end{pmatrix},$$

$b\tau' = b\sigma'$, $\chi' = \zeta'$ and $\psi' = \omega'$.

5.3. Lexicographic S-uniformization

We reach the point we were aiming at. The arbitrary choices that took place in the preceeding proof will be made intrinsic in the case of deterministic relations. For that purpose, we suppose that a (total) ordering $\preccurlyeq$ is chosen on the alphabet B and we complete it as an ordering on $B_\$$ with the convention that \$ is *smaller than every letter* of B. This ordering extends to the *lexicographic ordering* on $B_\$^*$.

The lexicographic ordering is "almost" compatible with the multiplication (in $B_\$^*$). Indeed, if $u = fxh$ and $u' = fyk$, with f, h, k in $B_\$^*$, x, y in B and $x \prec y$, then $uv \prec u'v'$ for any v, v' in $B_\$^*$. The reverse statement reads then:

Property 5.5. $\forall u, u', v, v' \in B_\$^* \quad (u'v' \prec uv$ *and* $u \prec u') \Rightarrow u$ *is a prefix of* u'.

As we work with matrix representations of rational relations, it is convenient to define the *lexicographic selection* of a matrix.

Definition 5.1. Let M be a $(P \times Q)$-matrix, with entries in $\mathfrak{P}(B^*)$. For every p in P, let w_p be the smallest element in the lexicographic ordering (if it exists) of $M_{p,\bullet}$. The *lexicographic selection* of M, denoted by M_{lex}, is the matrix defined by

$$\forall p \in P, \qquad \forall q \in Q \quad (M_{\text{lex}})_{p,q} = \begin{cases} w_p & \text{if } w_p \in M_{p,q}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that by definition, every entry of M_{lex} is *monomial, i.e.* either a word in B^* or 0, and that *if M is prefix, then M_{lex} is row monomial.*

In the sequel, M will be a $P \times Q$-matrix and N a $Q \times R$-matrix, both with entries in $\mathfrak{P}(B^*)$. If X and Y are matrices with entries in $\mathfrak{P}(B^*)$, we say that X is a submatrix of Y if for every entry (p,r), $X_{p,r} \subset Y_{p,r}$.

Lemma 5.3. *If M is* prefix, *then $M_{\text{lex}}N_{\text{lex}}$ is a submatrix of $(MN)_{\text{lex}}$.*

Proof. Let p in P, r in R and $w = (M_{\text{lex}}N_{\text{lex}})_{p,r} \neq 0$; then there exists q in Q such that

$$u = (M_{\text{lex}})_{p,q}, \quad v = (N_{\text{lex}})_{q,r} \quad \text{and} \quad w = uv.$$

By definition, u is the smallest element of $M_{p,\bullet}$ and v is the smallest element of $N_{q,\bullet}$. Assume that there exists an element w' of $(MN)_{p,\bullet}$, $w' \in (MN)_{p,r'}$, smaller than w. Then there exists q' in Q such that

$$u' = (M_{\text{lex}})_{p,q'}, \qquad v' = (N_{\text{lex}})_{q',r'} \quad \text{and} \quad w' = u'v'.$$

By Property 5.5, $w' \prec w$ implies that u is a prefix of u'. As $\{M_{p,s}\}_{s \in Q}$ is a prefix family, $u = u'$ and $q = q'$. Now, $w' \prec w$ and $u = u'$ imply $v' \prec v$, and since $q = q'$, $v = v'$. Hence $[(MN)_{\text{lex}}]_{p,r} = w$. $\square$

Proof of Theorem 5.2. Let $\theta : A^* \to B^*$ be a *deterministic* rational relation and let (λ, μ, ν) be a *prefix* representation *with endmarker* – of dimension Q – of θ:

$$v \in u\theta \iff v\$ \in \lambda \cdot u\mu \cdot \nu.$$

Since \$ is the smallest letter of $B_\$$,

$$v = u\theta_{\mathrm{lex}} \Leftrightarrow v\$ = (\lambda \cdot u\mu \cdot v)_{\mathrm{lex}}.$$

holds. Let $\mathscr{C}$ be the *real-time* 2-automaton defined by (λ, μ, v) and $\mathscr{A}$ its underlying input automaton. Let (η', κ', ξ') be the (Boolean) representation – of dimension 2^Q – of $\mathscr{A}_{\mathrm{co}\mathscr{S}}$.

As above, let $(\zeta', \sigma', \omega')$ be the S-co-covering of (λ, μ, v) and let (χ'', τ'', ψ'') be the representation of the "lexicographic" S-co-uniformization defined as follows:

$$\begin{aligned}
&\forall S \subset Q \quad \chi''_{(S,Q)} = [\zeta'_{(S,Q)}]_{\mathrm{lex}} = [\lambda^{|S|}]_{\mathrm{lex}},\\
&\forall a \in A,\ \forall P, S \subset Q \quad a\tau''_{(P,Q),(S,Q)} = [a\sigma'_{(P,Q),(S,Q)}]_{\mathrm{lex}} = [a\mu^{|S|}]_{\mathrm{lex}},\\
&\forall P \subset Q \quad \psi''_{(S,Q)} = [\omega'_{(P,Q)}]_{\mathrm{lex}}.
\end{aligned}$$

In other words, the choices that have to be performed for the construction of a S-uniformization are *determined* – they are not arbitrary any more – by the lexicographic selection *in every block* of the S-co-covering.

In order to conclude, we have to go slightly more into details than for the proof of Theorem 5.1

Let $f = a_1 a_2 \ldots a_n$ be a word of A^* and let

$$P_0 \xrightarrow{a_1} P_1 \xrightarrow{a_2} P_2 \cdots P_{n-1} \xrightarrow{a_n} P_n$$

be the *unique* successful path of label f in $\mathscr{A}_{\mathrm{co}\mathscr{S}}$. We have

$$\begin{aligned}
\lambda \cdot f\mu \cdot v &= \zeta' \cdot f\sigma' \cdot \omega'\\
&= \zeta'_{(P_0,Q)} \cdot a_1\sigma'_{(P_0,Q),(P_1,Q)} \cdot a_2\sigma'_{(P_1,Q),(P_2,Q)} \cdot \cdots \cdot a_n\sigma'_{(P_{n-1},Q),(P_n,Q)} \cdot \omega'_{(P_n,Q)}.
\end{aligned}$$

Since every matrix on the right-hand side of the equation is *prefix* (as submatrix of a prefix matrix), we have, by Lemma 5.3,

$$\begin{aligned}
[\lambda \cdot f\mu \cdot v]_{\mathrm{lex}} &\ni [\zeta'_{(P_0,Q)}]_{\mathrm{lex}} \cdot [a_1\sigma'_{(P_0,Q),(P_1,Q)}]_{\mathrm{lex}} \cdot \cdots \cdot [a_n\sigma'_{(P_{n-1},Q),(P_n,Q)}]_{\mathrm{lex}} \cdot [\omega'_{(P_n,Q)}]_{\mathrm{lex}}\\
&= \chi'' \cdot f\tau'' \cdot \psi''.
\end{aligned}$$

Since we know, by the proof of Theorem 5.1, that $\chi'' \cdot f\tau'' \cdot \psi''$ is different from 0, it follows then that

$$[\lambda \cdot f\mu \cdot v]_{\mathrm{lex}} = \chi'' \cdot f\tau'' \cdot \psi''.$$

Therefore, (χ'', τ'', ψ'') is a representation (with endmarker) of θ_{lex}; it is unambiguous as it is a S-co-immersion. $\square$

Once again, this proof shows the power of the S-co-covering: as (λ, μ, v) is a prefix representation, one could try to take directly its lexicographic selection. This yields of

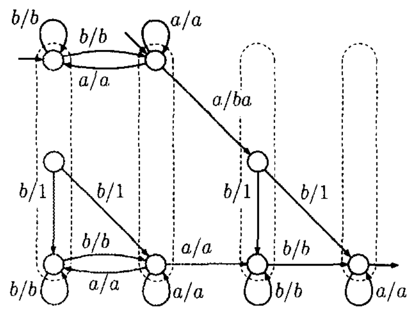


Fig. 23. The lexicographic S-uniformization of θ_1.

course a function whose graph is contained in the graph of θ but whose domain is, in general, stricly contained in the domain of θ: it is not a uniformization of θ.

Example 1.1 (*Continued*). The representation (χ'', τ'', ψ'') of the lexicographic S-uniformization of θ_1, shown in Fig. 23, is given by

$$a\tau'' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \begin{smallmatrix} \boxed{a} & \boxed{0} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{smallmatrix} & \begin{smallmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{smallmatrix} & \begin{smallmatrix} 0 & \boxed{ba} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{smallmatrix} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \end{pmatrix},$$

$b\tau'' = b\sigma'$, $\chi'' = \zeta'$ and $\psi'' = \omega'$.

The lexicographic selection of $(\lambda_1, \mu_1, \nu_1)$ is

$$\lambda'_1 = (1 \quad 0 \quad 0), \qquad a\mu'_1 = \begin{pmatrix} a & \boxed{0} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}, \qquad b\mu'_1 = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & b \end{pmatrix}, \qquad \nu'_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It realizes a function with empty domain.

Remark 5.1. In the preceding proof, we have used an "external" and powerful argument to assert that, roughly speaking, the lexicographical selection of a product (of prefix matrices) is the product of the lexicographical selection of the matrices. One can also give a sufficient condition on the matrices to achieve the same property.

Definition 5.2. We say that the matrix N *continues* the matrix M (or, is a *continuation* of M) if

$$\forall p \in P,\ \forall q \in Q \quad M_{p,q} \neq 0 \Rightarrow N_{q,\bullet} \neq 0.$$

Proposition 5.4. *If M is* prefix, *and if N continues M, then* $M_{\mathrm{lex}} N_{\mathrm{lex}} = (MN)_{\mathrm{lex}}$.

Proof. Let p in P, r in R and $w = [(MN)_{\text{lex}}]_{p,r} \neq 0$; then there exists q in Q such that

$$u = M_{p,q}, \quad v = N_{q,r} \quad \text{and} \quad w = uv.$$

By definition, w is the smallest element of $(MN)_{p,\bullet}$. Assume that there exists an element u' in $M_{p,\bullet}$, $u' \in M_{p,q'}$, smaller than u. Since N continues M, there exists r' in R such that $N_{q',r'}$ contains at least one element v' and $u'v'$ is in $(MN)_{p,\bullet}$. By Property 5.5, $w \prec w'$ implies that u' is a prefix of u. As $\{M_{p,s}\}_{s \in Q}$ is a prefix family, $u = u'$ and $q = q'$. Now, $w \prec w'$ and $u = u'$ imply $v \prec v'$. Thus, v is the smallest element in $N_{q,\bullet}$ and $v \in (N_{\text{lex}})_{q,r}$. Therefore $w \in (M_{\text{lex}} N_{\text{lex}})_{p,r}$.

The reverse inclusion is given by Lemma 5.3. □

Proposition 2.7 implies the following specialization of Theorem 5.2:

Corollary 5.5 (Sakarovitch [15]). *The lexicographic uniformization of the mapping equivalence of a morphism between free monoids is a rational function.*

5.4. Uniformization of synchronized relations

Since synchronized relations are deterministic, the lexicographic selection of any synchronized relation is a rational function. In this section, we prove that the lexicographic selection of any synchronized relation is in fact a synchronized function, and that the same is true of other selections than the lexicographic one. The proof relies on the closure of the set of synchronized relations by complement and by composition. To prove this last assertion, it is convenient to characterize the synchronized relations by their real-time representations.

Definition 5.3. A representation (λ, μ, v) from A^* into B^* of dimension Q is a *synchronized* representation if there exists a partition $Q_1 \cup Q_2$ of Q such that

$$\lambda \in \mathsf{Block}[\mathbb{B}\ 0] \quad \text{and} \quad a\mu \in \mathsf{Block}\begin{bmatrix} B & \mathbb{B} \\ 0 & \mathbb{B} \end{bmatrix} \quad \text{for every } a \text{ in } A.$$

Proposition 5.6. *A rational relation is synchronized if and only if it has a synchronized representation.*

Proof. Assume first that θ is realized by a letter-to-letter 2-automaton $\mathscr{A} = \langle Q, A^* \times B^*, E, I, \omega \rangle$ with terminal function ω taking its value in $\mathrm{Diff}_{\mathrm{Rat}}$. Without loss of generality, we may assume that there exists a unique t in Q such that $t\omega$ is different from $\emptyset$. Let $S \in \mathrm{Rat}\, A^*$ and $T \in \mathrm{Rat}\, B^*$ such that $t\omega = (S \times \{1\}) \cup (\{1\} \times T)$ and S does not contain 1_{A^*}. Let $\mathscr{M} = \langle P, A, G, j, F \rangle$ be a deterministic automaton recognizing S. Let

$$R_1 = Q \quad \text{and} \quad R_2 = P.$$

Let (λ,μ,ν) be the representation of dimension $R=R_1\cup R_2$ defined by

$$\lambda_p=\begin{cases}1 & \text{if } p\in I,\\ 0 & \text{otherwise,}\end{cases}$$

$$a\mu_{p,q}=\begin{cases}\{b\in B \mid (p,a,b,q)\in E\} & \text{if } p\in R_1 \text{ and } q\in R_1,\\ 1 & \text{if } p=t,\ q\in R_2 \text{ and } (j,a,q)\in G,\\ 1 & \text{if } p\in R_2,\ q\in R_2 \text{ and } (p,a,q)\in G,\\ 0 & \text{otherwise.}\end{cases}$$

$$\nu_q=\begin{cases}T_q & \text{if } q=t,\\ 1 & \text{if } q\in F,\\ 0 & \text{otherwise.}\end{cases}$$

Then (λ,μ,ν) is a synchronized representation of θ.

Conversely, assume that θ has a synchronized representation (λ,μ,ν) of dimension $R=R_1\cup R_2$. For every q in R_1 and r in R_2, let

$$S_{q,r}=\{u\in A^* \mid u\mu_{q,r}=1\}.$$

Let

$$Q=R_1,$$
$$E=\{(p,a,b,q)\in Q\times A\times B\times Q \mid b\in a\mu_{p,q}\},$$
$$I=\{q\in Q \mid \lambda_q=1\},$$
$$q\omega=\bigcup_{r\in R}(S_{q,r}\times \nu_r) \quad \text{for every } q\in Q.$$

Then $\mathscr{A}=\langle Q,A^*\times B^*,E,I,\omega\rangle$ is a letter-to-letter 2-automaton with terminal function ω taking its value in $\mathrm{Rec}(A^*\times B^*)$ and $\mathscr{A}$ realizes θ. □

Proposition 5.6 has been set up here in view of the following.

Proposition 5.7. *The composition of two synchronized relations is a synchronized relation.*

Proof. Let θ be a synchronized relation from A^* into B^* and let θ' be a synchronized relation from B^* into C^*. Let (λ,μ,ν) be a synchronized representation of θ of dimension $P=P_1\cup P_2$ and let (λ',μ',ν') be a synchronized representation of θ' of dimension $P'=P'_1\cup P'_2$. According to Proposition 1.2, $\theta\theta'$ is realized by the representation (α,β,γ) of dimension $Q=P\times P'$ defined in the following way:

$$\alpha_{(p,P')}=\lambda'.(\lambda_p\mu'),$$
$$a\beta_{(p,P'),(q,P')}=((a\mu)_{p,q})\mu',$$
$$\gamma_{(q,P')}=(\nu_q\mu')\cdot\nu'$$

for all p,q in P and for all a in A. Then

$$\alpha \in \mathsf{Block}\ \begin{array}{cccc} P_1\times P_1' & P_1\times P_2' & P_2\times P_1' & P_2\times P_2' \\ (\mathbb{B} & 0 & 0 & 0) \end{array}$$

and

$$a\beta \in \mathsf{Block}\ \begin{array}{c} \begin{array}{cccc} P_1\times P_1' & P_1\times P_2' & P_2\times P_1' & P_2\times P_2' \end{array} \\ \begin{pmatrix} C & \mathbb{B} & \mathbb{B} & \mathbb{B} \\ 0 & \mathbb{B} & \mathbb{B} & \mathbb{B} \\ 0 & 0 & \mathbb{B} & \mathbb{B} \\ 0 & 0 & \mathbb{B} & \mathbb{B} \end{pmatrix} \end{array}.$$

Let $Q_1 = P_1 \times P_1'$ and $Q_2 = Q \backslash Q_1$ then (α,β,γ) is a synchronized representation of dimension $Q_1 \cup Q_2$. $\square$

Let us now come to the uniformization of synchronized relations. It will be possible to give a statement that holds not only for the lexicographic ordering but for a whole familly of orderings.

In the sequel, let α be an ordering on B^*. Basically, α is a *relation* from B^* into itself: (u,v) is in α if, and only if, u is smaller than, or equal to, v for α. It is thus legitimate to say that α is "*synchronized*" if, as a relation, it is a synchronized (rational) relation. Note that, since ι, the identity relation, is a synchronized relation, it is equivalent to say that α or $\alpha\backslash\iota$, the *strict* ordering associated to α, is synchronized. In Section 2.4, we have seen that the lexicographic and military orderings are synchronized.

Let now θ be any relation from A^* into B^*. As we have done for the lexicographic ordering at the beginning of this section, we define the *α-selection of θ*, denoted by θ_α, to be the function that associates to every f in A^* the smallest element of $f\theta$ (in the ordering α) and *when this smallest element exists*. Obviously, $\mathrm{Dom}\,\theta_\alpha \subseteq \mathrm{Dom}\,\theta$; if $\mathrm{Dom}\,\theta_\alpha = \mathrm{Dom}\,\theta$, i.e. if $f\theta$ contains a smallest element for every f in $\mathrm{Dom}\,\theta$, then θ_α is a uniformization of θ.

Proposition 5.8. *Let α be a synchronized ordering on B^* and θ a synchronized rational relation from A^* into B^*. Then the α-selection θ_α is a synchronized rational function.*

Proof. For simplicity, let $\alpha' = \alpha\backslash\iota$. For every f in A^*, the set of words in B^* that are larger than one element of $f\theta$ is

$$f\theta \circ \alpha' = \{v \mid \exists u \in f\ \theta (u,v) \in \alpha'\}.$$

Then the smallest element of $f\theta$ is the unique element of $f\theta\backslash(f\theta \circ \alpha')$ if this set is non-empty. It follows that

$$\theta_\alpha = \theta\backslash(\theta \circ \alpha')$$

is a synchronized rational relations as these relations are closed under composition and difference. □

The specialization of Proposition 5.8 to lexicographic and military orderings yields:

Corollary 5.9. *The lexicographic selection and the military uniformization of a synchronized relation are synchronized functions.*

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