JOURNAL OF FUNCTIONAL ANALYSIS 33, 58–94 (1979)

Weak Interpolation in Banach Spaces

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Communicated by the Editors

Received September 6, 1977; revised April 17, 1978

1. INTRODUCTION

The uses of interpolation theory in various branches of analysis, especially Fourier analysis, are well known. For many operators a weak type interpolation theory is indispensible for accurately describing their mapping properties. Such for example is the case with the Hilbert transform, maximal operators, etc.

The early results in interpolation theory were for spaces of measurable functions. Subsequently, a strong type interpolation theory was developed for arbitrary Banach spaces by using various functionalizations of these spaces. The most widely known are the A. P. Calderón complex method [16] and the Lions–Peetre real method [13].

We will show in this paper that it is a straightforward matter to develop a generalized weak type interpolation theory for arbitrary Banach spaces by combining the Peetre functionalization with the maximal operators of Calderón [15]. As would be expected, this generalized weak type theory has many interesting applications.

* The research of the first named author was supported in part by the National Science Foundation Grant No. MCS 76-05847.

† The research of the second named author was partially supported by Canadian National Research Council Grant No. A7687.

‡ The research of the third named author was supported in part by the National Science Foundation Grant No. MCS 77-03666.

0022-1236/79/070058–37$02.00/0

It was shown by Calderón [15] that T is a linear operator which is simultaneously of weak type (p_1, q_1) and (p_2, q_2), $1 \leqslant p_1 < p_2 < \infty$, $1 \leqslant q_1$, $q_2 \leqslant \infty$ (in the classical sense) if and only if for each measurable f,

$$(Tf)^*(t) \leqslant c \left\{ t^{-1/q_1} \int_0^{t^m} s^{1/p_1} f^*(s) \frac{ds}{s} + t^{-1/q_2} \int_{t^m}^{\infty} s^{1/p_2} f^*(s) \frac{ds}{s} \right\}, \qquad t > 0 \tag{1.1}$$

where g^* denotes the decreasing rearrangement of $|g|$ and m is the slope of the line which passes through the points $(1/p_1, 1/q_1)$, $(1/p_2, 1/q_2)$. We use the term decreasing in the broader sense of nonincreasing.

It is an important observation that (1.1) can be used as the definition of weak type if now we talk about weak type for two pairs of indices (p_1, q_1) and (p_2, q_2) rather than the indices separately. It was shown by C. Bennett [4] that (1.1) serves as a natural definition for weak type even if p_2 is infinite. Hence, if T satisfies (1.1), we will say T is of weak type $\sigma(p_1, q_1; p_2, q_2)$.

Suppose an operator T satisfies (1.1). Function norms can be applied to both sides of (1.1) and then Hardy's inequality, or some variation thereof, can be used to obtain mapping properties of T (see [6, 15, 25]). This gives not only information for indices strictly interior to the segment $\sigma(p_1, q_1; p_2, q_2)$ (cf. Section 2) but also information at the endpoints (p_i, q_i), $i = 1, 2$ as well. So the inequality (1.1) automatically contains information on the mapping properties of T. In Section 2, we give an expanded discussion of weak type $\sigma(p_1, q_1; p_2, q_2)$ interpolation, and the resulting mapping properties.

It is a simple matter to replace the roles of f^* and $(Tf)^*$ in (1.1) by an appropriate functionalization of the Banach spaces to obtain a definition of weak type interpolation in arbitrary Banach spaces. For the majority of our applications, the K functional is the most suitable functionalization. In this case if (X_1, X_2) and (Y_1, Y_2) are two pairs of Banach spaces, we say that T is of generalized weak type $\sigma(p_1, q_1; p_2, q_2)$ with respect to the couples (X_1, X_2), (Y_1, Y_2) if for every $t > 0$,

$$\frac{K_Y(Tf, t)}{t} \leqslant c \left\{ t^{-1/q_1} \int_0^{t^m} s^{1/p_1} \left(\frac{K_X(f, s)}{s} \right) \frac{ds}{s} + t^{-1/q_2} \int_{t^m}^{\infty} s^{1/p_2} \left(\frac{K_X(f, s)}{s} \right) \frac{ds}{s} \right\} \tag{1.2}$$

where $K_Y(g, \cdot) = K(g, \cdot; Y_1, Y_2)$ and $K_X(f, \cdot) = K(f, \cdot; X_1, X_2)$. The notation of generalized weak type $\sigma[p_1, q_1; p_2, q_2)$ and $\sigma(p_1, q_1; p_2, q_2]$ means that the integral corresponding to the closed endpoint does not appear on the right hand side of (1.2). Of course, if an inequality such as (1.2) holds for an operator T, then only one endpoint and the slope of σ will be explicitly given. We take the remaining endpoint of σ to be the intersection of the boundary of the unit square with the straight line passing through the given endpoint with slope m. There are, in general, two points of intersection, but the requirement $1 \leqslant p_1 < p_2 \leqslant \infty$, $1 \leqslant q_1$, $q_2 \leqslant \infty$, $q_1 \neq q_2$ (which we will insist hold throughout the paper) determines the remaining endpoint uniquely. Hardy type inequalities will show that this is the optimal selection in our situation.

Using Hardy inequalities as mentioned above, we can deduce that if T is of generalized weak type $\sigma(p_1, q_1; p_2, q_2)$ and $(1/p, 1/q)$ is on the open line segment joining $(1/p_1, 1/q_1)$ to $(1/p_2, 1/q_2)$, then the operator T maps $(X_1, X_2)_{K,p,\alpha}$ continuously into $(Y_1, Y_2)_{K,q,\alpha}$ (notation as in Section 3). Finer results are also possible such as inclusion of logarithm factors in the definition of the spaces and a description of the mapping properties of T at the endpoints. These are spelled out in detail in Section 3.

In order to point out the usefulness of the formulation (1.2), we give several applications of generalized weak type interpolation to various problems in analysis. In many of these applications, the resulting weak type inequality (1.2) is a well known classical inequality, while in others the very formulation of (1.2) leads one to search for the correct weak type inequality. The reason that (1.2) usually has a classical formulation is that for the classical spaces there are descriptions of the K functional in terms of more familiar quantities. For example, $K(f, t; L_1, L_\infty) = tf^{**}(t)$ with $f^{**}(t) = t^{-1} \int_0^t f^*(s)\, ds$ and $K(f, t^r; L_p, W_p{}^r) \sim \omega_r(f, t)_p$ (cf. Section 4) when $W_p{}^r$ is the Sobolev space of r times differentiable functions in L_p and $\omega_r(f, \cdot)_p$ is the r-th order moduli of smoothness in L_p.

Perhaps the most familiar weak type inequality is Marchaud's inequality which compares the moduli of smoothness of two different orders. If Ω is a subset of R^n, then it is a simple matter to show that for each $f \in L_p(\Omega)$ and r and k positive integers, we have

$$\omega_{r+k}(f, t)_p \leqslant c\ \omega_r(f, t)_p, \qquad t > 0. \tag{1.3}$$

With certain structural assumptions on Ω, this inequality has a (weak) converse

$$\frac{\omega_r(f, t)_p}{t^r} \leqslant c \left\{ \|f\|_p + \int_t^\infty \frac{\omega_{r+k}(f, s)_p\, s^k}{s^{r+k}} \frac{ds}{s} \right\} \tag{1.4}$$

which is the famous Marchaud inequality (the term $\|f\|_p$ does not appear when $\Omega = R^n$). Using the equivalence of the K functional with ω_r, (1.4) can be rewritten as a weak type inequality of the form (1.2); namely, the identity operator is of generalized weak type $\sigma[1, 1; (r+k)/k, \infty)$ for the couples (L_p, W_p^{r+k}) and $(L_p, W_p{}^r)$. There are other inequalities for smoothness of derivatives (even for the semigroup setting) which are of generalized weak type. These are given in Section 4 along with their applications to equivalent characterizations of Besov spaces, reduction theorems for semi-groups, etc.

There are also weak type inequalities for moduli of smoothness in different L_p spaces. We examine this in Section 5, where we prove among other things that if Ω satisfies certain properties, then for $q > p$ and $f \in L_p(\Omega)$,

$$\omega_r(f, t)_q \leqslant c \int_0^t s^{-\theta} \omega_r(f, s)_p \frac{ds}{s}, \qquad t > 0$$

where $\theta = n/p - n/q$. This shows that the identity operator is of generalized weak type $\sigma(r/(r - \theta), 1; \infty, r/\theta]$ for the pairs $(L_p, W_p{}^r)$ and $(L_q, W_q{}^r)$. This implies classical embedding results for Besov spaces (cf. Section 5): for example that $B_p^{\lambda+\theta,a} \to B_q^{\lambda,a}$ if $\lambda > 0$, as well as $B_p^{\theta,1} \to L_q$.

There are finer descriptions of the mappings of Besov spaces into the L_q spaces and these rest on the weak type inequality

$$f^{**}(t) \leqslant c \left\{ \|f\|_p + \int_{t^{1/n}}^{\infty} s^{-n/p} \omega_n(f, s)_p \frac{ds}{s} \right\}$$

where again the term $\|f\|_p$ can be dropped when $\Omega = R^n$. This inequality leads to the embeddings $B_p^{\theta,a} \to L^{q,a}$ when $\theta = n/p - n/q$ as well as results for $q = \infty$ which exhibit a loss of a logarithm (see Corollary 5.5).

In Section 6, we show that the Hilbert transform Hf satisfies the weak type inequality

$$\omega_r(Hf, t)_p \leqslant c \left\{ \int_0^t \omega_r(f, s)_p \frac{ds}{s} + t^r \int_t^{\infty} \omega_r(f, s)_p \frac{ds}{s^{r+1}} \right\}.$$

Here, only the cases $p = 1, \infty$ are interesting since when $1 < p < \infty$, we have strong inequalities. This inequality together with the mapping theorems established in Section 3, show that if $f \in B_p^{\theta,a}$ then the conjugate function is also in $B_p^{\theta,a}$. This of course includes the classical result that if $f \in \operatorname{Lip} \alpha$, then $Hf \in \operatorname{Lip} \alpha$, $0 < \alpha < 1$. The endpoint mappings of Section 3 also give endpoint results for the Hilbert transform now with the anticipated loss of a logarithm (cf. Zygmund [30, p. 121]).

In Section 7, we give some applications to approximation theory. Following [8] and [20] we first show that inverse theorems for approximation on the circle by trigonometric polynomials can be written as weak type inequalities. Using this together with Jackson's direct theorem we see (see Corollary (7.1)) that approximation spaces can be characterized as Besov spaces while a loss of logarithm occurs in the endpoint embeddings of the approximation spaces and Favard classes. Next we prove a weak type inequality relating the growth of Fourier coefficients to the modulus of continuity: for $1 \leqslant p \leqslant 2$

$$(\hat{f})^{**}(t) \leqslant c \frac{1}{t} \left(\|f\|_1 + \int_{1/t}^{1} \frac{\omega(f, s)_p}{s^{1/p}} \frac{ds}{s} \right), \qquad t \geqslant 1 \qquad (1.5)$$

On R, we can drop the term $\|f\|_1$ and allow all t, but must integrate to ∞. This shows that the Fourier transform is of weak type $\sigma[1, p/(p-1); p/(p-1), 1)$ for (L_p, W_p^1) and (L_1, L_∞), $1 \leqslant p \leqslant 2$. Applying function norms, we get classical statements about the absolute convergence of Fourier transforms.

It is possible to derive the mapping results for many of our applications by avoiding the question of weak type inequalities completely. This has been done by Peetre in his many beautiful applications of interpolation theory in various problems of analysis. The weak type inequality is replaced by some sort of argument with the J-functional together with appropriate use of the reiteration and equivalence theorem for K and J interpolation. In this case, all results are stated in terms of mappings of the appropriate spaces.

Using weak type interpolation has not only the advantage of giving a unified approach, but one can also see the fundamental inequality behind the mapping properties. Such a weak type inequality carries more information than any statement about mapping properties deduced from it (or in some other way). However, there is no question that these two approaches are closely related. We mention these relationships in Remark 8.5. The essential point is that the J- and K-functionals are comparable by certain strong and weak type inequalities. We also discuss briefly there how the reiteration theorem can be viewed in the light of weak type inequalities.

As mentioned above, some of the inequalities are classical and most of the others can be found in the literature. However, we do supply the proofs of these inequalities if we have a more direct approach.

2. The Classical Theory

We want to begin with an overview of classical weak type interpolation for spaces of measurable functions. This will serve as an orientation for the formulation of general weak type interpolation in the next section and also introduce much of the needed notation.

It is important to begin with the "correct" formulation of weak type. For our purposes, this turns out to be the approach given by C. Bennett and K. Rudnick [6] based on the Calderón maximal operators.

If $1 \leqslant p_1 < p_2 \leqslant \infty$, $1 \leqslant q_1$, $q_2 \leqslant \infty$, $q_1 \neq q_2$, let $\sigma(p_1, q_1; p_2, q_2)$ denote the set of all (p, q) such that the point $(1/p, 1/q)$ belongs to the open line segment in R^2 with endpoints $(1/p_1, 1/q_1)$ and $(1/p_2, 1/q_2)$. The notation $\sigma[p_1, q_1; p_2, q_2)$ (resp. $\sigma(p_1, q_1; p_2, q_2]$) means that the endpoint $(1/p_1, 1/q_1)$ (resp. $(1/p_2, 1/q_2)$) is included in the segment. The restrictions imposed on p_1, q_1, p_2, and q_2 will be carried throughout the paper.

There are three maximal operators associated with segments depending on the form of the segment. Suppose σ is a segment of one of the three types given above.

For $t \in (0, \infty)$ and f measurable on $(0, \infty)$, let

$$
\begin{aligned}
S_{1\sigma}(f)(t) &= t^{-1/q_1} \int_0^{t^m} f(u)\, u^{1/p_1} \frac{du}{u} \\
S_{2\sigma}(f)(t) &= t^{-1/q_2} \int_{t^m}^{\infty} f(u)\, u^{1/p_2} \frac{du}{u}
\end{aligned}
\tag{2.1}
$$

where $m = (1/q_1 - 1/q_2)/(1/p_1 - 1/p_2)$ is the slope of the segment σ.

We now define the maximal operator S_σ for a segment σ as

$$
S_\sigma = \begin{cases} S_{1\sigma} & \text{if} \quad \sigma = \sigma(p_1, q_1; p_2, q_2] \\ S_{2\sigma} & \text{if} \quad \sigma = \sigma[p_1, q_1; p_2, q_2) \\ S_{1\sigma} + S_{2\sigma} & \text{if} \quad \sigma = \sigma(p_1, q_1; p_2, q_2) \end{cases}
\tag{2.2}
$$

If (X, μ) is a totally σ-finite measure space and f a measurable function which is finite a.e., then denote by f^* the decreasing rearrangement of $|f|$. Suppose T is a quasilinear operator. We say that T is σ-weak type if

$$
(Tf)^*(t) \leqslant c\, S_\sigma(f^*)(t), \qquad t \in (0, \infty),
\tag{2.3}
$$

for all f for which the right hand side is finite. For a discussion of the relation of σ-weak type with the classical definitions of weak type as well as several examples of weak type operators in Fourier analysis, we refer the reader to [6].

It is clear from (2.3) that any mapping property of S_σ will in turn give a statement on mapping properties of T. For example, consider the Lorentz–Zygmund spaces $L^{pa}(\log L)^\alpha$ defined for $1 \leqslant p \leqslant \infty$, $1 \leqslant a \leqslant \infty$, $-\infty < \alpha < \infty$ as the set of all functions f for which $\| f^* \|_{p,a,\alpha} < \infty$, where for any decreasing ψ

$$
\| \psi \|_{p,a,\alpha} = \begin{cases} \left\{ \displaystyle\int_0^\infty (t^{1/p} \psi(t)(1 + |\ln t|)^\alpha)^a \frac{dt}{t} \right\}^{1/a}, & 1 \leqslant a < \infty \\ \displaystyle\sup_{0<t<\infty} [t^{1/p}(1 + |\ln t|)^\alpha\, \psi(t)], & a = \infty \end{cases}
\tag{2.4}
$$

For $p > 1$, or $p = 1$ with $a = 1$ and $\alpha \geqslant 0$, these spaces are Banach spaces under a norm equivalent to $\| \psi \|_{p,a,\alpha}$. In order to obtain mapping results for σ-weak type operators on the spaces $L^{pa}(\log L)^\alpha$, we are led to examine integral inequalities for S_σ. We state two such theorems. The first applies to interior indices, while the second deals with the endpoint case. In these theorems, ψ denotes an arbitrary nonnegative measurable function, but of course $\psi = f^*$ is the choice of present interest. For proofs we refer the reader to Theorems 6.4 and 6.5 of [6].

THEOREM 2.1 (Intermediate inequalities). *Let* $1 \leqslant a \leqslant \infty$ *and* $-\infty < \alpha < \infty$. *If* $(p, q) \in \sigma$, *then*

$$\left\{\int_0^\infty (S_\sigma(\psi)(t)\, t^{1/q}(1 + |\ln t|)^\alpha)^a \frac{dt}{t}\right\}^{1/a} \leqslant c \left\{\int_0^\infty (\psi(t)\, t^{1/p}(1 + |\ln t|)^\alpha)^a \frac{dt}{t}\right\}^{1/a} \tag{2.5}$$

where the integrals are replaced by a supremum norm when $a = \infty$ *and* c *is a constant independent of* ψ.

THEOREM 2.2 (Endpoint inequalities). *Let* I *be either the interval* $(0, 1)$ *or* $(1, \infty)$ *and define* $\operatorname{sgn} I$ *as the sign of the function* $\ln t$ *on* I. *For a segment* σ *with slope* m *define*

$$J = J(\sigma, I) = \begin{cases} I & \text{if } m > 0 \\ (0, \infty) - I & \text{if } m < 0. \end{cases}$$

Let $1 \leqslant b \leqslant a$, $\beta - 1 + 1/b = \alpha + 1/a$ *and* $i = 1$ *or* 2. *If* $\operatorname{sgn}(\alpha + 1/a) = (-1)^i \operatorname{sgn} I \operatorname{sgn} m$ *when* (p_i, q_i) *is not in* σ, *the operator* S_σ *satisfies*

$$\left\{\int_I (S_\sigma(\psi)(t)\, t^{1/q_i}(1 + |\ln t|)^\alpha)^a \frac{dt}{t}\right\}^{1/a} \leqslant c \left\{\left(\int_J (\psi(t)\, t^{1/p_i}(1 + |\ln t|)^\beta)^b \frac{dt}{t}\right)^{1/b} + S_\sigma(\psi)(1)\right\} \tag{2.6}$$

with the change to the supremum norm if a *or* b *are* ∞ *and* c *a constant independent of* ψ. *When* $\alpha + 1/a > 0$, *the term* $S_\sigma(\psi)(1)$ *in* (2.6) *can be dropped. Inequality* (2.6) *is also valid for* $\alpha = \beta = 0$, $a = \infty$, *and* $b = 1$.

Taking $\psi = f^*$ in (2.5), it follows that if T is of weak type σ and $(p, q) \in \sigma$, then T is a continuous map from the Banach space $L^{pa}(\log L)^\alpha$ into $L^{qa}(\log L)^\alpha$. At an endpoint (p_i, q_i) not in σ, the inequality (2.6) gives a similar result except there is a loss of a logarithm.

3. Generalized Weak Type Inequalities

In order to extend the definition of weak type to the Banach space setting, it is enough to functionalize the Banach spaces. This can be done in several ways and the method that should be chosen depends on the problem at hand. For most of our applications, it will be convenient to work with the Peetre K-functional representation.

A pair of Banach spaces (X_1, X_2) continuously embedded in some Hausdorff topological vector space $\mathscr{X}$ is called a Banach couple. The sum of X_1 and X_2, denoted by $X_1 + X_2$, is the set of all $f = f_1 + f_2$, with $f_i \in X_i$, $i = 1, 2$. For each $f \in X_1 + X_2$, we define the Peetre K-functional for f by

$$K(f, t; X_1, X_2) = \inf_{f=f_1+f_2} \{\| f_1 \|_{X_1} + t \| f_2 \|_{X_2}\}. \tag{3.1}$$

If there is no chance of confusion we do not indicate the dependence on X_1 and X_2 and write instead $K(f, t)$. Also, in some specific settings, it is customary to work with a modified K-functional in which $\| \ \|_{X_2}$ is a seminorm. This is the case for example with interpolation between Sobolev spaces which we use in some of our applications. All the results of this section hold as well for such a modified K functional.

The K method generates interpolation spaces (see [13, Chapter 3]) by applying function norms to $K(f, t)/t$. For example, if $1 \leqslant p, a \leqslant \infty$ and $-\infty < \alpha < \infty$, then let us denote by $X_{p,a,\alpha} = (X_1, X_2)_{K,p,a,\alpha}$ the set of all $f \in X_1 + X_2$ for which

$$\| f \|_{K,p,a,\alpha} = \| f \|_{X_{p,a,\alpha}} = \| K(f, t)/t \|_{p,a,\alpha} < +\infty, \tag{3.2}$$

where $\| \ \|_{p,a,\alpha}$ is defined in (2.4). Note that $K(f, t)$ is concave and so $K(f, t)/t$ is decreasing. Thus $K(f, t)/t$ in (3.2) takes the place of f^* in (2.4).

In what follows we will generalize the weak type interpolation of Section 2 by placing $K(f, t)/t$ in the role of f^*. As we have mentioned, in some instances it is better to work with decreasing functionalizations other than $K(f, t)/t$, such as $k(f, t)$, the derivative of $K(f, t)$. For example $k(f, t)$ would exactly recover the Bennett Rudnick setting. In some problems in approximation theory the approximation functional $E(f, t)$ (see Section 7) seems to be most convenient. All in all, which functionalization to use depends on the application in mind and it is easy to rework our results for that setting.

Definition. A quasi-linear operator T is of generalized weak type σ with respect to the Banach couples (X_1, X_2) and (Y_1, Y_2) if whenever $S_\sigma[K(f, \cdot; X_1, X_2)/(\cdot)](t_0)$ is finite for some t_0, then Tf belongs to $Y_1 + Y_2$ and there holds

$$\frac{K(Tf, t; Y_1, Y_2)}{t} \leqslant c\, S_\sigma \left[\frac{K(f, \cdot; X_1, X_2)}{(\cdot)}\right](t), \qquad 0 < t < \infty. \tag{3.3}$$

As is the case for weak type operators for spaces of measurable functions, this definition is equivalent to the property that T map $(X_1, X_2)_{K,p_i,1,0}$ to $(Y_1, Y_2)_{K,q_i,\infty,0}$ for $i = 1, 2$ (see Theorem (3.4)). For notational convenience, in the remainder of this section we shall drop the dependence on the spaces X_1, X_2, Y_1, Y_2. It is understood that the domain of the operator T is associated with a pair (X_1, X_2) and the range of T is associated with a pair (Y_1, Y_2).

When T is of generalized weak type σ, Theorem 2.1 gives almost immediately an interpolation theorem for the spaces generated as in (3.2).

THEOREM 3.1 (Intermediate interpolation). *If T is of generalized weak type σ and $(p, q) \in \sigma$, then*

$$\| Tf \|_Y \leqslant c \| f \|_X , \tag{3.4}$$

where $Y = (Y_1, Y_2)_{K,q,a,\alpha}$ and $X = (X_1, X_2)_{K,p,a,\alpha}$ with $1 \leqslant a \leqslant \infty$, $-\infty < \alpha < \infty$.

Proof. Let $a < \infty$ (the case $a = \infty$ is handled similarily). From (3.3), it follows that

$$\begin{aligned} &\| K(Tf, t)/t \|_{q,a,\alpha} \\ &\quad \leqslant c \left\{ \int_0^\infty (S_\sigma(K(f, \cdot)/(\cdot))(t)\, t^{1/q}(1 + |\ln t|)^\alpha)^a \frac{dt}{t} \right\}^{1/a} \end{aligned} \tag{3.5}$$

Since $S_\sigma(K(f, \cdot)/(\cdot))$ may not be decreasing, we avoid the quasinorm notation on the right hand side. Applying (2.5) for $\psi(t) = K(f, t)/t$ and $(p, q) \in \sigma$ to the right hand side of (3.5), we get (3.4), as desired.

The endpoint case is much more complicated as the variety of inequalities within (2.6) illustrates. Because of the nature of the inequalities, it is generally not possible to state results just in terms of the spaces $X_{p,a,\alpha}$ and $Y_{q,a,\alpha}$ but instead we introduce the following spaces.

For $1 \leqslant p_1 < p_2 \leqslant \infty$, $1 \leqslant a, b \leqslant \infty$ and $-\infty < \alpha, \beta < \infty$, we denote by $(X_1, X_2)_{K,p_1,a,\alpha} + (X_1, X_2)_{K,p_2,b,\beta}$ the space generated by the norm

$$\begin{aligned} &\left\{ \int_0^1 \left[\frac{K(f, t)}{t}\, t^{1/p_1}(1 + |\ln t|)^\alpha \right]^a \frac{dt}{t} \right\}^{1/a} \\ &\quad + \left\{ \int_1^\infty \left[\frac{K(f, t)}{t}\, t^{1/p_2}(1 + |\ln t|)^\beta \right]^b \frac{dt}{t} \right\}^{1/b} \end{aligned} \tag{3.6}$$

with the usual change if a or b are ∞. Note that the ordering $p_1 < p_2$ has an essential role in the nature of these spaces. The intersection $(X_1, X_2)_{K,p_1,a,\alpha} \cap (X_1, X_2)_{K,p_2,b,\beta}$ is generated by

$$\max(\| K(f, t)/t \|_{p_1,a,\alpha} , \| K(f, t)/t \|_{p_2,b,\beta}). \tag{3.7}$$

The definition in (3.7) is equivalent to interchanging the indices p_1 and p_2 in (3.6). It follows from Theorem 2.2 that

THEOREM 3.2 (Endpoint interpolation). *Suppose $1 \leqslant b \leqslant a \leqslant \infty$, $1 \leqslant d \leqslant c \leqslant \infty$, and $-\infty < \alpha, \beta, \gamma, \delta < \infty$. If T is of generalized weak type $\sigma(p_1, q_1;$*

$p_2, q_2)$ *and* $\alpha + 1/a = \beta - 1 + 1/b > 0$, $\gamma + 1/c = \delta - 1 + 1/d > 0$, *then*

$$T : (X_1, X_2)_{K,p_1,b,\beta} + (X_1, X_2)_{K,p_2,d,\delta} \to (Y_1, Y_2)_{K,q_1,a,\alpha} + (Y_1, Y_2)_{K,q_2,c,\gamma} .$$

When $\alpha + 1/a = \beta - 1 + 1/b < 0$, $\gamma + 1/c = \delta - 1 + 1/d < 0$, *then*

$$T : (X_1, X_2)_{K,p_1,b,\beta} \cap (X_1, X_2)_{K,p_2,d,\delta} \to (Y_1, Y_2)_{K,q_1,a,\alpha} \cap (Y_1, Y_2)_{K,q_2,c,\gamma} .$$

In some of our applications, we will be in the situation that only small values of t are important in the definitions of the spaces $(X_1, X_2)_{K,p,a,\alpha}$. For this reason, we want to give the corresponding versions of Theorems 3.1 and 3.2.

THEOREM 3.3. *Let* $X_2 \subseteq X_1$, $Y_2 \subseteq Y_1$ *and let* σ *be a segment with positive slope. Further let* $1 \leqslant b \leqslant a$, *and* $\beta - 1 + 1/b = \alpha + 1/a$. *If* T *is of generalized weak type* σ (*where* (3.3) *only need hold for* $0 < t < 1$) *and* (p, q) *is in* σ, *then*

$$\begin{aligned}\left\{\int_0^1 \left[\frac{K(Tf, t)}{t}\, t^{1/q}(1 + |\ln t|)^{\alpha}\right]^a \frac{dt}{t}\right\}^{1/a}& \\ \leqslant c \left\{\int_0^1 \left[\frac{K(f, t)}{t}\, t^{1/p}(1 + |\ln t|)^{\alpha}\right]^a \frac{dt}{t}\right\}^{1/a}&.\end{aligned} \tag{3.8}$$

If (p_i, q_i) *is an endpoint of* σ *with* $\operatorname{sgn}(\alpha + 1/a) = (-1)^{i+1}$, *then*

$$\begin{aligned}\left\{\int_0^1 \left(\frac{K(Tf, t)}{t}\, t^{1/q_i}(1 + |\ln t|)^{\alpha}\right)^a \frac{dt}{t}\right\}^{1/a}& \\ \leqslant c \left(\left\{\int_0^1 \left[\frac{K(f, t)}{t}\, t^{1/p_i}(1 + |\ln t|)^{\beta}\right]^b \frac{dt}{t}\right\}^{1/b} + \|f\|_{X_1}\right)&.\end{aligned} \tag{3.9}$$

In the case $i = 1$ *and* $S_\sigma = S_{1\sigma}$, *the term* $\|f\|_{X_1}$ *can be dropped from inequality* (3.9).

We mention one additional result before we turn to the applications of generalized weak type inequalities. Using weak type notions which we outline in Remark 8.5, Calderón's theorem [15, Theorem 8] can be extended to Banach spaces in the following manner.

THEOREM 3.4. *Suppose* $1 < p_1 < p_2 < \infty$, $1 \leqslant q_1, q_2 \leqslant \infty$, *and* $q_1 \neq q_2$, *then a necessary and sufficient condition that an operator* T *be of generalized* $\sigma(p_1, q_1; p_2, q_2)$ *weak type for* (X_1, X_2) *and* (Y_1, Y_2) *is that* $T\colon X_{K,p_i,1,0} \to Y_{K,q_i,\infty,0}$, $i = 1, 2$.

4. Moduli of Smoothness

Our applications of the weak type theory introduced in Section 3 will center for the most part around inequalities for moduli of smoothness and their uses in obtaining embedding theorems and mapping properties of some operators on smoothness spaces.

Let Ω denote a subset of R^n of one of the following two types: in the case of one dimension, $n = 1$, Ω is either a closed interval $[a, b]$ or the whole line R; in the case $n \geqslant 2$, Ω is either R^n or a closed bounded subset for which

> there exists a finite number of open sets U_i covering the boundary of Ω and corresponding finite open cones C_i such that $x + C_i$ is contained in the interior of Ω for each x in $U_i \cap \Omega$. (4.0)

Our results could also be given for semi-infinite cases.

If Ω is as above and $h \in R^n$, we let $\Omega_h = \{x: x + \alpha h \in \Omega, \text{ for all } 0 \leqslant \alpha \leqslant 1\}$. When $r > 0$ and $x \in \Omega_{rh}$, let $\Delta_h^r(f, x)$ denote the r-th difference of f

$$\Delta_h^r(f, x) = \sum_0^r (-1)^{r+k} \binom{r}{k} f(x + kh)$$

and define the r-th order modulus of smoothness for functions f in $L_p(\Omega)$ by

$$\omega_r(f, t)_p = \sup_{|h| \leqslant t} \| \Delta_h^r(f, x) \|_{p(x)} (\Omega_{rh}).$$

Here, the notation $\| \; \|_{p(x)}(\Omega_{rh})$ indicates that the norm is taken with respect to the variable x and is the L_p norm over Ω_{rh}. Similar notation is used throughout. In the definition of $\omega_r(f, t)_p$ in the case of 2π-periodic functions, the norm is taken over all of $(-\pi, \pi]$.

Let $W_p^r(\Omega)$ denote the Sobolev space consisting of all functions f which have partial derivatives (in the distributional sense) of order α, for all $|\alpha| \leqslant r$ and for which

$$\| f \|_{p,r}(\Omega) = \max_{0 \leqslant |\alpha| \leqslant r} \| D^\alpha f \|_p (\Omega) < \infty.$$

Sometimes, we will need the seminorm

$$| f |_{p,r}(\Omega) = \max_{|\alpha| = r} \| D^\alpha f \|_p (\Omega).$$

We do not indicate the dependence on Ω if there is no confusion.

The modulus of smoothness $\omega_r(f, \cdot)_p$ is intimately connected to the modified K-functional

$$K_r(f, t)_p = \inf_{g \in W_p^r} \{ \| f - g \|_p + t \, | g |_{p,r} \}, \qquad t \geqslant 0.$$

In fact, for our purposes $K_r(f, t^r)_p$ and $\omega_r(f, t)_p$ are equivalent because there are constants C_1, $C_2 > 0$ such that

$$C_1\, \omega_r(f, t)_p \leqslant K_r(f, t^r)_p \leqslant C_2\, \omega_r(f, t)_p, \qquad t \geqslant 0. \tag{4.1}$$

The inequalities in (4.1) are quite easy to prove when $\Omega = R^n$, $n \geqslant 1$, by using Stecklov averages. The general case of Ω is more complicated and was given only recently by H. Johnen and K. Scherer [19, Theorem 1, Corollary 3].

Let us now consider the fundamental question of comparing different order moduli of smoothness. If $k \geqslant 0$, it is a trivial matter to show that

$$\omega_{r+k}(f, t)_p \leqslant c\, \omega_r(f, t)_p, \qquad t \geqslant 0, \tag{4.2}$$

with c a constant depending only on k and r. As a partial converse to (4.2), we have the famous Marchaud type inequalities

$$\frac{\omega_r(f, t)_p}{t^r} \leqslant c \left\{\|f\|_p + \int_t^\infty \frac{\omega_{r+k}(f, s)_p\, s^k}{s^{r+k}} \frac{ds}{s}\right\} \tag{4.3}$$

with c depending only on k and r. When $\Omega = R^n$ or we work with 2π periodic functions, the term $\|f\|_p$ can be dropped. For proofs, see H. Johnen [18] for the case $n = 1$ and H. Johnen–K. Scherer [19] for $n \geqslant 2$.

The inequalities in (4.1) can be used to rewrite (4.3) as a weak type inequality. Indeed, replacing $\omega_r(f, t)_p$ by $K_r(f, t^r)_p$ and changing variables gives for $\sigma = \sigma[1, 1; (r+k)/k, \infty)$ and any $t > 0$

$$\frac{K_r(f, t)}{t} \leqslant c \left\{\|f\|_p + S_\sigma\left(\frac{K_{r+k}(f, \cdot)_p}{(\cdot)}\right)(t)\right\} \tag{4.4}$$

which is the key weak type inequality. Recall, $\|f\|_p$ does not appear when $\Omega = R^n$ and, since we are always working within the space L_p, the term $\|f\|_p$ has no effect for other Ω.

The inequality (4.4) gives information about embeddings of the spaces $X_{\mu,q,\alpha} = (L_p, W_p^{(r+k)})_{K,\mu,q,\alpha}$ into the spaces $Y_{\mu,q,\alpha} = (L_p, W_p{}^r)_{K,\mu,q,\alpha}$. Before stating these results, it is appropriate for us at this stage to introduce the classical notation for these spaces.

If $\theta > 0$, $1 \leqslant p \leqslant \infty$, and $1 \leqslant q \leqslant \infty$, and r is any integer $> \theta$, then the Besov space $B_p^{\theta,q}$ is the space of all functions f in $L_p(\Omega)$ for which

$$\|f\|_p^{\theta,q} = \left\{\int_0^1 (t^{-\theta}\omega_r(f, t)_p)^q \frac{dt}{t}\right\}^{1/q} < \infty \tag{4.5}$$

when $q < \infty$. The usual change is made when $q = \infty$. Since $\omega_r(f, t)_p \leqslant 2^r \|f\|_p$, the integral in (4.5) could also be taken over $(0, \infty)$. Hence, $B_p^{\theta,q} =$

$Y_{\mu,q,0}$ with $\mu = r/(r - \theta)$. The embedding results from Theorem 3.3 in the case of bounded domain and Theorem 3.1 in the case of R^n together with the inequalities (4.2) and (4.4) show that the definition of the Besov space $B_p^{\theta,q}$ does not depend on the choice of the integer r. Usually, the choice $r = [\theta] + 1$ is made for definitiveness which we will also do.

Let us also introduce the spaces $B_p^{\theta,q,\alpha}$ as the set of functions $f \in L^p(\Omega)$ for which

$$\|f\|_p^{\theta,q,\alpha} = \left\{\int_0^1 (t^{-\theta}\omega_r(f, t)_p(1 + |\ln t|)^\alpha)^q \frac{dt}{t}\right\}^{1/q} < \infty \tag{4.6}$$

again with the usual change for $q = \infty$. In (4.6) $-\infty < \alpha < \infty$. We can even let $\theta = 0$ in which case only the values of $\alpha \geqslant -1/q$ give something different than L_p. We should remark that in the case that $\theta = 0$, $\alpha \geqslant -1/q$, the space $B_p^{0,q,\alpha}$ is not the same as the space $X_{1,q,\alpha}$ since the latter space is trivial (polynomials of degree $\leqslant r$) due to the fact that the integral over $(1, \infty)$ is not negligible.

The results of Section 3 also give endpoint results with the expected loss of a logarithm. For example, among many other things, Theorem 3.3 shows that for $1 \leqslant p \leqslant \infty$, $\alpha + 1/q < 0$, the space $(L_p, W_p^{r+1})_{K,\infty,q,\alpha+1}$ is continuously embedded in $(L_p, W_p^{r})_{K,\infty,q,\alpha}$ (for $\alpha = -1$, $q = \infty$, compare p. 107 of [28]).

There are also comparisons for the smoothness of derivatives of f and the smoothness of f. A strong inequality is

$$\omega_{r+k}(f, t)_p \leqslant c\,\{t^k \sup_{|\beta|=k} \omega_r(D^\beta f, t)_p\}, \qquad t > 0. \tag{4.7}$$

As a partial converse to (4.7), we have [19] that if β is an n-tuple of non-negative integers with $|\beta| = k$, then

$$\omega_r(D^\beta f, t)_p \leqslant c\left\{\int_0^t \frac{\omega_{r+k}(f, s)_p}{s^{r+k}}\, s^r \frac{ds}{s}\right\}, \qquad t > 0 \tag{4.8}$$

Changing over to the K functionals shows that D^β is of generalized weak type $\sigma((r + k)/r, 1; \infty, \infty]$. Thus Theorem 3.3 (or Theorem 3.1 for $\Omega = R^n$) shows that the space $B_p^{\theta,q,\alpha}$ with $\theta > k$ could equally well be described as the set of all f for which

$$\int_0^1 [\sup_{|\beta|=k} \omega_r(D^\beta f, t)_p\, t^{-\theta+k}(1 + |\ln t|)^\alpha]^q \frac{dt}{t} < \infty.$$

At the endpoint we have that if $f \in B_p^{k,q,\alpha+1}$, $\alpha + 1/q > 0$, then for any $|\beta| = k$, $D^\beta f \in B_p^{0,q,\alpha}$.

The development given above can also be carried out for semi-groups of operators. We follow the treatise in Butzer–Berens [13]. If X is a Banach space and $\{T(t); 0 \leqslant t < \infty\}$ is an equibounded C_0 semi-group, we can define the r-th order modulus of smoothness for $f \in X$ and $t > 0$ by

$$\omega_r(f, t)_T = \sup_{0 \leqslant s \leqslant t} \|(T(s) - I)^r f\|_X .$$

Likewise, if A is the infinitesimal generator of the semi-group, let

$$K_r(f, t)_T = \inf_{g \in D(A^r)} (\|f - g\|_X + t \| A^r g \|_X)$$

with $D(A^r)$ the domain of A^r. As before one shows

$$c_1 \, \omega_r(f, t)_T \leqslant K_r(f, t^r)_T \leqslant c_2 \, \omega_r(f, t)_T . \tag{4.9}$$

Analogous to (4.2), we have that for any $k \geqslant 0$

$$\omega_{r+k}(f, t)_T \leqslant c \, \omega_r(f, t)_T \tag{4.10}$$

with c a constant depending only on r and k. Marchaud's inequality is proved for semi-groups in the same way as it is proved for ordinary differences on R, so that

$$\omega_r(f, t)_T \leqslant c \int_t^\infty \frac{\omega_{r+k}(f, s)_T}{s^{r+k}} \, s^k \, \frac{ds}{s}, \qquad t > 0. \tag{4.11}$$

This last result can be stated in terms of K-functionals as

$$\frac{K_r(f, t)_T}{t} \leqslant c \, S_\sigma \left[\frac{K_{r+k}(f, \cdot)_T}{(\cdot)} \right] (t) \tag{4.12}$$

with $\sigma = \sigma[1, 1; (r + k)/k, \infty)$. Hence, now from Theorem 3.3, we have that $(X, D(A^r))_{K,q,a,\alpha}$ is equivalent to $(X, D(A^{r+k}))_{K,p,a,\alpha}$ provided (p, q) is in $\sigma[1, 1; (r + k)/k, \infty)$, i.e. $1/q = (r + k)/rp - k/r$. Here, in the definition of the intermediate spaces, we take the integral in (2.3) only over the interval (0, 1) and we also use the modified K-functional as defined above. Theorem 3.3 also gives among many other things the embeddings $(X, D(A^{r+k}))_{K,(r+k)/k,a,\alpha+1} \subset (X, D(A^r))_{K,\infty,a,\alpha}$ provided $\alpha + 1/a < 0$.

It is also possible to establish weak inequalities for powers of A analogous to (4.8). These in turn give reduction theorems for semi-groups (see [13, Section 3.4]). The appropriate weak inequality for A^j is

$$\omega_{r-j}(A^j f, t)_T \leqslant c \int_0^t \frac{\omega_r(f, s)_T}{s^j} \, \frac{ds}{s}, \qquad 0 < j < r. \tag{4.13}$$

The inequality (4.13) can be derived from the inequality

$$\| A^j f \|_X \leqslant c \, \{t^{-j} \| f \|_X + t^{r-j} \| A^r f \|_X\} \tag{4.14}$$

which holds for all $f \in D(A^r)$ and $0 \leqslant j \leqslant r$. To prove (4.14) let M_j be the B-spline of degree $j-1$ (order j) with knots $\{0, 1,\ldots, j\}$ (see [17]). Then M_j has the properties: $M_j \geqslant 0$ and M_j vanishes outside of the interval $(0, j)$, $\int_{-\infty}^{\infty} M_j(t)\, dt = 1$, and most importantly M_j is the Peano kernel for j-th differences

$$t^{-j}(T(t) - I)^j f = \int_{-\infty}^{\infty} (T(s)\, A^j f)\, M_j(t^{-1}s)\, t^{-1}\, ds \tag{4.15}$$

where the integral is vector valued.

Now write

$$\begin{aligned}(-1)^{r-j+1} A^j f = \int_0^{\infty} [[(T(s) - I)^{r-j} + (-1)^{r-j+1} I]\, A^j f]\, M_j(t^{-1}s)\, t^{-1}\, ds \\ - \int_0^{\infty} [(T(s) - I)^{r-j} A^j f]\, M_j(t^{-1}s)\, t^{-1}\, ds = f_1 + f_2 .\end{aligned}$$

Because of (4.15), f_1 can be written as a sum of terms of the form $t^{-j}(T(t/k) - I)^{r-j} f$, $1 \leqslant k \leqslant r-j$. Hence

$$\| f_1 \|_X \leqslant c\, t^{-j} \| f \|_X . \tag{4.16}$$

On the other hand, in order to estimate f_2, we use the fact that

$$\begin{aligned}&\|(T(s) - I)^{r-j} A^j f \|_X \\ &= \left\| \int_0^s \cdots \int_0^s \int_0^s T(s_1 + s_2 + \cdots + s_{r-j})\, A^r f\, ds_1\, ds_2 \cdots ds_{r-j} \right\|_X \\ &\leqslant c s^{r-j} \| A^r f \|_X\end{aligned}$$

and find

$$\| f_2 \|_X \leqslant c \| A^r f \|_X \int_{-\infty}^{\infty} s^{r-j} M_j(t^{-1}s)\, t^{-1}\, ds \leqslant c\, t^{r-j} \| A^r f \|_X$$

where we used the fact that M_j has integral one and is supported on $(0, j)$. This last inequality together with (4.16) shows that (4.14) holds.

The proof that (4.14) gives (4.13) is the same as in the classical case (see [18]) and we do not give the details. The weak inequality (4.13) shows that A^j is of generalized weak type $\sigma(r/(r-j), 1;\, \infty, \infty]$ for the pairs $(X, D(A^r))$ and $(X, D(A^{r-j}))$.

Hence we get the following reduction theorems (see Theorems 3.4.6 and 3.4.10 of [13]): For $j = 0, 1,\ldots, r-1$ and $0 \leqslant 1/p < (r-j)/r$ the space $(X, D(A^r))_{K,p,a,\alpha}$ can be described as the set of f such that

$$\int_0^1 [\omega_{r-j}(A^j f, t)_T\, t^{j+r(1/p-1)} (1 + |\ln t|)^{\alpha}]^a \frac{dt}{t}$$

is finite. In particular, for $j = r - 1$ and $q = p/r > 1$ we obtain: $f \in (X, D(A^r))_{K,p,a,\alpha}$ if and only if $f \in D(A^{r-1})$ and $A^{r-1}f \in (X, D(A))_{K,q,a,\alpha}$. Thus, we obtain Theorem 3.4.6 of [13] when $q < \infty$ and Theorem 3.4.10 of [13] when $q = \infty$.

5. Embeddings for Besov and Sobolev Spaces

Besides the inequalities for moduli of smoothness of different orders given in Section 4, there are also inequalities relating moduli of smoothness in different L_p spaces. These inequalities together with the weak type interpolation theory of Section 3 combine to give simple proofs of the fundamental embeddings for Besov and Sobolev spaces. We continue to work in the setting of Section 4 except that now we assume that Ω is a bounded set. This is only for convenience. The case of infinite Ω has similar results with slightly different proofs. So, throughout this section, we assume that Ω has the properties given in relation (4.0).

The following lemma summarizes some of the fundamental inequalities which lie at the heart of proving weak type estimates for moduli of smoothness in different L_p spaces.

Lemma 5.1. *If r is a positive integer and $1 \leqslant p \leqslant q \leqslant \infty$, then the following hold for $\theta = n/p - n/q$ and for some $h_0 > 0$ (depending on Ω):*

(5.1) *If $g \in W_p{}^r(\Omega)$, $r \geqslant n$, $h_0 > h > 0$, then*

$$\| g \|_q \leqslant c\, h^{-\theta}\{\| g \|_p + h^r \,| g |_{p,r}\}.$$

(5.2) *If $g \in W_p{}^r(\Omega)$, $h_0 > h > 0$, $r \geqslant n$, then*

$$\omega_r(g, h)_q \leqslant c\, h^{r-\theta}\, | g |_{p,r}\,.$$

(5.3) *For each $h_0 > h > 0$ and $f \in L_1(\Omega)$, there is a g (independent of p) such that whenever f in addition belongs to $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$, then*

$$\text{(i)}\quad \| f - g \|_p \leqslant c\, \omega_r(f, h)_p$$
$$\text{(ii)}\quad | g |_{p,r} \leqslant c\, h^{-r}\omega_r(f, h)_p$$

where constants depend at most on p, q, r, n, and Ω and all norms are assumed to be taken over Ω.

In order to move more quickly to the weak type inequalities, we postpone the somewhat technical proof of this lemma until the end of this section. The next theorem gives the fundamental weak type inequality for moduli of smoothness in different L_p spaces.

THEOREM 5.2. *If $r \geqslant n$, $1 \leqslant p \leqslant q \leqslant \infty$, $t > 0$ and $f \in L_p(\Omega)$, then*

$$\omega_r(f, t)_q \leqslant c \int_0^t s^{-\theta} \omega_r(f, s)_p \frac{ds}{s} \tag{5.4}$$

with the constant independent of f and t.

Proof. For each $k \geqslant k_0$, let g_k be a function which satisfies (5.3) for $h = 2^{-k} < h_0$ and let $\psi_k = g_{k+1} - g_k$. If $f \in L_p(\Omega)$ is such that the right hand side of (5.4) is finite, then for any $j \geqslant k_0$

$$\begin{aligned} \sum_{k \geqslant j} \| \psi_k \|_q &\leqslant c \sum_{k \geqslant j} 2^{k\theta}(\| \psi_k \|_p + 2^{-kr} | \psi_k |_{p,r}) \\ &\leqslant c \sum_{k \geqslant j} 2^{k\theta} \omega_r(f, 2^{-k})_p \leqslant c \int_0^{2^{-j}} s^{-\theta} \omega_r(f, s)_p \frac{ds}{s} \end{aligned} \tag{5.5}$$

where the first inequality uses (5.1) and the second uses the triangle inequality along with (5.3). In particular, taking $j = k_0$ in (5.5) and writing $f = g_{k_0} + \sum_{k \geqslant k_0} \psi_k$, we see that $f \in L_q(\Omega)$.

Now, if $0 < t < 2^{-k_0}$, take j so that $2^{-j} \leqslant t < 2^{-j+1}$, and write $f = g_j + \sum_{k \geqslant j} \psi_k$. Then,

$$\begin{aligned} \omega_r(f, t)_q &\leqslant \omega_r(g_j, t)_q + c \sum_{k \geqslant j} \| \psi_k \|_q \\ &\leqslant c\, (t^{r-\theta} | g_j |_{p,r} + \sum_{k \geqslant j} 2^{k\theta}(\| \psi_k \|_p + 2^{-kr} | \psi_k |_{p,r})) \\ &\leqslant c\, (2^{j\theta} \omega_r(f, 2^{-j})_p + \sum_{k \geqslant j} 2^{k\theta} \omega_r(f, 2^{-k})_p) \\ &\leqslant c \int_0^t s^{-\theta} \omega_r(f, s)_p \frac{ds}{s} \end{aligned}$$

where the second sum was estimated as in (5.5) and the term g_j was estimated by (5.2) and (5.3)(ii). This proves (5.4) for $t < 2^{-k_0}$. The inequality then automatically holds for $t \geqslant 2^{-k_0}$ from standard properties of ω_r.

If we use (4.1) to replace the moduli of smoothness by the corresponding K-functionals, then Theorem 5.1 shows that the identity operator is of weak type $\sigma(r/(r - \theta), 1; \infty, r/\theta]$ for the pairs $(L_p(\Omega), W_p{}^r(\Omega))$ and $(L_q(\Omega), W_q{}^r(\Omega))$. We then have the following embeddings for the Besov spaces $B_p^{\theta,a}$ and their generalizations $B_p^{\theta,a,\alpha}$ (cf. Section 4).

COROLLARY 5.3. *Let $1 \leqslant p \leqslant q \leqslant \infty$ and $\theta = n/p - n/q$. For any $1 \leqslant a \leqslant \infty$ and $-\infty < \alpha < \infty$, we have*

(i) $B_p^{\lambda+\theta,a,\alpha} \to B_q^{\lambda,a,\alpha}$, $0 < \lambda$,

(ii) $B_p^{\theta,a,\alpha+1} \to B_q^{0,a,\alpha}$, $\alpha + 1/a > 0$,

(iii) $B_p^{\theta,1} \to L_q$.

Proof. For (i), take $r = [\lambda + \theta] + 1$ and recall that if $0 < \mu < r$, then $B_p^{\mu,a,\alpha} = (L_p, W_p^r)_{r/(r-\mu),a,\alpha}$ with equivalent norms. Hence (i) and (ii) follow from Theorem 3.3. Finally, if $f \in B_p^{\theta,1}$, then the right hand side of (5.4) is finite for all t and hence as we have shown in the proof of Theorem 5.2, this gives $f \in L_q$.

The embedding (iii) of Corollary 5.3 can be refined and this in turn will give more information as to which Besov spaces can be embedded in L_q. These refinements depend on a weak type inequality between the K-functional

$$K^*(f, t) = K(f, t; L_1(\Omega), L_\infty(\Omega))$$

and the K-functional $K_n(f, \cdot)_p$, or equivalently, $\omega_n(f, \cdot)_p$. This is given in the following theorem.

THEOREM 5.4. *If $1 < p < \infty$ and p' is the conjugate index to p, $(1/p + 1/p' = 1)$, then the identity operator is of generalized weak type $\sigma[1, p; p', \infty)$ with respect to the pairs (L_p, W_p^r), (L_1, L_∞). More precisely, for any $f \in L_1(\Omega)$, $t > 0$,*

$$\frac{K^*(f, t)}{t} \leqslant c \left(\|f\|_p + \int_{t^{1/n}}^{\infty} s^{-n/p} \omega_n(f, s)_p \frac{ds}{s} \right). \tag{5.6}$$

For $p = 1$, we have the strong inequality

$$K^*(f, t) \leqslant c\,(K_n(f, t)_1 + \min(1, t)\,\|f\|_1), \qquad t > 0. \tag{5.7}$$

Proof. Consider first the case $1 < p < \infty$. We use the same notation as in Theorem 5.3. If $2^{-j} \leqslant t < 2^{-j+1}$, and f is a function so that the right hand side of (5.6) is finite, then we decompose f as $f = g_j + (f - g_j)$. According to (5.3) we have

$$\|f - g_j\|_p \leqslant c\,\omega_n(f, 2^{-j})_p \leqslant c\,\omega_n(f, t)_p. \tag{5.8}$$

To estimate the L_∞ norm of g_j, write $g_j = \sum_{k_0}^{j} \psi_k$, with $\psi_k = g_k - g_{k-1}$ for $k > k_0$ and $\psi_{k_0} = g_{k_0}$. From (5.1) with $h = 2^{-k}$, we have

$$\|\psi_k\|_\infty \leqslant c\, 2^{kn/p} \max(\|\psi_k\|_p, 2^{-kn} \,|\psi_k|_{p,r}) \leqslant c\, 2^{kn/p} \omega_n(f, 2^{-k})_p, \tag{5.9}$$

where we used (5.3) (writing $\psi_k = (f - g_{k-1}) - (f - g_k)$). For $k = k_0$, we have

$$\|\psi_{k_0}\|_\infty = \|g_{k_0}\|_\infty \leqslant c\,(\|g_{k_0}\|_p + |g_{k_0}|_{p,r}) \leqslant c\,\|f\|_p, \tag{5.10}$$

where the first inequality used (5.1) and the second used (5.3) and the fact that $\omega_n(f, h)_p \leqslant 2^n \|f\|_p$.

The inequalities (5.9) and (5.10) combine to show that

$$\begin{aligned} \|g_j\|_\infty &\leqslant \sum_{k_0}^{j} \|\psi_k\|_\infty \leqslant c\left(\|f\|_p + \sum_{k_0+1}^{j} 2^{kn/p}\omega_n(f, 2^{-k})_p\right) \\ &\leqslant c\left(\|f\|_p + \int_t^1 s^{-n/p}\omega_n(f, s)_p \frac{ds}{s}\right) \end{aligned} \tag{5.11}$$

Now let $\psi = f - g_j$ and ψ^* denote the decreasing rearrangement of ψ. Then,

$$\begin{aligned} \frac{K^*(\psi, t^n)}{t^n} &\leqslant t^{-n}\int_0^{t^n} \psi^*(s)\, ds \leqslant (t^n)^{-1+1/p'} \left\{\int_0^{t^n} (\psi^*(s))^p\, ds\right\}^{1/p} \\ &\leqslant t^{-n/p}\|\psi^*\|_p = t^{-n/p}\|\psi\|_p \leqslant c\, t^{-n/p}\omega_n(f, t)_p \qquad (5.12) \\ &\leqslant c\int_t^\infty s^{-n/p}\omega_n(f, s)_p \frac{ds}{s}. \end{aligned}$$

Finally,

$$\begin{aligned} K^*(f, t^n) &\leqslant K^*(\psi, t^n) + K^*(g_j, t^n) \leqslant K^*(\psi, t^n) + t^n \|g_j\|_\infty \\ &\leqslant c\left(t^n\int_t^\infty s^{-n/p}\omega_n(f, s)_p \frac{ds}{s} + t^n\|f\|_p\right) \end{aligned}$$

where the last inequality uses (5.12) and (5.11). Replacing t^n by t gives the desired result (5.6) for $1 < p < \infty$.

When $p = 1$, $q = \infty$, taking $h = 2^{-k_0}$ in (5.1) gives that whenever $g \in W_1^n$, g is in L_∞ and

$$\|g\|_\infty \leqslant c\,(\|g\|_1 + |g|_{1,n}).$$

If $t > 0$, let $g \in W_1^n$ be such that

$$\|f - g\|_1 + t\,|g|_{1,n} \leqslant 2K_n(f, t)_1.$$

From the definition of K^*, we have for $0 < t < 1$

$$\begin{aligned} K^*(f, t) &\leqslant \|f - g\|_1 + t\|g\|_\infty \leqslant \|f - g\|_1 + ct(\|g\|_1 + |g|_{1,n}) \\ &\leqslant c[K_n(f, t)_1 + t\|f\|_1] \end{aligned}$$

where we used the fact that $\|g\|_1 \leqslant \|f - g\|_1 + \|f\|_1$. This shows (5.7) for $0 < t \leqslant 1$. But (5.7) holds trivially for $t > 1$, and so the theorem is proved.

COROLLARY 5.5. *If* $1 \leqslant p < q < \infty$, $1 \leqslant a \leqslant \infty$, $-\infty < \alpha < \infty$, *and* $\theta = n/p - n/q$, *then*

(i) $B_p^{\theta,a,\alpha} \to L^{q,a}(\log L)^\alpha$.

(ii) *If* $1 \leqslant p < \infty$, $1 \leqslant a \leqslant \infty$, $\alpha + 1/a < 0$, *then* $B_p^{n/p,a,\alpha+1} \to L^{\infty,a}(\log L)^\alpha$.

(iii) $B_p^{n/p,1,0} \to L^\infty$.

Proof. The embedding (i) follows from Theorem 3.2. For (ii) one need only apply Theorem (2.2) in what is by now an obvious fashion. The embeddings for $\Omega = R^n$ have a slightly different form (see 8.3).

We now turn to the proof of Lemma 5.1. In order to avoid technical difficulties we will give the proofs only for the case $\Omega = I^n$, with $I = [-1, 1]$. The same ideas carry over to the general case of Ω using arguments similar to that in [19]. The proof of (5.3) is already given in [19], since the function g constructed there does not depend on p. We therefore concentrate on the proofs of (5.1) and (5.2).

The proof of (5.1) for $p = 1$, $q = \infty$ is given in [7], so we restrict ourselves further to the remaining cases. The cube I^n can be written as the union of 2^n cubes $I_1, \dots, I_{2^n}$, with each I_k of the form $I_k = \{x\colon 0 \leqslant (-1)^{\alpha_i} x_i \leqslant 1\}$ with the α_i's either 0 or 1. We will show that for each $g \in W_p^r(I^n)$, $\epsilon > 0$, and $1 \leqslant k \leqslant 2^n$, we have

$$\| g \|_q(I_k) \leqslant c\, \epsilon^{-\theta}\{\| g \|_p(I^n) + \epsilon^r \mid g \mid_{p,r}(I^n)\}. \tag{5.13}$$

This of course gives (5.1).

In order to establish (5.13), let $J_k = \{x\colon 0 \leqslant (-1)^{\alpha_i+1} x_i \leqslant 1\}$. Thus, if $x \in I_k$ and $y \in J_k$, then $x + y \in I^n$. Further, let $T_{\epsilon,k} = \{x\colon 0 \leqslant (-1)^{\alpha_i+1} x_i < \epsilon\}$ and for $x \in I_k$ define

$$g_{\epsilon,k}(x) = \epsilon^{-n} \int_{T_{\epsilon,k}} ((-1)^{r+1} \Delta_t^r(g, x) + g(x))\, dt. \tag{5.14}$$

Now,

$$\Delta_t^r(g, x) = \int_{-\infty}^{\infty} \sum_{|\alpha|=r} D^\alpha(g, x + \xi t)\, t^\alpha M_r(\xi)\, d\xi, \tag{5.15}$$

where M_r is as before the B-spline of order r with knots $0, 1, \dots, r$. From (5.15), and Fubini's theorem we find that

$$| g(x) - g_{\epsilon,k}(x)| \leqslant \epsilon^{r-n} \int_{-\infty}^{\infty} \int_{R^n} \Big(\sum_{|\alpha|=r} | D^\alpha(g, x + \xi t)|\, \chi_{T_{\epsilon,k}}(t)\, M_r(\xi)\, dt\Big)\, d\xi. \tag{5.16}$$

For notational convenience, let us set $G_\alpha(x) = | D^\alpha(g, x)|\, \chi_{I^n}(x)$. Changing variables gives for each $x \in I_k$ and each α,

$$\int_{R^n} | D^\alpha(g, x + \xi t)|\, \chi_{T_{\epsilon,k}}(t)\, dt = \xi^{-n}(G_\alpha * h_\xi)(x).$$

with $h_\xi(x) = \chi_{T_{\epsilon\xi,k}}(-x)$. The right hand side makes sense for all x. Thus, taking an L_q norm in (5.16) and using Minkowski's inequality to take the norm inside the integral gives

$$\begin{aligned}
&\| g - g_{\epsilon,k} \|_q(I_k) \\
&\quad\leqslant \epsilon^{r-n} \int_{-\infty}^{\infty} \Big(\sum_{|\alpha|=r} \| G_\alpha * h_\xi \|_q(I_k) \Big) \xi^{-n} M_r(\xi)\, d\xi \\
&\quad\leqslant \epsilon^{r-n} \int_{-\infty}^{\infty} \Big(\sum_{|\alpha|=r} \| G_\alpha * h_\xi \|_q(R^n) \Big) \xi^{-n} M_r(\xi)\, d\xi \\
&\quad\leqslant \epsilon^{r-n} \int_{-\infty}^{\infty} \Big(\sum_{|\alpha|=r} \| G_\alpha \|_p(R^n)\, \| h_\xi \|_s(R^n)\, \xi^{-n} M_r(\xi)\, d\xi \\
&\quad\leqslant | g |_{p,r}(I^n)\, \epsilon^{r-n+n/s} \int_{-\infty}^{\infty} \xi^{-n+n/s} M_r(\xi)\, d\xi \leqslant c\epsilon^{r-\theta} \,| g |_{p,r}(I^n)
\end{aligned} \tag{5.17}$$

where in the third inequality, we used Young's inequality with $1/s = 1 + 1/q - 1/p > 0$; in the fourth inequality we used the definition of G and $\| h_\xi \|_s = (\xi\epsilon)^{n/s}$; and in the last inequality we used the fact that M_r has a zero of order $r - 1 \geqslant n - 1$ at the origin to conclude that the integral is finite.

We also need to estimate $\| g_{\epsilon,k} \|_q(I_k)$. We have

$$| g_{\epsilon,k}(x)| = \epsilon^{-n} \left| \sum_{\nu=1}^{r} \int_{T_{\epsilon,k}} (-1)^\nu \binom{r}{\nu} g(x + \nu t)\, dt \right|$$

and so with $G = g\chi_{I^n}$, $h_\nu(x) = \chi_{T_{\nu\epsilon,k}}(-x)$

$$\begin{aligned}
\left\| \int_{T_{\epsilon,k}} g(\cdot + \nu t)\, dt \right\|_q (I_k) &= \nu^{-n} \| G * h_\nu \|_q(R^n) \\
&\leqslant \nu^{-n} \| G \|_p(R^n) \| h_\nu \|_s(R^n) \\
&\leqslant \nu^{-n} \| g \|_p(I^n)\, (\nu\epsilon)^{n/s}.
\end{aligned}$$

Summing over ν, we get

$$\| g_{\epsilon,k} \|_q(I_k) \leqslant c\epsilon^{-\theta} \| g \|_p(I^n). \tag{5.18}$$

Thus the estimates (5.17) and (5.18) show that (5.1) is true.

The proof of (5.2) is very similar to that of (5.1). Again, the case $p = 1$, $q = \infty$ is argued separately, so we prove only the case $1/p - 1/q < 1$. Using the same notation as in the proof of (5.1) and letting $\tilde{I}_k = \{x; -\frac{1}{2} \leqslant (-1)^{\alpha_i} x_i \leqslant 1\}$, then the exact same estimate as in (5.17) shows that

$$\| g - g_{\epsilon,k} \|_q(\tilde{I}_k) \leqslant c\epsilon^{r-\theta} \,| g |_{p,r}(I^n). \tag{5.19}$$

In the same way that we have argued in the proof of (5.18) we can show that for $|\alpha| = r$,

$$\| D^\alpha g_{\epsilon,k} \|_q(\tilde{I}_k) \leqslant c\epsilon^{-\theta} \| D^\alpha g \|_p(I_n) \leqslant c\epsilon^{-\theta} | g |_{p,r}(I^n).$$

Since α is arbitrary, we have

$$| g_{\epsilon,k} |_{q,r}(\tilde{I}_k) \leqslant c\epsilon^{-\theta} | g |_{p,r}(I^n). \tag{5.20}$$

If we denote by $K_r(f, \cdot, J)_p$, the usual K-functional for interpolation between $L_p(J)$ and $W_p{}^r(J)$, then the inequalities (5.19) and (5.20) show that

$$K_r(f, \epsilon^r, \tilde{I}_k)_q \leqslant c\epsilon^{r-\theta} | g |_{p,r}(I^n).$$

We now use the modified subadditivity of the K-functional (see [19, p. 8]) to find

$$\begin{aligned} K_r(f, \epsilon^r, I^n)_q = K_r\left(f, \epsilon^r, \bigcup_k \tilde{I}_k\right)_q &\leqslant c \sum_{k=1}^{2^n} K_r(f, \epsilon^r, \tilde{I}_k)_q \\ &\leqslant c\epsilon^{r-\theta} | g |_{p,r} (I^n). \end{aligned}$$

Finally, using the equivalence of $K_r(f, \epsilon^r)_q$ with $\omega_r(f, \epsilon)_q$ (see (4.1)) in this last inequality gives (5.2) as desired.

6. Smoothness of the Hilbert Transform

In this section we want to consider the smoothness properties of the Hilbert transform on R:

$$Hf(x) = \text{p.v.} \frac{1}{\pi} \int f(x - y) \frac{dy}{y}. \tag{6.1}$$

It is well known that H maps $L_p(R)$ and $W_p{}^r(R)$ boundedly into themselves when $1 < p < \infty$ and therefore satisfy strong estimates. For this reason only the cases $p = 1, \infty$ are interest. On L^∞ one must redefine the Hilbert transform to get almost everywhere existence, so in order that we not obscure the ideas we only consider the case $p = 1$ and leave to Remark 8.4 the comments about generalization to other spaces, to other singular integrals, and the modifications necessary when $p = \infty$.

Before proceeding to estimate the smoothness of Hf we derive some inequalities involving the smoothness of a partition of unity applied to the kernel of the transform (6.1). Let $\{\psi_k\}_{-\infty}^{\infty}$ be a partition of unity of the line generated by a nonnegative infinitely differentiable ψ in such a manner that

$$\psi_k(y) = \psi\left(\frac{|y|}{2^k t}\right), \qquad \sum_{-\infty}^{\infty} \psi_k(y) = 1 \tag{6.2}$$

and $\{y: \psi(y) > 0\} = (\frac{1}{2}, 2)$. Notice that this forces ψ_k and its derivatives to have support in $(-2^{k+1}t, -2^{k-1}t) \cup (2^{k-1}t, 2^{k+1}t)$. Define

$$a_k(y) = \frac{\psi_k(y)}{y} \tag{6.3}$$

and

$$\begin{aligned} b_k(y) &= \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} \frac{\psi_k(y/j)}{y} \\ &= \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} \frac{1}{j}\, a_k(y/j) \end{aligned} \tag{6.4}$$

for fixed positive integer r.

Now, obviously

$$\int |a_k(y)|\, dy \leqslant 2 \int_{2^{k-1}t}^{2^{k+1}t} \psi_k(y) \frac{dy}{y} \leqslant 8 \,\|\psi\|_\infty \leqslant c \tag{6.5}$$

and thus there is a constant c independent of k so that

$$\int |b_k(y)|\, dy \leqslant c. \tag{6.6}$$

Similarly, using Leibnitz's rule of differentiation we can estimate

$$\begin{aligned} \int |D^r a_k(y)|\, dy &\leqslant \sum_{i=0}^{r} \binom{r}{i} (2^k t)^{-i} (r-i)! \int \left| D^i \psi \left(\frac{|y|}{2^k t}\right) \frac{dy}{|y|^{r-i+1}} \right. \\ &\leqslant c(2^k t)^{-r}, \qquad \text{all } k \end{aligned} \tag{6.7}$$

and so there is a c so that

$$\int |D^r b_k(y)|\, dy \leqslant c(2^k t)^{-r}, \qquad \text{all } k. \tag{6.8}$$

For each f in L_1, we let

$$H_k f(x) = f * b_k(x). \tag{6.9}$$

Since Hf exists almost everywhere and $\sum_{j=1}^{r} (-1)^{j+1}\binom{r}{j} = 1$, we have

$$Hf(x) = \sum_{k=-\infty}^{\infty} H_k f(x) \quad \text{a.e.}$$

The following lemma is an essential part of our estimation of smoothness of the Hilbert transform.

LEMMA 6.1. *There is a constant c depending only on r such that*

$$K_r(H_k f, t^r)_1 \leqslant c \min(1, 2^{-kr})\, K_r(f, (2^k t)^r)_1\,, \qquad \textit{all } k. \tag{6.10}$$

Proof. First consider the case $k \leqslant 0$. By a change of variables, we see

$$H_k f(x) = \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} \int f(x - jy)\, a_k(y)\, dy = \int -\Delta_{-y}^r f(x)\, a_k(y)\, dy$$

since $\int a_k(y)\, dy = 0$ allows us to insert the term for $j = 0$. Hence we obtain

$$\begin{aligned} K_r(H_k f, t^r)_1 &\leqslant \| H_k f \|_1 \leqslant \int | a_k(y) |\, dy \sup_{|y| \leqslant 2^{k+1} t} \| \Delta_y{}^r f \|_1 \\ &\leqslant c\, \omega_r(f, 2^{k+1} t)_1 \leqslant c\, K_r(f, (2^k t)^r)_1 \end{aligned}$$

where we have used the inequalities (6.5), $\omega_r(f, 2s) \leqslant 2^r \cdot \omega_r(f, s)$, and the equivalence inequality (4.1).

Now consider the case $k > 0$ and fix such a k. Select $f_1 \in L_1$ and $f_2 \in W_1{}^r$ so that

$$\| f_1 \|_1 + (2^k t)^r \| D^r f_2 \|_1 \leqslant 2 K_r(f, (2^k t)^r)_1\,. \tag{6.11}$$

Let $g = b_k * f_1$ and $h = b_k * f_2$, then $H_k f = g + h$. Using inequalities (6.8) and (6.11), we see that

$$\begin{aligned} K_r(g, t^r)_1 &\leqslant t^r \| D^r g \|_1 \leqslant t^r \int | D^r b_k(y) |\, dy\, \| f_1 \|_1 \\ &\leqslant c 2^{-kr} \| f_1 \|_1 \leqslant c 2^{-kr} K_r(f, (2^k t)^r)_1\,. \end{aligned} \tag{6.12}$$

Similarly, using (6.6) and (6.11), we obtain

$$\begin{aligned} K_r(h, t^r)_1 &\leqslant t^r \| D^r h \|_1 \leqslant t^r \int | b_k(y) |\, dy\, \| D^r f_2 \|_1 \\ &\leqslant c t^r \| D^r f_2 \|_1 \leqslant c 2^{-kr} K_r(f, (2^k t)^r)_1\,. \end{aligned} \tag{6.13}$$

Using the subadditivity of the modified K functional together with the estimates (6.12) and (6.13) establishes the lemma for $k > 0$.

Now using the lemma we have

$$\begin{aligned} \frac{K_r(Hf, t^r)_1}{t^r} &\leqslant \sum_{-\infty}^{\infty} \frac{K_r(H_k f, t^r)_1}{t^r} \\ &\leqslant c \sum_{-\infty}^{\infty} \frac{K_r(f, (2^k t)^r)_1}{t^r} \min(1, 2^{-kr}) \\ &\leqslant c \int_0^\infty \frac{K_r(f, s^r)_1}{s^r} \min\left(\frac{s^r}{t^r}, 1\right) \frac{ds}{s} \end{aligned} \tag{6.14}$$

since

$$K_r(f, (2^k t)^r)_1 \leqslant ct^r \int_{2^{k-1}t}^{2^k t} K_r(f, s^r)_1 \frac{ds}{s} \qquad \text{for} \quad k \leqslant 0$$

and

$$K_r(f, (2^k t)^r)_1 \leqslant c(2^k t)^r \int_{2^{k-1}t}^{2^k t} \frac{K_r(f, s^r)_1}{s^r} \frac{ds}{s} \qquad \text{for} \quad k \geqslant 1.$$

Making the change of variables $t^r \to t$ and $s^r \to s$ in (6.14), we obtain

THEOREM 6.2. *If r is a positive integer, then the Hilbert transform is of generalized weak type $\sigma(1, 1; \infty, \infty)$ for the couples (L_1, W_1^r) and (L_1, W_1^r), that is*

$$\frac{K_r(Hf, t)_1}{t} \leqslant c \int_0^\infty \frac{K_r(f, s)_1}{s} \min\left(\frac{s}{t}, 1\right) \frac{ds}{s}.$$

An immediate application of Theorem 6.2 is the following mapping result for the Hilbert transform.

COROLLARY 6.3. *If $\theta > 0$, $a \geqslant 1$, and $-\infty < \alpha < \infty$, then whenever $f \in B_1^{\theta,a,\alpha}$ the Hilbert transform Hf is also in $B_1^{\theta,a,\alpha}$. The standard loss of logarithm occurs when $\theta = 0$.*

7. FURTHER APPLICATIONS

In this section we examine two additional areas where weak type inequalities play an important role: inverse theorems of approximation and absolute convergence of Fourier transforms. We consider one example only from each area.

Interpolation theory has for some time been recognized as an important tool in the approximation of functions. It's potential was recognized early by Butzer and Berens [13], Berens [8], Butzer and Scherer [14], Peetre [24] and their collaborators. In the first part of this section we point out that the classical inverse theorem of Bernstein can be formulated as a weak type inequality involving the modulus of smoothness and error of approximation. We consider periodic functions on the unit circle and approximate by trigonometric polynomials. The error of approximation in L_p is given by

$$E(f, t)_p = \inf\{\|f - g\|_p \mid g \in \mathscr{P}, \deg(g) \leqslant [t]\} \tag{7.1}$$

where $\mathscr{P}$ is the set of trigonometric polynomials. For each f, the functionalization $E(f, \cdot)_p$ is a positive decreasing function on $(0, \infty)$ which is constant on each interval $[i, i+1)$. The approximation space $A_p^{\theta,a,\alpha}$ is defined as the set of all functions f in L_p for which the functional

$$\left\{\int_1^\infty [E(f,t)_p\, t^\theta(1+\ln t)^\alpha]^a \frac{dt}{t}\right\}^{1/a} \tag{7.2}$$

is finite, where $0 \leqslant \theta < \infty$, $-\infty < \alpha < \infty$, and $1 \leqslant a < \infty$. The usual modification is made for $a = \infty$. Actually, (7.2) is equivalent to the sequence space norm $l^{p,a}(\log l)^\alpha$ applied to $\{E(f,n)_p\}_{n=1}^\infty$.

To estimate the r-th modulus of smoothness, we need to consider

$$E_r(f,t)_p = E(f, t^{1/r})_p, \qquad t \geqslant 1 \tag{7.3}$$

which provides the necessary approximation functionalizations to carry out our discussion. We notice by a simple change of variable that the spaces $A_p^{\theta,a,\alpha}$ could be defined as the set of f in L_p where

$$\left\{\int_1^\infty [E_r(f,t)_p\, t^{\theta/r}(1+\ln t)^\alpha]^a \frac{dt}{t}\right\}^{1/a}$$

is finite. Here any integral $r > \theta$ will do, but $r = [\theta]+1$ is usually chosen.

Jackson's theorem (cf. [20, p. 56], [28, p. 260]) gives a direct estimate of the form

$$E(f, 1/t)_p \leqslant c\, \omega_r(f,t)_p, \qquad 0 < t \leqslant 1.$$

Using (4.1), we rewrite this to get

$$E_r(f, 1/t)_p \leqslant c\, K_r(f,t)_p, \qquad 0 < t \leqslant 1. \tag{7.4}$$

The "weak" converse is commonly called Bernstein's inequality

$$\frac{K_r(f,t)_p}{t} \leqslant c \int_0^{1/t} E_r(f,s)_p\, ds, \qquad 0 < t \leqslant 1 \tag{7.5}$$

(cf. [20, p. 59], [28, p. 331]). In our terminology, this is just the statement that the identity operator is of weak type $\sigma(1, \infty; \infty, 1]$ for the pairs $(L_p, \mathscr{P})_r$ and (L_p, W_p^r) where $(L_p, \mathscr{P})_r$ gives rise to the functionalization $E_r(f, \cdot)_p$. Using relations (7.4) and (7.5), we can now state:

COROLLARY 7.1. *If* $1 \leqslant a \leqslant \infty$, *then*

(i) $A_p^{\theta,a,\alpha} = B_p^{\theta,a,\alpha}$ *for* $0 \leqslant \theta$, $-\infty < \alpha < \infty$.

(ii) $A_p^{r,a,\alpha+1} \to (L_p, W_p^r)_{K,\infty,a,\alpha} \to A_p^{r,a,\alpha}$ *for* $r \geqslant 1$, $\alpha + 1/a < 0$.

(iii) $A_p^{r,1,0} \to (L_p, W_p^r)_{K,\infty,\infty,0}$ *for* $r \geqslant 1$.

Proof. The embeddings $A_p^{\theta,a,\alpha} \to B_p^{\theta,a,\alpha}$ follow from the intermediate mapping properties of S_σ (Theorem 2.1) applied to (7.5), while the first embedding in (ii) is just the endpoint result of Theorem 2.2. The remainder of the embeddings in (i) and (ii) follow from relation (7.4). Part (iii) follows directly from (7.5).

A similiar analysis can be carried out for approximation on a closed interval by algebraic polynomials (cf. [28, pp. 344, 521], [20, pp. 65, 73]) or for approximation on the line by integral functions of exponential type (cf. [28, pp. 259, 340]). Combining (7.5) with the results of Section 4, one obtains differentiability properties of functions according to their rates of approximation. Indeed, weak type inequalities relating these concepts have a long history (cf. pp. 406, 347, 365 of [28] and pp. 57, 61 of [20]).

Now we turn to our final application: the absolute convergence of Fourier transforms. We choose to work with Fourier series but the methods presented are adequate for Fourier transforms on R^n (see (7.10)). Our choice was influenced by the cumbersome nature of the endpoint results on R^n (see Theorem 3.2).

The Fourier transform of an integrable function f on $[-\pi, \pi]$ is defined as $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$ where

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)\, e^{-ins}\, ds.$$

The weak type control of the growth of the Fourier transform by smoothness properties is given by

LEMMA 7.2. *For each $f \in L_p[-\pi, \pi]$, $1 \leqslant p \leqslant 2$, and $1/p' = 1 - 1/p$, we have*

$$1/t \int_0^t (\hat{f})^*(s)\, ds \leqslant c\, 1/t \left(\|f\|_1 + \int_{1/t}^1 \frac{K_1(f, s)_p}{s}\, s^{1/p'} \frac{ds}{s} \right), \qquad t \geqslant 1. \quad (7.6)$$

Proof. We assume that $1 < p \leqslant 2$. The case $p = 1$ follows similarly. Since $\|(\hat{f})^*\|_{p'}^{p'} = \|\hat{f}\|_{p'}^{p'}$ and

$$\sum_{|k| \leqslant 2^{m-1}} |\hat{f}(k)|^{p'} \leqslant \sum_{k=0}^{2^m} [(\hat{f})^*(k)]^{p}, \qquad m \geqslant 0$$

we have

$$\sum_{2^m < k} [\hat{f}^*(k)]^{p'} \leqslant \sum_{2^{m-1} < |k|} |\hat{f}(k)|^{p'}. \tag{7.7}$$

But if $f = f_1 + f_2$ with $f_1 \in L_p$ and $f_2 \in W_p^1$, then

$$\begin{aligned}\Big(\sum_{2^{m-1} < |k|} |\hat{f}(k)|^{p'} \Big)^{1/p'} &\leqslant \|\hat{f}_1\|_{p'} + 2^{-m+1} \Big(\sum_{2^{m-1} < |k|} |k\hat{f}_2(k)|^{p'} \Big)^{1/p'} \\ &\leqslant \|f_1\|_p + 2^{-m+1} |f_2|_{p,1} \\ &\leqslant 2(\|f_1\|_p + 2^{-m} |f_2|_{p,1})\end{aligned}$$

where we've used the Hausdorff–Young theorem to get $\|\hat{f}\|_{p'} \leqslant \|f\|_p$ $(1 \leqslant p \leqslant 2)$ and the fact that $|(Df_2)\hat{}(k)| = |k\hat{f}_2(k)|$. Combining this with (7.7) and minimizing over all selections of f_1 and f_2, we get

$$\Big(\sum_{2^m<k} [(\hat{f})^*(k)]^{p'}\Big)^{1/p'} \leqslant 2K_1(f, 2^{-m})_p, \qquad m = 0, 1, 2,\dots . \tag{7.8}$$

Using Hölder's inequality and (7.8), we then obtain

$$\begin{aligned}\sum_{2^m<k\leqslant 2^{m+1}} \hat{f}^*(k) &\leqslant 2^{m/p}\Big(\sum_{2^m<k} |\hat{f}^*(k)|^{p'}\Big)^{1/p'} \\ &\leqslant c\, 2^{m/p}K_1(f, 2^{-m})_p \qquad\qquad (7.9)\\ &\leqslant c\int_{2^{-m}}^{2^{-m+1}} \frac{K_1(f, s)_p}{s}\, s^{1/p'}\, \frac{ds}{s}, \qquad m = 0, 1, 2,\dots .\end{aligned}$$

For $t > 2$, pick m_0 such that $2^{m_0} < t \leqslant 2^{m_0+1}$, then summing (7.9) from $m = 1$ to m_0, we get

$$\begin{aligned}\int_2^t (\hat{f})^*(s)\, ds \leqslant \sum_{k=2}^{2^{m_0+1}} \hat{f}^*(k) &\leqslant c\int_{2^{-m_0}}^1 \frac{K_1(f, s)_p}{s}\, s^{1/p'}\, \frac{ds}{s} + \hat{f}^*(2)\\ &\leqslant c\int_{1/t}^1 \frac{K_1(f, s)_p}{s}\, s^{1/p'}\, \frac{ds}{s} + \|f\|_1 .\end{aligned}$$

But

$$\int_0^2 (\hat{f})^*(s)\, ds = \hat{f}^*(0) + \hat{f}^*(1) \leqslant 2\, \|f\|_1 .$$

Adding these last two inequalities and dividing by t leads to inequality (7.6).

The proof which we have presented is actually an adapted proof of Bernstein's theorem due to Peetre [23]. On R^n, the weak type statement reads: for $r \geqslant n$ and $1 \leqslant p \leqslant 2$

$$\frac{K(\hat{f}, t; L_1(R^n), L_\infty(R^n))}{t} \leqslant c\, 1/t \int_{t^{-r/n}}^{\infty} \frac{K_r(f, s)_p}{s}\, s^{1-n/pr}\, \frac{ds}{s} \tag{7.10}$$

which shows that the Fourier transform is of weak type $\sigma[1, p'; pr/(pr - n), 1)$ for the pairs $(L_p(R^n), W_p{}^r(R^n))$ and $(L_1(R^n), L_\infty(R^n))$.

Applying function norms to (7.6) in the usual way produces

COROLLARY 7.3. *For $1 \leqslant p \leqslant 2$, $1/p' = 1 - 1/p$, $1 < q \leqslant p'$, and $\theta = 1/q - 1/p'$ the Fourier transform on the circle satisfies*:

(i) $\|\hat{f}\|_{q,a,\alpha} \leqslant \|\hat{f}^{**}\|_{q,a,\alpha} \leqslant c\|f\|_{B_p^{\theta,a,\alpha}}$, $1 \leqslant a \leqslant \infty$, $-\infty < \alpha < \infty$.

(ii) $\|\hat{f}\|_{1,a,\alpha} \leqslant \|\hat{f}^{**}\|_{1,a,\alpha} \leqslant c\|f\|_{B_p^{1/p,a,\alpha+1}}$, $1 \leqslant a \leqslant \infty$, $\alpha + 1/a < 0$.

(iii) $\|\hat{f}\|_1 \leqslant c\|f\|_{B_p^{1/p,1,0}}$.

Proof. The first inequalities of (i) and (ii) follow since

$$\hat{f}^*(t) \leqslant 1/t \int_0^t \hat{f}^*(s)\, ds = \hat{f}^{**}(t).$$

The second inequalities follow from the intermediate theorem 2.1 and the endpoint theorem 2.2, respectively. Part (iii) follows directly from (7.6) upon multiplying by t and taking the limit as $t \to \infty$.

One should keep in mind that the norms of the spaces on the left hand side of each of the inequalities in Corollary 7.3 are actually sequence space norms $l^{p,a}(\log l)^\alpha$. Using the fact that $\|\psi\|_{1,a,\alpha+1/a} \leqslant c\|\psi^{**}\|_{1,a,\alpha}$ for $\alpha + 1/a < 0$ (see Theorem 12.1 of [6]), we could actually strengthen part (ii) of Corollary 7.3 to read

$$(B_p^{1/p,a,\alpha+1})^\wedge \to l^{1,a}(\log l)^{\alpha+1/a}$$

where $1 \leqslant p \leqslant 2$, $1 \leqslant a \leqslant \infty$, and $\alpha + 1/a < 0$.

8. Additional Results and Remarks

Remark 8.1. The interpolation results of this paper can be extended to function norms [21] other than those of the Lorentz–Zygmund spaces. We only need to determine pairs of measurable function spaces L_{ρ_1} and L_{ρ_2} such that $S_\sigma: L_{\rho_1} \to L_{\rho_2}$ where S_σ appears in the weak type inequality (3.3). Then each operator T of generalized weak type σ will satisfy

$$\|Tf\|_{K,\rho_2} \leqslant c\|f\|_{K,\rho_1}.$$

A particular example of this situation is obtained from results of Boyd [10]: Suppose T is of generalized weak type $\sigma(p_1, p_1; p_2, p_2)$ for (X_1, X_2) and (Y_1, Y_2) then

$$T: (X_1, X_2)_{K,\rho} \to (Y_1, Y_2)_{K,\rho}$$

when the Boyd indices α_ρ and β_ρ of L_ρ satisfy

$$1/p_2 < \alpha_\rho, \qquad \beta_\rho < 1/p_1.$$

Here $\|f\|_{(X_1,X_2)_{K,\rho}} = \rho(K(f,\cdot)/(\cdot))$ and $\|\psi\|_{L_\rho} = \rho(\psi)$ (see [10, 3, 4]). Boyd gives necessary and sufficient conditions on function norms ρ such that Calderón operators S_σ of the type above map L_ρ to L_ρ. We only need to invoke the sufficiency of this theorem for our result, the proof of which is straight forward.

In the same way, we may generate other interpolation results once we know the classical mapping properties of S_σ (see [25]).

Endpoint results involving iterated logarithms could be established as well and depend upon proving the appropriate Hardy inequalities. We have avoided this type of extension since the added notational complexity would only obscure the ideas.

Remark 8.2. The use of weak type inequalities in determining embedding properties for Besov spaces and other Lipschitz type spaces has been implicit in the literature for some time. The importance of operators $S_{1\sigma}$ and $S_{2\sigma}$ in such embedding theorems is present already in A. P. Calderón [16]. Brudnyi and Shalashov [12] present embedding theorems for a variety of Lipschitz type spaces by explicitly applying function norms to both sides of Marchaud type inequalities. Again, operators of the form $S_{1\sigma}$ and $S_{2\sigma}$ play a major role in determining the embedding properties. In another work, Brudnyi [11] states an inequality which is stronger than our inequality (5.4) and remarks that embedding results involving the space B.M.O. also follow from his techniques.

Remark 8.3. For $\Omega = R^n$ in Section 5 the proofs become somewhat simpler because we can use Steklov averages in proving Lemma 5.1. In this case, the statements in Lemma 5.1 will hold for any $h > 0$. Theorem 5.2 and Corollary 5.3 carry over unchanged.

In equations (5.6) and (5.7), the terms $\|f\|_p$ and $\min(1, t)\|f\|_1$, respectively, can be omitted on the right hand sides. This may be seen by using $h = 2^{-k_0}$ in (5.1) to estimate $\|g_{k_0}\|_\infty$ in (5.10) and noting that k_0 is arbitrary. In the proof of (5.7) we use (5.1) with arbitrarily large h.

The endpoint embeddings of Corollary 5.5 now take on the variety suggested by Theorems 2.2 and 3.2. Since the endpoint $(1, p)$ is contained in σ, Theorem 3.2 does not give all the desired information. However, we can apply Theorem 2.2 and the results in Bennett and Rudnick to obtain the proper theorem. Since such an exercise is instructive and illustrates the complexity in the case of functionalizations on $[0, \infty)$, we sketch the analogue of the second half of Theorem 3.2. Applying (5.5) and Lemma 2.2 to the norm (3.6) with $p_1 = p'$, $p_2 = 1$ (equivalent to the norm of the intersection (3.7)), we obtain

$$\begin{aligned} &\left\{\int_0^1 \left[\frac{K^*(f,t)}{t}(1+|\ln t|)^\alpha\right]^a \frac{dt}{t}\right\}^{1/a} \\ &\qquad + \left\{\int_1^\infty \left[\frac{K^*(f,t)}{t}\, t^{1/p}(1+|\ln t|)^\delta\right]^d \frac{dt}{t}\right\}^{1/d} \\ &\leqslant c\left(\left\{\int_0^1 \left[\frac{K_n(f,t)_p}{t}\, t^{1/p'}(1+|\ln t|)^\beta\right]^b \frac{dt}{t}\right\}^{1/b}\right. \\ &\qquad \left. + \left\{\int_1^\infty \left[\frac{K_n(f,t)_p}{t}\, t(1+\ln t)^\delta\right]^d \frac{dt}{t}\right\}^{1/d}\right) \end{aligned} \tag{8.1}$$

for $\alpha + 1/a = \beta - 1 + 1/b < 0$, where we have used the fact that S_σ is of strong type $(1, p)$ in estimating the second term (see also [6, Theorem 9.1]). From (8.1) and the remarks following (4.6), we can deduce

$$B_p^{n/p,b,\beta} \to (L^p, W_p^n)_{K,p',b,\beta} \cap (L^p, W_p^n)_{K,1,d,\delta} \to L^{\infty,a}(\log L)^\alpha \cap L^{p,d}(\log L)^\delta \tag{8.2}$$

for $\alpha + 1/a = \beta - 1 + 1/b < 0$, $1 \leqslant d \leqslant \infty$, $\delta + 1/d < 0$. Using the fact that $L^{\infty,\infty}(\log L)^\alpha$ is the Orlicz space $\exp(L^{-1/\alpha})$ for $\alpha < 0$ [6, Theorem 10.3], (8.1) and (8.2) give in particular

$$B_p^{n/p,b,0} \to \exp(L^{1/b'}) \cap L^{p+\epsilon,d} \tag{8.3}$$

and

$$B_p^{n/p,1,0} \to B_p^{n/p,1,-\beta} \to \exp(L^{1/\beta}) \cap L^{p+\epsilon,d} \tag{8.4}$$

$1 < p < \infty$, $1 < b \leqslant \infty$, $1/b + 1/b' = 1$, $\beta > 0$, $\epsilon > 0$ and $1 \leqslant d \leqslant \infty$, where in the first embedding of (8.4) we used [6, Theorem 9.3]. After some scrutiny one can see that these results contain Theorems 4.4 and 4.2 respectively of Blozinski [9] since his class $\exp L(p, q)$ is $\exp(L^q) \cap L^{p,q}$.

The complete endpoint result generated by (5.5) in the case $\Omega = R^n$ is given in the following theorem.

THEOREM 8.1. *If* $1 < p < \infty$, *then*

(i) $B_p^{n/p,b,\beta} \to L^{\infty,a}(\log L)^\alpha \cap L^{p,d}(\log L)^\delta$ *for* $\alpha + 1/a = \beta - 1 + 1/b < 0$, $1 \leqslant d \leqslant \infty$, $\delta + 1/d < 0$;

(ii) $B_p^{0,d,\delta} \to L^{p,d}(\log L)^\delta + L^{\infty,a}(\log L)^{1/\alpha}$ *for* $\alpha + 1/a > 0$, $1 \leqslant d \leqslant \infty$, $-\infty < \delta < \infty$.

Remark 8.4. In Section 6 we determined weak type inequalities for the Hilbert transform involving $L_1(R)$ and $W_1^r(R)$. One can extend these results to Sobolev spaces $W_X^r(R)$ which are based on translation invariant function spaces X other than L_1. The only requirement is that Hf exist in the principle value sense for each $f \in X$. Along this same line one may redefine the Hilbert transform in order to get almost everywhere convergence for functions in $L_\infty(R)$:

$$Hf(x) = \lim_{\epsilon \to 0^+} \int_R f(y)[a_\epsilon(x - y) - a_1(-y)]\, dy$$

where

$$a_\epsilon(y) = \begin{cases} \dfrac{1}{y}, & |y| > \epsilon \\ 0, & \text{otherwise.} \end{cases}$$

With this definition and the proof of Theorem 6.2 one can show that the Hilbert transform is of generalized weak type $\sigma(1, 1; \infty, \infty)$ for the couples $(C, W_\infty{}^r)$ and $(C, W_\infty{}^r)$, where C is the space of continuous functions.

In another direction one can obtain weak type inequalities for smoothness of singular integral operators on other groups. For example, a natural operator on the circle group is the conjugate transform

$$\tilde{f}(x) = \text{p.v.}\ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \cot(t/2)\, dt.$$

It is a standard argument to write

$$\tilde{f} = H'f + Mf.$$

Here $H'f(x) = \text{p.v.}\ 1/\pi \int_{-\pi}^{\pi} f(x+t)\, dt/t$ and $Mf = f * \phi$ where ϕ is a continuous periodic function on $[-\pi, \pi)$. After altering slightly the partition of unity in the proof of Theorem 6.2 near $\pm\pi$, one can use the proof to establish corresponding theorems for the conjugate operator. On R^n one can establish the weak type relationships for singular integrals whose kernel is of the form $a(y) = \Omega(y)/|\, y\,|^{n+1}$ where $\Omega(y)$ is homogeneous of degree zero and satisfies the necessary differentiability properties. An obvious example would be the Riesz transforms given by

$$R_j f(x) = c_n\ \text{p.v.} \int_{R^n} f(x-y) \frac{y_j}{|\, y\,|^{n+1}}\, dy, \qquad 1 \leqslant j \leqslant n.$$

The actual weak type inequality for the conjugate function is not new. For $r = 1$ it appears in Zygmund [30, p. 121], and the general case was given by Bari and Stechkin [2] (see [28, p. 163]).

Remark 8.5. We want to consider the connections between the J- and K-functionals given in the standard development of the interpolation theory as set forth in Chapter 3 of Butzer–Berens [13]. The important equivalence and reiteration theorems follow from certain strong and generalized weak type inequalities.

The J-functional for the Banach couple (X_1, X_2) is defined by

$$J(f, t) = J(f, t; X_1, X_2) = \max\{\|f\|_{X_1}, t\,\|f\|_{X_2}\}$$

for each f in $X_1 \cap X_2$. A basic relation for the J- and K-functionals for the same Banach couple is

$$K(f, t) \leqslant \min(1, t/s)\, J(f, s). \tag{8.5}$$

The space $(X_1, X_2)_{J,p,\alpha,0}$ generated by the J-functional is the set of all elements f in $X_1 + X_2$, which have a representation $f = \int_0^\infty u(t)\, dt/t$, where $u(t)$ is a

$X_1 \cap X_2$ valued strongly measurable function on $(0, \infty)$ and the integral converges in the usual vector valued sense in the $X_1 + X_2$ norm. The norm in this space is given by

$$\|f\|_{J,p,a,0} = \inf_{f=\int_0^\infty u(t)\,dt/t} \| J(u(t), t)/t \|_{p,a,0}.$$

The equivalence theorem [13, Sect. 3.2.3] states that the spaces generated by the K- and J-functionals coincide for the same choice of parameters $1 < p < \infty$, $a \geqslant 1$. The direct inclusion

$$(X_1, X_2)_{K,p,a,0} \subset (X_1, X_2)_{J,p,a,0}, \qquad 1 < p < \infty \tag{8.6}$$

is obtained by constructing an appropriate $u(t)$ so that $f = \int_0^\infty u(t)\,dt/t$ and for which the strong inequality

$$J(u(t), t) \leqslant 4eK(f, t), \qquad 0 < t \tag{8.7}$$

holds.

In order to get the reverse inclusions to (8.6), one seeks a weak type inequality as a partial converse to (8.7). Since $K(\cdot, t)$ is a norm, using (8.5), we find that for $t > 0$

$$\begin{aligned} \frac{K(f,t)}{t} &\leqslant \frac{1}{t}\int_0^\infty K(u(s), s)\,\frac{ds}{s} \\ &\leqslant \frac{1}{t}\int_0^\infty \min(1, t/s)\, J(u(s), s)\,\frac{ds}{s} = S_\sigma\left[\frac{J(u(\cdot), \cdot)}{\cdot}\right](t) \end{aligned} \tag{8.8}$$

with $\sigma = \sigma(1, 1; \infty, \infty)$ which is the desired weak type inequality that is the converse to (8.7). Applying norms $\|\cdot\|_{p,a,0}$ to both sides of (8.8), taking an infimum over all u, and using Theorem 3.1 gives the reverse inclusions in (8.6). It is clear that the above embedding could be extended to spaces with the inclusion of a logarithm and then in this setting derive endpoint results as well.

The standard development of the K method has as a fundamental component the "reiteration" theorem. This theorem says in effect that the construction of intermediate spaces of intermediate spaces can be identified as intermediate spaces of the original pairs. This helps in the identification of intermediate spaces in classical settings and provides the essential interpolation property. We want to give a brief description of how the weak type theory comes into play at this point.

Let $X_{\theta_i} = (X_1, X_2)_{K,p_i,a,0}$ where $\theta_i = 1 - 1/p_i$, $i = 1, 2$ and $1 < p_1 < p_2 < \infty$. For the pair $(X_{\theta_1}, X_{\theta_2})$, we have the K-functional

$$\bar{K}(f, t) = K(f, t; X_{\theta_1}, X_{\theta_2}).$$

From the definition of the K-functionals, we have [13, p. 177] the strong inequality

$$K(f, t) \leqslant c\, t^{\theta_1} \bar{K}(f, t^{\theta_2-\theta_1}) \tag{8.9}$$

which holds for every f in $X_{\theta_1} + X_{\theta_2}$. This gives the direct inclusion

$$(X_{\theta_1}, X_{\theta_2})_{\bar{K},q,\alpha,0} \subseteq (X_1, X_2)_{K,p,\alpha,0} \qquad \text{for all} \quad (p, q) \in \sigma(p_1, 1; p_2, \infty). \tag{8.10}$$

As in the equivalence theorem, we look for the appropriate weak type inequality to reverse the inclusions in (8.10). This inequality can be derived from the proof in [13, Sect. 3.2.4] and (8.5) and it reads

$$\frac{\bar{K}(f, t)}{t} \leqslant c\, S_\sigma\left(\frac{K(f, \cdot)}{\cdot}\right)(t), \qquad t > 0, \tag{8.11}$$

where $\sigma = \sigma(p_1, 1; p_2, \infty)$. This shows that the inclusion in (8.10) can be reversed.

The weak type inequality (8.11) allows us to extend Calderón's theorem [15, Theorem 8] to Banach spaces (see Theorem 3.4).

Proof of Theorem 3.4. The necessity is, of course, the last remark in Theorem 2.2 with $\psi(t) = K(f, t)/t$.

If T maps $X_{K,p_i,1,0}$ to $Y_{K,q_i,\infty,0}$, $i = 1, 2$, then we have

$$\frac{K(Tf_i, t)}{t} \leqslant c\, t^{-1/q_i} \|f_i\|_{K,p_i,1,0} \tag{8.12}$$

for $f_i \in X_{K,p_i,1,0}$. Letting $X_{\theta_i} = X_{K,p_i,1,0}$, $\theta_i = 1 - 1/p_i$, and using the subadditivity and definition of the K-functional, we obtain for $f \in X_{\theta_1} + X_{\theta_2}$

$$\begin{aligned} \frac{K(Tf, t)}{t} &\leqslant c \inf_{f=f_1+f_2} \{t^{-1/q_1} \|f_1\|_{\theta_1} + t^{-1/q_2} \|f_2\|_{\theta_2}\} \\ &= c\, t^{-1/q_1} \bar{K}(f, t^{1/q_1-1/q_2}). \end{aligned} \tag{8.13}$$

Now the result follows from (8.13) by applying inequality (8.11) and changing variables.

Remark 8.6. We indicate in this remark the simplicity of multilinear interpolation for the K method [29] when viewed in terms of multilinear Calderón operators [26]. Suppose, for convenience, T is a bilinear operator from $(X_1 \cap X_2) \oplus (Y_1 \cap Y_2)$ into $Z_1 \cap Z_2$ such that

$$\| T(f, g)\|_{Z_i} \leqslant c \|f\|_{X_i} \|g\|_{Y_i} \qquad i = 1, 2.$$

It then follows that

$$J(T(u, v), rs; Z_1, Z_2) \leqslant c\, J(u, r; X_1, X_2)\, J(v, s; Y_1, Y_2)$$

for each $u \in X_1 \cap X_2$, $v \in Y_1 \cap Y_2$, and so one easily obtains

$$\frac{K(T(f, g), t)}{t} \leqslant c \int_0^\infty \int_0^\infty \frac{K(f, r)}{r} \frac{K(g, s)}{s} \min\left(\frac{rs}{t}, 1\right) \frac{dr}{r} \frac{ds}{s}$$
$$= c\, S_\sigma\left(\frac{K(f, \cdot)}{(\cdot)}, \frac{K(g, \cdot)}{(\cdot)}\right)(t)$$

where S_σ is the multilinear Calderón operator [26] corresponding to the interpolation segment $\sigma = ((1, 1), 1; (\infty, \infty), \infty)$. Now applying function norms to both sides (Theorem 3.4 of [26]) one obtains the multilinear theorem for the K method:

$$\| T(f, g)\|_{K,p,r} \leqslant c \| f \|_{K,p\ s_1} \| g \|_{K,p,s_2}$$

where $1/r = 1/s_1 + 1/s_2 - 1$. As illustrated in [26] one may even obtain the logarithmic convexity of the operator norms in the above inequality. Selections of σ other than $((1, 1), 1; (\infty, \infty), \infty)$ may also prove useful in applications when strong type initial estimates do not hold. One example appears in a special case of interpolation of several spaces: Suppose T is a bilinear operator satisfying

$$\| T(f, g)\|_{Z_1} \leqslant c \| f \|_{X_1} \| g \|_{Y_1}$$

$$\| T(f, g)\|_{Z_2} \leqslant c \| f \|_{X_1} \| g \|_{Y_2}$$

$$\| T(f, g)\|_{Z_2} \leqslant c \| f \|_{X_2} \| g \|_{Y_1}$$

where (X_1, X_2), (Y_1, Y_2) and (Z_1, Z_2) are Banach couples, then one can show that $J(T(u, v), s + t) \leqslant c\, J(u, s)\, J(v, t)$ and so

$$\frac{K(T(f, g), t)}{t} \leqslant c \int_0^\infty \int_0^\infty \frac{K(f, r)}{r} \frac{K(g, s)}{s} \min\left(r, s, \frac{rs}{t}\right) \frac{dr}{r} \frac{ds}{s}$$
$$= c\, S_\sigma\left[\frac{K(f, \cdot)}{(\cdot)}, \frac{K(g, \cdot)}{(\cdot)}\right](t)$$

where S_σ is the multilinear Calderón operator [26] for the segment $\sigma = ((1, 1), 1; (\infty, 1), \infty; (1, \infty), \infty)$. Applying function norms to this inequality will produce norm estimates [26]. One example of an operator of this type is convolution with $X_1 = Y_1 = Z_1 = L_1$ and $X_2 = Y_2 = Z_2 = L_\infty$.

Remark 8.7. At the beginning of Section 7 we mentioned that interpolation theory has long been recognized as a useful tool in approximation theory. In particular, Butzer and Scherer [14] developed an extensive theory using Jackson and Bernstein type estimates. Berens [8] has also noted that Hardy inequalities can be put to effective use in our situation. The book of Timan [28] contains weak type inequalities for many approximation processes.

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