JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 196, 314–331 (1995)

Local Uniqueness and Convergence of Iterative Methods for Nonsmooth Variational Inequalities*

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Submitted by Philip D. Loewen

Received July 6, 1993

In this paper, we study nonsmooth variational inequalities. Using nonsmooth analysis, we characterize various monotonicity properties of locally Lipschitzian functions and give some sufficient conditions to guarantee the local uniqueness of solutions to variational inequalities. Two iterative algorithms are also presented to solve nonsmooth variational inequalities. © 1995 Academic Press, Inc.

1. INTRODUCTION

Throughout this paper, we assume that K is a nonempty, closed, and convex subset of R^n. Consider the *nonsmooth variational inequality* which is to find an $x \in K$ such that

$$(y - x)^T F(x) \geq 0, \qquad \forall y \in K, \tag{1}$$

where $F: R^n \to R^n$ is a given function. This problem is denoted by VI(K, F). If F is smooth (or continuously differentiable), then VI(K, F) is a smooth variational inequality, which has been extensively studied in the past two decades and has many applications in mathematical programming, complementarity, and economic equilibrium problems. See [4, 21, 14] for more details.

King and Rockafellar [6] recently obtained some results about Lipschitzian and differentiability properties of solutions of generalized equations (VI(K, F) is a special case of generalized equations) by using new differenti-

* This work was supported by the Australian Research Council.

0022-247X/95 $12.00

ation concepts for multivalued maps. Also, Mordukhovich [11] obtained some necessary and sufficient conditions for stability of solutions of nonsmooth generalized equations under very general conditions by using the *coderivative*, which he introduced.

In this paper, we use nonsmooth analysis to study the monotonicity properties of nonsmooth vector functions and the local uniqueness of the solutions of nonsmooth VI problems, and give some algorithms for solving nonsmooth VI problems. The remainder of the paper is organized as follows. First, we give some preliminary results which will be used in the subsequent sections. We also characterize the different monotonicity properties by positive definiteness or positive semidefiniteness of the generalized Jacobians of functions. In Section 3, we derive several sufficient conditions for the local uniqueness of the solutions of VI(K, F). Finally, we construct in Section 4 two locally convergent iterative algorithms for solving nonsmooth VI problems. The first converges superlinearly under the key assumption of semismoothness, while the second converges linearly under a certain splitting condition.

2. Preliminary Results

For any $x \in R^n$ and $\delta > 0$, let $\delta(x) = \{y \in R^n : \|y - x\| \leq \delta\}$. A function $F: R^n \to R^n$ is said to be *locally Lipschitzian* on a set D if for any $x \in D$, there exist $L_x > 0$ and $\delta > 0$ such that for all $y, z \in \delta(x) \cap D$,

$$\|F(y) - F(z)\| \leq L_x\|y - z\|.$$

For a locally Lipschitzian function F, Clarke [3] introduced the *generalized subdifferential* ∂F, which is defined as

$$\partial F(x) = \text{conv } \partial_B F(x),$$

where

$$\partial_B F(x) = \{\lim_{x_k \to x} \nabla F(x_k) : F \text{ is differentiable at } x_k\}.$$

The notation ∂_B was introduced in [16, 15]. The function F is said to be *semismooth* at x if F is locally Lipschitzian at x and

$$\lim_{V \in \partial F(x + td'), d' \to d, t \downarrow 0} \{Vd'\}$$

exists for any $d \in R^n$.

Semismoothness was introduced by Mifflin [9, 10] for functionals and was extended by Qi and Sun [19] to vector-valued functions. The class of semismooth functions is very broad. We refer the reader to [9, 15, 19] for more discussion. Semismoothness plays a key role in establishing the superlinear convergence of Newton methods for nonsmooth equations. We shall see that it also plays an important role in proving the superlinear convergence of an iterative algorithm which solves nonsmooth VI problems.

The following two propositions relate the fundamental properties of locally Lipschitzian functions and semismooth functions.

PROPOSITION 2.1 [3]. *If F is locally Lipschitzian on an open convex set D containing the point x, then*

(a) *$\partial F(x)$ is a nonempty convex compact subset of $R^{n\times n}$;*

(b) *∂F is closed at x, i.e., if $x_k \to x$, $V_k \in \partial F(x_k)$, and $V_k \to V$, then $V \in \partial F(x)$;*

(c) *(the mean-value theorem) for $y, z \in D$,*

$$F(y) - F(z) \in \operatorname{conv} \partial F([y, z])(y - z),$$

where $\operatorname{conv} \partial F([y, z]) = \operatorname{conv} \{V \in \partial F(w) : w \in [y, z]\}$.

PROPOSITION 2.2 [16]. *Suppose that F is semismooth at x. Then*

(a) *$F(x)$ is directionally differeniable at x and for any direction $d \in R^n$*

$$F'(x, d) = \lim_{V \in \partial F(x + td'), d' \to d, t \downarrow 0} \{Vd'\};$$

(b) *as $d \to 0$,*

$$F(x + d) - F(x) - F'(x, d) = o(\|d\|);$$

(c) *as $d \to 0$, for any $V \in \partial F(x + d)$,*

$$Vd - F'(x, d) = o(\|d\|).$$

It is well known [4] that various monotonicity properties of the function F are closely related to the existence and uniqueness of solutions to (1).

DEFINITION 2.1 [4]. The function $F: R^n \to R^n$ is said to be

(a) monotone on a set D if

$$(F(x) - F(y))^T(x - y) \geq 0 \qquad \forall x, y \in D;$$

(b) strictly monotone on D if

$$(F(x) - F(y))^T(x - y) > 0 \qquad \forall x, y \in D, x \neq y;$$

(c) strongly monotone on D if there exists an $\alpha > 0$ such that

$$(F(x) - F(y))^T(x - y) \geq \alpha \|x - y\|^2 \qquad \forall x, y \in D, x \neq y.$$

If F is smooth, then the above monotonicity properties of F are closely related to positive semidefiniteness or positive definiteness of the Jacobian matrix $\nabla F(x)$ [12]. The following proposition will establish a relationship between monotonicity properties of F and positive semidefiniteness (or positive definiteness) of its generalized Jacobians.

PROPOSITION 2.3. *Let D be an open convex set. Let $F: D \subseteq R^n \to R^n$ be locally Lipschitzian. Then*

(a) *F is monotone on D if and only if all generalized Jacobians $V \in \partial_B F(x)$ (or $V \in \partial F(x)$) are positive semidefinite for all $x \in D$;*

(b) *F is strongly monotone on D if and only if there exists $\alpha > 0$ such that for all $x \in D$, $d \in R^n$, and all $V \in \partial F(x)$ (or $V \in \partial_B F(x)$)*

$$d^T V d \geq \alpha d^T d; \tag{2}$$

(c) *if all $V \in \partial F(x)$ are positive definite for all $x \in D$, then F is strictly monotone and injective on D;*

(d) *if F is monotone on D and A is a positive definite matrix in $R^{n \times n}$, then $F(\cdot) + A \cdot$ is strongly monotone on D.*

Proof. We only prove (b), since we can prove the other statements similarly. For any $V \in \partial F(x)$ there exist $V_1, V_2, ..., V_r \in \partial_B F(x)$ such that $V = \sum_{i=1}^{r} \lambda_i V_i$, where $\lambda_i \geq 0$ and $\sum_{i=1}^{r} \lambda_i = 1$. So it suffices to prove that (b) is true for any $V \in \partial_B F(x)$. Suppose that F is strongly monotone on the set D, i.e., there exists $\alpha > 0$ such that for all $x, y \in D$

$$(F(x) - F(y))^T(x - y) \geq \alpha \|x - y\|^2.$$

By the definition of $\partial_B F(x)$, for any $V \in \partial_B F(x)$, there exist $x_k (\in D) \to x$ such that F is differentiable at x_k and for all $d \in R^n$

$$\begin{aligned} d^TVd &= d^T \lim_{x_k \to x} \nabla F(x_k)d \\ &= \lim_{x_k \to x} d^T \nabla F(x_k)d \\ &= \lim_{x_k \to x} d^T \lim_{t \downarrow 0} \frac{F(x_k + td) - F(x_k)}{t} \\ &= \lim_{x_k \to x} \lim_{t \downarrow 0} \frac{td^T(F(x_k + td) - F(x_k))}{t^2} \\ &\geq \lim_{x_k \to x} \lim_{t \downarrow 0} \frac{\alpha \|td\|^2}{t^2} \\ &\geq \alpha \|d\|^2, \end{aligned}$$

which is (2).

Conversely, by the mean-value theorem, for any $x, y \in D$ there exists $V \in \text{conv}\, \partial F[x, y]$ such that $F(x) - F(y) = V(x - y)$. Then (2) implies that $(x - y)^T(F(x) - F(y)) = (x - y)^TV(x - y) \geq \alpha \|x - y\|^2$. Hence, F is strongly monotone on D. Q.E.D.

The next result is a consequence of the closedness of the generalized Jacobians of locally Lipschitzian functions. It implies that the positive definiteness of the generalized Jacobians is an open property for locally Lipschitzian functions. This result can also be interpreted as follows: if the generalized Jacobian of F at x^* is positive definite, then F is locally strongly monotone at x^*. This will be used later.

LEMMA 2.1. *Suppose that $F: R^n \to R^n$ is locally Lipschitzian around x^* and all $V^* \in \partial_B F(x^*)$ are positive definite. Then there exist $\alpha > 0$, $\delta > 0$ such that for all $x \in \delta(x^*)$ and $V \in \partial F(x)$, V is positive definite and for all $d \in R^n$*

$$d^TVd \geq \alpha \|d\|^2. \tag{3}$$

Proof. By $\partial F(x^*) = \text{conv}\, \partial_B F(x^*)$, it is easy to establish that all $V^* \in \partial F(x^*)$ are positive definite if and only if all $V^* \in \partial_B F(x^*)$ are positive definite. Suppose that the conclusion is not true. Then there exist $d_k \in R^n$ and $x_k \to x^*$ such that $d_k^T V_k d_k < (1/k)\|d_k\|^2$, where $V_k \in \partial F(x_k)$. Without loss of generality, we may assume that $\|d_k\| = 1$, $d_k \to d$ $(\neq 0)$, $V_k \to$

$V^* \in \partial F(x^*)$. Hence, $d^T V^* d \leq 0$, which contradicts that all $V^* \in \partial F(x^*)$ are positive definite. The result follows. Q.E.D.

3. Uniqueness of Solutions

In this section, we derive sufficient conditions for the local uniqueness of solutions of nonsmooth VI problems. A solution x^* of VI(K, F) is said to be *locally unique* if there exists a neighbourhood $\delta(x^*)$ of x^* such that x^* is the only solution of VI(K, F) in $\delta(x^*)$. To generalize some local uniqueness results of [14], recall the *critical cone* [14] of VI(K, F) at a point x in K,

$$\mathscr{C}(x, K, F) = \{d \in R^n : d^T F(x) = 0,\ d \in \mathscr{T}(x, K)\},$$

where

$$\mathscr{T}(x, K) = \{d \in R^n : d = \lim_{k \to \infty} t_k^{-1}(x_k - x),\ t_k \geq 0,\ t_k \to 0,\ x_k (\in K) \to x\}$$

is the tangent cone of K at x.

A matrix $V \in R^{n \times n}$ is said to be *strictly copositive* on a cone Q if $d^T V d > 0$ for all $d \in Q \setminus \{0\}$. See [20, 14]. The dual of a given cone Q is $Q^* = \{d : d^T x \geq 0,\ \forall x \in Q\}$.

Proposition 3.1. *Let K be a nonempty polyhedron in R^n and let $F: R^n \to R^n$ be locally Lipschitzian. Suppose that x^* is a solution of* VI(K, F). *Consider the following three statements*:

(a) *the homogeneous system*

$$d \in \mathscr{C}(x^*, K, F), \qquad V^* d \in \mathscr{C}(x^*, K, F)^*, \qquad d^T V^* d = 0 \tag{4}$$

has the unique solution $d = 0$ for all $V^* \in \partial F(x^*)$;

(b) *for all $V^* \in \partial F(x^*)$, the implication below holds,*

$$\left.\begin{array}{ll} x^* + d \in K,\ d^T F(x^*) = 0, & d \neq 0 \\ (y - x^*)^T (F(x^*) + V^* d) \geq 0, & \forall y \in K \end{array}\right\} \Rightarrow d^T V^* d > 0;$$

(c) *x^* is a locally unique solution of* VI(K, F).

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. (a) $\Rightarrow$ (b). Suppose that (b) is not true, i.e., there exist $V^* \in \partial F(x^*)$ and $d^* \in R^n$ such that

$$\begin{aligned} &x^* + d^* \in K, (d^*)^T F(x^*) = 0, \qquad d^* \neq 0, \\ &(y - x^*)^T (F(x^*) + V^* d^*) \geq 0, \qquad \forall y \in K, \end{aligned} \tag{5}$$

but $(d^*)^T V^* d^* \leq 0$. We claim that in this case the nonzero vector $d = d^*$ satisfies (4). It is obvious that $d^* \in \mathscr{C}(x^*, K, F)$. By the polyhedral property of K, for all $z \in \mathscr{C}(x^*, K, F)$, there exist $t_z > 0$ and $y \in K$ such that $t_z z = y - x^*$. Hence, (5) and $d^* \in \mathscr{C}(x^*, K, F)$ imply that

$$z^T V^* d^* = (F(x^*) + V^* d^*)^T z = \frac{1}{t_z}(F(x^*) + V^* d^*)^T (y - x^*) \geq 0,$$

i.e., $V^* d^* \in \mathscr{C}(x^*, K, F)^*$. This, together with $(d^*)^T V^* d^* \leq 0$, gives $(d^*)^T V^* d^* = 0$, as required.

(b) $\Rightarrow$ (c). If x^* is not locally unique, then there exists a sequence $\{x_k\}$ ($x_k \neq x^*$) solving VI(K, F), which converges to x^*. We may assume that the unit vectors $(x_k - x^*)/\|x_k - x^*\|$ converge to a nonzero vector d^*.
Since K is polyhedral, for all sufficiently small $t > 0$

$$x^* + td^* \in K. \tag{6}$$

Since both x^* and x_k solve VI(K, F), we have $F(x^*) \in \mathscr{T}(x^*, K)^*$, $F(x_k) \in \mathscr{T}(x_k, K)^*$, and

$$(x_k - x^*)^T F(x^*) \geq 0, \qquad (x^* - x_k)^T F(x_k) \geq 0, \tag{7}$$

which imply that

$$(d^*)^T F(x^*) = 0. \tag{8}$$

The polyhedral property of K implies that $\mathscr{T}(x^*, K) \subseteq \mathscr{T}(x^k, K)$ for all sufficiently large k and hence $F(x^*)$, $F(x_k) \in \mathscr{T}(x^*, K)^*$. By the polyhedral property of $\mathscr{T}(x^*, K)^*$ again, for all sufficiently large k and small $t > 0$,

$$F(x^*) + t\frac{F(x_k) - F(x^*)}{\|x_k - x^*\|} \in \mathscr{T}(x^*, K)^*. \tag{9}$$

From the mean-value theorem and the Carathéodory theorem (Theorem 17.1 of [24]), there exist $t_k^i \in [0, 1]$, $\lambda_k^i \in [0, 1]$, $V_k^i \in \partial F(x^* + t_k^i(x_k - x^*))$, $\sum_{i=1}^{n+1} \lambda_k^i = 1$, for $i = 1, \ldots, n + 1$ such that

$$F(x_k) - F(x^*) \in \text{conv}\, \partial F[x^*, x_k](x_k - x^*),$$

$$F(x_k) - F(x^*) = \sum_{i=1}^{n+1} \lambda_k^i V_k^i (x_k - x^*).$$

By passing to a subsequence, we can assume that $\lambda_k^i \to \lambda^i$ and $V_k^i \to V^i \in \partial F(x^*)$ as $k \to \infty$ by the closedness of ∂F at x^*. We have $\lambda^i \in [0, 1]$ for $i = 1, ..., n + 1$ and $\sum_{i=1}^{n+1} \lambda^i = 1$. Then, together with the convexity of ∂F at x^*,

$$\lim_{k\to\infty} \frac{F(x_k) - F(x^*)}{\|x_k - x^*\|} = \sum_{i=1}^{n+1} \lambda^i V^i d^* = V^* d^*,$$

where $V^* = \sum_{i=1}^{n+1} \lambda^i V^i \in \partial F(x^*)$. Hence, (9) gives that for all sufficiently small $t > 0$,

$$F(x^*) + tV^*d^* \in \mathcal{T}(x^*, K)^*. \tag{10}$$

It follows from (7) that

$$0 \geq (x_k - x^*)^T(F(x_k) - F(x^*)),$$

together with the mean-value theorem, which implies that

$$(d^*)^T V^* d^* \leq 0. \tag{11}$$

Combining (6), (8), (10), (11), we can prove that $\hat{d} = \hat{t}d^*$, where $\hat{t} > 0$ is sufficiently small, satisfies the conditions of (b) but $(\hat{d})^T V^* \hat{d} \leq 0$, which is a contradiction. Q.E.D.

If the polyhedral assumption is removed from Proposition 3.1, then a stronger condition is needed for ensuring the local uniqueness of the solution. The following is such a result.

COROLLARY 3.1. *Let K be a nonempty closed convex set in R^n and let $F: R^n \to R^n$ be locally Lipschitzian. Suppose that x^* is a solution of* VI(K, F). *Then x^* is a locally unique solution of* VI(K, F), *if any one of the following conditions holds*:

(a) *for all $V^* \in \partial F(x^*)$, V^* is strictly copositive on the critical cone $\mathcal{C}(x^*, K, F)$, or*

(b) *F is locally strongly monotone at the solution point x^*.*

Proof. Clearly, (b) implies (a). We may prove the result under condition (a) in a manner analogous to the proof of Proposition 3.1. Q.E.D.

PROPOSITION 3.2. *Let K be a polyhedral cone in R^n and let $F: R^n \to R^n$ be locally Lipschitzian. If x^* is a locally unique solution of* VI(K, F_{V^*}) *for all $V^* \in \partial F(x^*)$, where $F_{V^*}(x) = F(x^*) + V^*(x - x^*)$, then x^* is a locally unique solution to* VI(K, F).

Proof. It is obvious that x^* is a solution to VI(K, F) by the conditions, i.e., for all $y \in K$,

$$(y - x^*)^T F(x^*) \geq 0, \qquad x^* \in K, \qquad F(x^*) \in K^*, \qquad (x^*)^T F(x^*) = 0. \tag{12}$$

Suppose that x^* is not a locally unique solution to VI(K, F), i.e., there exists a sequence $\{x_k\} \in K$ such that $x_k \neq x^*$, $x_k \to x^*$, and x_k solves VI(K, F):

$$(y - x_k)^T F(x_k) \geq 0, \qquad \text{for all } y \in K, x_k \in K, F(x_k) \in K^*, x_k^T F(x_k) = 0. \tag{13}$$

Let $s_k = (x_k - x^*)/\alpha_k$, $\alpha_k = \|x_k - x^*\|$. Without loss of generality, we assume that $s_k \to s$ with $\|s\| = 1$. We shall prove that for all sufficiently small $t > 0$, $z_t = x^* + ts$ are the solutions of VI(K, F_{V^*}) for some $V^* \in \partial F(x^*)$.

Let $\mathcal{F}_{x^*}(K)$ denote the feasible direction cone of K at x^*. Then $s_k \in \mathcal{F}_{x^*}(K)$. The closedness of $\mathcal{F}_{x^*}(K)$ implies $s \in \mathcal{F}_{x^*}(K)$, i.e., for all sufficiently small $t > 0$,

$$z_t = x^* + ts \in K. \tag{14}$$

By $F(x^*) \in K^*$ and $F(x_k) \in K^*$, $(F(x_k) - F(x^*))/\alpha_k \in \mathcal{F}_{F(x^*)}(K^*)$. Since K^* is a polyhedral cone, we have that the feasible direction cone $\mathcal{F}_{F(x^*)}(K^*)$ of K^* at $F(x^*)$ is closed. Hence, any accumulation point of $\{(F(x_k) - F(x^*))/\alpha_k\}$ is in $\mathcal{F}_{F(x^*)}(K^*)$. By the mean-value theorem, there exists $V_k \in \text{conv}\, \partial F[x^*, x_k]$ such that $F(x_k) - F(x^*) = \alpha_k V_k s_k$. Let $k \to \infty$. By passing to a subsequence, it follows from the closedness of ∂F at x^* that there exists $V^* \in \partial F(x^*)$ such that $V^* s \in \mathcal{F}_{F(x^*)}(K^*)$, i.e., for all sufficiently small $t > 0$,

$$F(x^*) + V^* ts = F(x^*) + V^*(x^* + ts - x^*) \in K^*. \tag{15}$$

By (12) and (13), we have for all x_k

$$(x_k - x^*)^T F(x^*) \geq 0, \tag{16}$$

$$(x^* - x_k)^T F(x_k) \geq 0, \tag{17}$$

which imply

$$s^T F(x^*) = 0 \tag{18}$$

by taking $k \to \infty$. From (12), (13), and the mean-value theorem,

$$\begin{aligned} 0 = x_k^T F(x_k) &= x_k^T (F(x^*) + \alpha_k V_k s_k) \\ &= (x_k - x^*)^T F(x^*) + x_k^T \alpha_k V_k s_k \\ &= \alpha_k s_k^T F(x^*) + \alpha_k x_k^T V_k s_k, \end{aligned}$$

which implies

$$(x^*)^T V^* s = 0. \tag{19}$$

Since $z_t \in K$, by (13), (12), (18), and (19), it follows

$$\begin{aligned} 0 \leq (z_t - x_k)^T F(x_k) &= z_t^T F(x_k) \\ &= (x^* + ts)^T (F(x^*) + \alpha_k V_k s_k) \\ &= (x^* + ts)^T \alpha_k V_k s_k, \end{aligned}$$

which implies

$$s^T V^* s \geq 0. \tag{20}$$

From (16) and (17), we have

$$\begin{aligned} 0 \geq s_k^T F(x_k) &= s_k^T (F(x^*) + \alpha_k V_k s_k) \\ &\geq s_k^T \alpha_k V_k s_k, \end{aligned}$$

which implies

$$s^T V^* s = 0 \tag{21}$$

by invoking (20). From (12), (18), (19), and (21), we obtain

$$z_t^T (F(x^*) + V^* ts) = (x^* + ts)^T (F(x^*) + V^*(z_t - x^*)) = 0. \tag{22}$$

Combining (14), (15), and (22), it follows that $z_t = x^* + ts$ is a solution to VI(K, F_{V^*}) for all sufficiently small $t > 0$ and fixed $V^* \in \partial F(x^*)$, which contradicts the assumptions of the proposition. Q.E.D.

Remark 3.1. The above proposition generalizes the result of Kyparisis [7] for linear complementarity problems when F is smooth.

In some cases, the set K is defined by inequalities and equalities of the form

$$K = \{x \in R^n : g(x) \leq 0, h(x) = 0\}, \tag{23}$$

where $g: R^n \to R^m$ and $h: R^n \to R^p$ are differentiable. Suppose that x^* is a solution of VI(K, F). Then under a certain constraint qualification (for example, Mangasarian and Fromovitz constraint qualification) for K at x^* (see [14]), there exist multiplier vectors $\pi \in R^m$, $\mu \in R^p$ such that

$$\begin{aligned} F(x^*) + \nabla g(x^*)^T\pi + \nabla h(x^*)^T\mu &= 0, \\ \pi^T g(x^*) &= 0, \\ h(x^*) &= 0, \\ \pi &\geq 0. \end{aligned} \tag{24}$$

The following uniqueness result generalizes Theorem 2.3 of [25], in which F is differentiable, to the case where F is only locally Lipschitzian. Its proof parallels that of Proposition 3.2, and is also similar to that of Theorem 2.3 of [25]. For the sake of completeness, we sketch the proof.

PROPOSITION 3.3. *Let $F: R^n \to R^n$ be locally Lipschitzian. Suppose that x^* solves* VI(K, F) *satisfying Mangasarian and Fromovitz constraint qualification, where K is defined by* (23). *If $d^TV^*d > 0$ holds for any $V^* \in \partial_B F(x^*)$, any $d \neq 0$, and any multiplier vector π such that*

$$\begin{aligned} &\nabla g_i(x^*)^T d \leq 0, && \text{all } i \text{ such that } g_i(x^*) = 0, \\ &\nabla g_i(x^*)^T d = 0, && \text{all } i \text{ such that } \pi_i > 0, \\ &\nabla h_j(x^*)^T d = 0, && j = 1, \ldots, p, \end{aligned} \tag{25}$$

then x^* is a locally unique solution to VI(K, F).

Proof. Suppose that x^* is not locally unique. Then there exists a sequence $\{x_k\}$ solving VI(K, F) and x_k ($\neq x^*$) $\to x^*$. From this sequence, we can construct a vector $s \neq 0$, which satisfies (25) but violates $s^T V^* s > 0$ for some $V^* \in \partial_B F(x^*)$. This is a contradiction. Q.E.D.

4. Convergence of Iterative Methods

In this section, we study two local iterative methods for solving nonsmooth VI problems and their convergence. Unlike the classical Newton methods, differentiability of F is not assumed. As in [16] for nonsmooth equations, in order to obtain a superlinear convergence result, we need the assumption of semismoothness of F at the solution point. When F is smooth, the following algorithm reduces to the classical Newton method for smooth VI problems [4, 5, 21].

Algorithm 1. Step 1. Choose $x_0 \in K$ sufficiently close to x^* and set $k = 0$.

Step 2. If x_k solves VI(K, F), then stop. Otherwise, go to Step 3.

Step 3. Choose $V_k \in \partial F(x_k)$ and solve the following subproblem, which is a linearized variational inequality,

$$(y - x)^T(F(x_k) + V_k(x - x_k)) \geq 0, \qquad \forall y \in K. \tag{26}$$

Suppose that x_{k+1} is a solution of (26). Let $k \leftarrow k + 1$ and go o Step 2.

Theorem 4.1. *Let K be a nonempty closed convex set in R^n and let F be locally Lipschitzian on K. Suppose that x^* is a solution to* VI(K, F), *that V^* is positive definite for all $V^* \in \partial F(x^*)$, and that F is semismooth at x^*. Then there exists $\delta > 0$ such that for any $x_0 \in \delta(x^*)$, Algorithm 1 is well defined and the sequence of $\{x_k\}$ generated by Algorithm 1 is convergent superlinearly to x^*.*

Proof. We prove the theorem by induction. From Lemma 2.1, it follows that there exists $\delta_1 > 0$ such that for any $x \in \delta_1(x^*)$, $V \in \partial F(x)$, (3) holds. Since F is semismooth at x^*, from Proposition 2.2, there exist $\delta_2 > 0$ and $0 < r < 1$ such that for any $x \in \delta_2(x^*)$,

$$\|F(x) - F(x^*) - F'(x^*, x - x^*)\| < \frac{r\alpha}{2}\|x - x^*\| \tag{27}$$

and

$$\|F'(x^*, x - x^*) - V(x - x^*)\| < \frac{r\alpha}{2}\|x - x^*\|, \tag{28}$$

where α satisfies (3).

Let $\delta = \min\{\delta_1, \delta_2\}$. Choose $x_0 \in \delta(x^*)$. It follows from the positive definiteness of V_0 that (26) has a unique solution x_1 for $x_k = x_0$. Since x^* is a solution of VI(K, F),

$$(y - x^*)^T F(x^*) \geq 0, \qquad \forall y \in K,$$
$$(y - x_1)^T (F(x_0) + V_0(x_1 - x_0)) \geq 0, \qquad \forall y \in K.$$

Hence,

$$(x_1 - x^*)^T F(x^*) \geq 0,$$
$$(x^* - x_1)^T (F(x_0) + V_0(x_1 - x_0)) \geq 0.$$

Adding these inequalities, we obtain

$$(x^* - x_1)^T (F(x_0) - F(x^*) + V_0(x_1 - x^* + x^* - x_0)) \geq 0.$$

By (27), (28)

$$\begin{aligned}(x^* - x_1)^T V_0 (x^* - x_1) &\leq (x^* - x_1)^T (F(x_0) - F(x^*) - F'(x^*, x_0 - x^*) \\ &\quad + F'(x^*, x_0 - x^*) - V_0(x_0 - x^*)) \\ &\leq r\alpha \|x^* - x_1\| \|x^* - x_0\|.\end{aligned}$$

Therefore, (3) implies that

$$\|x^* - x_1\| < r\|x^* - x_0\|,$$

i.e., $x_1 \in \delta(x^*)$. Generally, we can prove by induction that

$$\|x^* - x_{k+1}\| < r\|x^* - x_k\|, \tag{29}$$

which implies that $x_k \to x^*$ as $k \to \infty$.

Since F is semismooth at x^* and x_0 is sufficiently close to x^*, it follows from (b) and (c) of Proposition 2.2 that for all k,

$$(x^* - x_{k+1})^T V_k(x^* - x_{k+1}) \le (x^* - x_{k+1}{}^T(F(x_k) - F(x^*) - F'(x^*, x_k - x^*)$$
$$+ F'(x^*, x_k - x^*) - V_k(x_k - x^*))$$
$$= \|x^* - x_{k+1}\| \, o(\|x^* - x_k\|),$$

where $V_k \in \partial F(x_k)$. Hence, (3) gives

$$\|x^* - x_{k+1}\| = o(\|x^* - x_k\|)$$

which implies that $\{x_k\}$ converges superlinearly to the solution point x^*.
Q.E.D.

Remark 4.1. In Step 3 of Algorithm 1, the linearized subproblem has to be solved. It should be noted that most of the linear approximation methods for solving smooth VI problems have the same problem. We refer the reader to [4]. If K is a polyhedron, the linearized subproblem (26) is tractable. If K is defined by (23), then we may transfer VI(K, F) into another VI problem with a polyhedron constraint. Following Robinson [21], we set $u = (x, \pi, \mu)$,

$$H(u) = \begin{pmatrix} F(x) + \nabla g(x)^T \pi + \nabla h(x)^T \mu \\ -g(x) \\ -h(x) \end{pmatrix},$$

and $C = R^n \times R^m_+ \times R^p$. Suppose that $g_i(x)$ is convex for $i = 1, 2, ..., m$ and $h_j(x)$ is affine for $j = 1, 2, ..., p$. Then, under an appropriate constraint qualification at x^*, that (x^*, π^*, μ^*) solves VI(C, H) is equivalent to x^* solves VI(K, F). Thus, we may solve VI(K, F) by solving VI(C, H), where C is a polyhedron. If we further assume that g_i and h_j are twice continuously differentiable, then the strong monotonicity property of F on a set D implies the monotonicity property of H on the set $D \times R^m_+ \times R^p$. Unfortunately, H may not be strongly monotone on $D \times R^m_+ \times R^p$. See the following example. Let $F(x) = x$ and $g(x) = x$ for $x \in R^1$. Suppose there is no equality constraint. Clearly, F is strongly monotone on R^1 and g is convex and twice continuously differentiable on R^1. But $H = \binom{x+\pi}{-x}$ is not strongly monotone on $R^1 \times R^1_+$. Hence, we cannot directly apply Theorem 4.1 to VI(C, H). One possible way to remedy this is to reduce the assumption of the positive definiteness of the generalized Jacobians of F at the solution point in Theorem 4.1 to an assumption that they are positive semidefinite and nonsingular. This possibility remains for further investigations.

Remark 4.2. As pointed out before Algorithm 1, when F is smooth Algorithm 1 reduces to the classical Newton method for smooth VI problems. To obtain the local convergence of Algorithm 1 for smooth VI problems, Robinson [22] introduced a key assumption called strong regularity at the solution point, and Josephy [5] gave a proof of the local convergence of Algorithm 1 under the strong regularity condition. One referee suggests that this strong regularity could be generalized in a suitable way to nonsmooth VI problems. Bonnans [1] has given a variation of Algorithm 1 by using two concepts for nonsmooth VI problems and established the local convergence of his algorithm. The first concept is *semistability* introduced by Bonnans in the same paper. The second concept is called *point-based approximation* which is a modification of the concept introduced by Robinson [23]. We also note that the point-based approximation used by Bonnans [1] is different from the linearization given in (26) which is based on the generalized Jacobians.

Remark 4.3. Also as the referee suggested, an inexact strategy could be useful when it is difficult to find an exact solution of (26). Pang [13] employed this strategy for smooth VI problems. Martinez and Qi [8] used it for the solution of nonsmooth equations. To describe an inexact version of Algorithm 1, we follow Pang [13] to define a measure function $\theta: R^n \to R^n$ by

$$\theta(x) = x - \mathrm{Proj}_K(x - F(x)),$$

where $\mathrm{Proj}_K(\cdot)$ denotes the projection operator onto the set K. Then in the inexact version of Algorithm 1, x_{k+1} is generated by the approximate rule

$$\|\theta_k(x_{k+1})\| \leq \eta_k \|\theta(x_k)\|,$$

where $\eta_k \geq 0$ is some given scaler and

$$\theta_k(x) = x - \mathrm{Proj}_K(x - (F(x_k) + V_k(x - x_k))),$$

for some $V_k \in \partial F(x_k)$. Clearly, if $\eta_k = 0$ for all k, then x_{k+1} is an exact solution of (26) and the inexact version of Algorithm 1 reduces to Algorithm 1 itself. If $\eta_k > 0$, then we guess that one possible sufficient condition for the superlinear convergence of the inexact version of Algorithm 1 is that $\lim_{k\to\infty} \eta_k = 0$.

In Algorithm 1, if F is not semismooth at the solution point, Theorem 4.1 cannot guarantee convergence. Then more general methods are needed. One possible method is the *splitting* method which was introduced originally for solving systems of nonsmooth equations. See [2, 17, 18, 26] for details.

Given a locally Lipschitzian function $F: R^n \to R^n$, $F(x) = f(x) + g(x)$ is said to be a *decomposition* of $F(x)$, if f is smooth and g is locally Lipschitzian at x^*. We propose the splitting algorithm.

ALGORITHM 2. Step 1. Let $F(x) = f(x) + g(x)$. Choose $x_0 \in K$ sufficiently close to x^* and set $k = 0$.

Step 2. If x_k solves VI(K, F), then stop. Otherwise, go to Step 3.

Step 3. Solve the following linearized variational inequality

$$(y - x)^T(F(x_k) + \nabla f(x_k)(x - x_k)) \geq 0, \qquad \forall y \in K. \tag{30}$$

Suppose that x_{k+1} is a solution of (30). Let $k \leftarrow k + 1$ and go to Step 2.

THEOREM 4.2. *Let K be a nonempty closed convex set in R^n and let $F(x)$ be locally Lipschitzian on K. Suppose that x^* is a solution to* VI(K, F), *that $f(x) + g(x)$ is a decomposition of $F(x)$ with constants $L > 0$, $\alpha > 0$, and $\delta_0 > 0$ such that for all $x, y \in \delta_0(x^*)$, $\|g(x) - g(y)\| \leq L\|x - y\|$. If $\nabla f(x^*)$ is positive definite and $L/\alpha < 1$, where α satisfies* (3) *when letting $V = \nabla f(x)$, then there exists $\delta > 0$ such that for any $x_0 \in \delta(x^*)$, Algorithm* 2 *is well defined and the sequence $\{x_k\}$ generated by Algorithm* 2 *is convergent linearly to x^*.*

Proof. We mimic the proof of Theorem 4.1. First, there exist $0 < r < 1$ and $0 < \delta < \delta_0$ such that (30) has a unique solution x_1 when $x_k = x_0 \in \delta(x^*)$, and for any $x \in \delta(x^*)$ and any $d \in R^n$

$$d^T \nabla f(x) d \geq \alpha\|d\|, \tag{31}$$

$$\|f(x) - f(x^*) - \nabla f(x^*)(x - x^*)\| < (\alpha r - L)\|x - x^*\|. \tag{32}$$

Then, since x^* and x_1 solve VI(K, F) and (30), respectively, we get

$$\begin{aligned}
&(x^* - x_1)^T \nabla f(x_0)(x^* - x_1) \\
&\quad \leq (x^* - x_1)^T(F(x_0) - F(x^*) - \nabla f(x^*)(x_0 - x^*)) \\
&\quad \leq (x^* - x_1)^T(f(x_0) - f(x^*) - \nabla f(x^*)(x_0 - x^*) \\
&\qquad + g(x_0) - g(x^*)) \\
&\quad \leq r\alpha\|x^* - x_1\| \, \|x^* - x_0\|.
\end{aligned}$$

Therefore, (31) implies $\|x_1 - x^*\| < r\,\|x_0 - x^*\|$. By induction we have

$$\|x_{k+1} - x^*\| < r\,\|x_k - x^*\|,$$

which completes the proof. Q.E.D.

Acknowledgments

The authors are thankful to associate editor Professor Philip Loewen and a referee for their helpful suggestions and comments which greatly improved the presentation of the paper.

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