Areas of Lattice Figures in the Planar Tilings with Congruent Regular Polygons

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For the well-known Pick theorem on the area of simple lattice polygons in the planar tiling by squares there exists an extension to nonsimple regions. All of the authors who have considered such an extension employ the Euler characteristic of the region to modify Pick's formula. In this paper we give a new formula applicable to finding the area of nonsimple polygons. In our formula the Euler characteristic is replaced by the boundary characteristic introduced by Ding and Reay. We also apply the boundary characteristic to similar Pick-type formulas in the case of the planar tilings by regular triangles and hexagons.

1. INTRODUCTION AND NOTATION

The discrete sets of vertices generated by the face-to-face tilings of the plane using regular squares, triangles, and hexagons of unit area (see Fig. 1) are denoted by $L(S)$, $L(T)$, and $L(H)$, respectively. We call uniformly the three sets lattices, although we are aware that only the sets $L(S)$ and $L(T)$ satisfy the definition of lattices in the sense of Grünbaum and Shephard [5, p. 29]. We hope that it will not cause any confusion.

Throughout the paper $V$ stands for an element of the set \{S, T, H\}. By $\mathcal{P}(V)$ we denote the set of all simple lattice polygons, that is, planar polygons whose boundary is a simple closed curve, and whose vertices lie in $L(V)$. For a polygon $P \in \mathcal{P}(V)$ we define

$$b = b(P) = |\partial P \cap L(V)| \quad \text{and} \quad i = i(P) = |\text{int} P \cap L(V)|,$$

where $\partial P$, int $P$, and $|P|$ represent the boundary of $P$, the interior of $P$, and the cardinality of $P$, respectively.

The classical theorem of Pick [1, 7, 10] asserts that if $P$ is any element of $\mathcal{P}(S)$, then the area, $A(P)$, of $P$ is given by

$$A(P) = \frac{b}{2} + i - 1. \quad (1.1)$$
It is easy to show (see [3]) that if $P \in \mathcal{P}(T)$, then
\[ A(P) = b + 2i - 2. \tag{1.2} \]

However, the example of triangles $XYZ$, $XZW$, and $XZU$ having the same $b = 3$ and $i = 0$ (see Fig. 2) shows that in $L(H)$ can be no expression for the area in terms of $b$ and $i$ only.

Ding and Reay [3, 4] have noticed that an additional parameter, called the boundary characteristic, is useful in finding the area of certain polygons from $\mathcal{P}(H)$. Let us recall the definition of the boundary characteristic. Each lattice point $x \in \partial P \cap L(V)$ is incident to exactly 4 ($V = S$), or 6 ($V = T$), or 3 ($V = H$) edges of the tiling. Each of these edges either

1. lies in $\partial P$, or
2. extends locally into the exterior of $P$ near $x$, or
3. extends locally into the interior of $P$ near $x$.

We will denote these three disjoint sets of edges by $B(x)$, $F(x)$, and $G(x)$, respectively. We will define the boundary characteristic at the boundary point $x$ as $c(x, P) = |F(x)| - |G(x)|$, and the boundary characteristic $c(P)$ of $P$ is defined by
\[ c(P) = \sum_{x \in \partial P \cap L(V)} c(x, P). \]

The boundary characteristic of any $P$ in $\mathcal{P}(S)$ and $\mathcal{P}(T)$ is equal to 8 and 12, respectively (see [3, 4]), and may vary in the case of the hexagonal

\[ A(\text{XZW}) = \frac{1}{3} \quad c(\text{XZW}) = 7 \]
\[ A(\text{XZU}) = \frac{1}{2} \quad c(\text{XZU}) = 9 \]
tiling even if $b$ and $i$ remain the same (see Fig. 2). Using the boundary characteristic Ding and Reay [3, 4], Ding, Kołodziejczyk, and Reay [2] have shown that for some classes of lattice polygons in $L(H)$ the formula

$$A(P) = \frac{b}{4} + \frac{i}{2} + \frac{c}{12} - 1$$  \hspace{1cm} (1.3)

is valid.

Formulas (1.1)–(1.3) are not true for nonsimple regions. Reeve [8] and others (see [6, 9–11]) have considered the following extension of the formula (1.1) whose range of validity covers all polygonal regions with vertices in $L(S)$:

$$A(P) = \frac{b}{2} + i - \chi(P) + \frac{1}{2} \chi(\partial P),$$  \hspace{1cm} (1.4)

where $\chi(P)$ and $\chi(\partial P)$ denote the Euler characteristic of $P$ and $\partial P$, respectively. It is fairly evident that analogous formula applicable to polygonal regions with vertices in $L(T)$ has the form

$$A(P) = b + 2i - 2\chi(P) + \chi(\partial P).$$  \hspace{1cm} (1.5)

The whole information about the area of polygons from $\mathcal{P}(S)$ and $\mathcal{P}(T)$ is focused at lattice points and in this sense formulas (1.1) and (1.2) are of purely combinatorial character. This character seems to be—to a certain extent—lost in formulas (1.4) and (1.5) which need additional, partly topological, information. The aim of this paper is to replace the Euler characteristic by a purely combinatorial parameter proper to a given tiling. It turns out that in the case of $L(S)$ and $L(T)$ the boundary characteristic can be applied as the desired parameter; namely, we show that the formula

$$A(P) = \frac{b}{2} + i - \frac{c}{8}$$  \hspace{1cm} (1.6)

is valid for all polygonal regions with vertices in $L(S)$ and the formula

$$A(P) = b + 2i - \frac{c}{6}$$  \hspace{1cm} (1.7)

\footnote{$\chi(\partial P) = W - E$, $\chi(P) = W - E' + F$, where $W$ and $E$ stand for the number of vertices and edges of $P$, while $E'$ and $F$ denote the number of edges and triangles of a decomposition of $P$ into triangles.}
holds for polygonal regions with vertices in $L(T)$. As to the hexagonal tiling we first prove the formula

$$A(P) = \frac{b}{4} + \frac{i}{2} + \frac{c}{12} - \chi(P) - \frac{1}{2} \chi(\partial P)$$ (1.8)

and next we use it to obtain another one involving centres of the hexagons.

2. LATTICE REGIONS

By $R(V)$ we denote the class of all lattice regions, i.e., polygonal figures which can be expressed as the union of finitely many such elements from $\mathcal{P}(V)$ such that if two edges of a region $R$ intersect, their intersection belongs to $L(V)$. We show that formulas (1.6), (1.7), and (1.8) measure areas of figures from $\mathcal{R}(S)$, $\mathcal{R}(T)$, and $\mathcal{R}(H)$, respectively. Our method of proving these formulas employs special description of the class $R(V)$ given below.

If $P \in R(V)$ then the degree of a vertex $x \in \partial P \cap L(V)$, denoted by $\deg x$, is the number of all edges of $P$ adjacent to $x$. Obviously $\deg x$ is an even number. A vertex $x$ is said to be a node if $\deg x > 2$. The degree of a region $P$, denoted by $D(P)$, is defined by

$$D(P) = \max \{ \deg x : x \in \partial P \cap L(V) \}.$$

We clearly have $D(P) = 2$ for each $P$ in $\mathcal{R}(V)$. One can observe the following characterization.

**Proposition.** A simply-connected lattice polygon $P$ is simple if and only if $D(P) = 2$.

By $R_k(V)$ we denote the family of all simply-connected regions $P$ for which $D(P) \leq 2k$. Thus we have $R_1(V) = \mathcal{P}(V)$ and $R_k(V) \subset R_{k'}(V)$ for $k \leq k'$.  

![Fig. 3. A region in $R(S)$ with a hole which is not a simple polygon.](image-url)
Using this notation we can describe the class of all simply-connected regions in the following way

$$\mathcal{R}_N(V) = \bigcup \{ \mathcal{R}_k(V) : k \in \mathbb{N} \}.$$  

Now by \( \mathcal{R}_{N,t}(V) \), \( t \geq 0 \), we denote the class of elements in \( \mathcal{R}(V) \) having at most \( t \) holes. Note that a hole need not be a simple lattice polygon as it was often assumed (see the polygon \( M \) in Fig. 3).

We end this section with the decomposition of the class of nonsimple lattice polygons

$$\mathcal{R}(V) = \bigcup \{ \mathcal{R}_{N,t}(V) : t \in \mathbb{N} \},$$  

which will be very useful in our method of proving formulas (1.6), (1.7), and (1.8).

3. AREAS OF REGIONS IN \( L(S) \)

In this section we prove the validity of formula (1.6). We show it in several steps, beginning with the lemma.

**Lemma 1.** If formula (1.6) holds for lattice regions \( P_1 \) and \( P_2 \) such that \( P_1 \cap P_2 = \{ y \} \), \( y \in L(S) \), then it also holds for the region \( P_1 \cup P_2 \).

**Proof.** First we observe the following three obvious relations

$$A(P_1 \cup P_2) = A(P_1) + A(P_2) \quad (3.1)$$
$$i(P_1 \cup P_2) = i(P_1) + i(P_2) \quad (3.2)$$
$$b(P_1 \cup P_2) = b(P_1) + b(P_2) - 1. \quad (3.3)$$

To prove the lemma we need also the relation between \( c(P_1) \), \( c(P_2) \), and \( c(P_1 \cup P_2) \). It is clear that we will have it if we know the relation between \( c(y, P_1) \), \( c(y, P_2) \), and \( c(y, P_1 \cup P_2) \). So, we prove now that the required relation has the form

$$c(y, P_1) + c(y, P_2) - 4 = c(y, P_1 \cup P_2).$$

To this end it is enough to check that the equality

$$e(y, P_1) + e(y, P_2) - 1 = e(y, P_1 \cup P_2) \quad (3.4)$$

is true, where \( e(y, P) \) denotes the contribution of the tiling edge \( e \) adjacent
to y to \( c(y, P) \). Because of the symmetric role of \( P_1 \) and \( P_2 \) it is enough to consider the following three possibilities, either

\[
\begin{align*}
  \begin{cases}
    e(y, P_1) &= 1 \\
    e(y, P_2) &= 1 \\
    e(y, P_1 \cup P_2) &= 1 
  \end{cases}
  \quad \text{and} \quad
  \begin{cases}
    e(y, P_1) &= 1 \\
    e(y, P_2) &= 0 \\
    e(y, P_1 \cup P_2) &= 0 
  \end{cases}
\]

In either case (3.4) holds; hence we have

\[ c(P_1) + c(P_2) - 4 = c(P_1 \cup P_2). \]  

(3.5)

Now substituting relations (3.2), (3.3), and (3.5) into formula (1.6) and using (3.1) we obtain the desired result. 1

**Lemma 2.** Formula (1.6) is valid for \( \mathcal{P}_S \).

**Proof.** The proof proceeds by induction. As we have already mentioned, the boundary characteristic is equal to 8 for any element \( P \) of \( \mathcal{P}_S = \mathcal{P}(S) \). Thus the validity of formula (1.6) for \( \mathcal{P}_S \) simply follows from Pick's theorem. Suppose formula (1.6) is valid for \( \mathcal{P}_k \), \( k \geq 1 \), and consider the class \( \mathcal{P}_{k+1} \). Now let \( m = m(P) \) be the number of all such nodes \( z \) of a region \( P \in \mathcal{P}_{k+1} \) such that \( \deg z = D(P) \). If for some \( P, m(P) = 0 \), then by the induction hypothesis we have that (1.6) can be applied to \( P \). We again use induction (but now on \( m \)) to establish the lemma for \( \mathcal{P}_{k+1} \). Suppose formula (1.6) is true for each \( P \) with \( m(P) < m_0, m_0 > 0 \), and take a region \( P \) with \( m(P) = m_0 + 1 \). Let \( y_0 \) be one of the nodes for which \( \deg y_0 = 2k + 2 \). Since \( 2k + 2 \geq 4 \), and since \( P \) is simply-connected, there are two regions \( P_1 \) and \( P_2 \) such that \( P_1 \cup P_2 = P \) and \( P_1 \cap P_2 = \{ y_0 \} \). Obviously \( m(P_i) \leq m_0 \), \( i = 1, 2 \), so by induction (on \( m \)) we have that formula (1.6) can be applied to both \( P_1 \) and \( P_2 \). Now Lemma 1 implies that formula (1.6) holds for \( P \) and, in consequence, for \( \mathcal{P}_{k+1} \).

**Lemma 3.** If formula (1.6) is true for any polygon with simple holes, then it is also true for polygons with nonsimple holes.

**Proof.** We use induction on the number \( k \) of nonsimple holes. The lemma is obviously true for \( k = 0 \). Suppose it holds for \( k_0 > 0 \) and we take a region \( P \) with \( k_0 + 1 \) nonsimple holes. Let \( M \) be one of the holes. Since \( M \) is nonsimple it has a hole, say \( P_1 \), which is a part of \( P \) (see Fig. 3). Thus polygons \( P' = \text{cl}(P \setminus P_1) \) and \( P_1 \) have at most \( k_0 \) nonsimple holes and,
by induction, formula (1.6) can be applied to them. The validity of formula (1.6) for \( P = P_1 \cup P' \) follows from Lemma 1.

Now we can formulate the main result of this section.

**Theorem 1.** The area of any element of \( \mathcal{R}(S) \) is given by

\[
A(P) = \frac{b}{2} + i - \frac{c}{8},
\]

(1.6)

*Proof.* The validity of formula (1.6) for the class \( \mathcal{R}_{n,0}(S) = \mathcal{R}_{n}(S) \) follows from Lemma 2. Suppose formula (1.6) is true for \( \mathcal{R}_{n,t}(S) \), \( t \geq 0 \), and we will show its validity for \( \mathcal{R}_{n,t+1}(S) \). Take any region \( P \) with \( t + 1 \) holes and consider a new polygon \( R = P \cup M \), where \( M \) is one of the holes of \( P \) (by virtue of Lemma 3 we can assume that \( M \) is a simple polygon). Obviously \( M \) and \( R \) belong to \( \mathcal{P}(S) \) and \( \mathcal{R}_{n,t}(S) \), respectively. Our induction hypothesis implies formula (1.6) can be applied to \( R \) and \( M \) (Because of later references we will write \( c(M) \) instead of \( \mathcal{R} \)).

Let \( Z(M) \) denote the set of all vertices of \( P \) which do not belong to \( \partial R \). Using this notation we have the relations

\[
b(R) = b(P) - |Z(M)|
\]

(3.6)

and

\[
i(R) = i(P) + i(M) + |Z(M)|.
\]

(3.7)

Now we show that formula (1.6) is true for \( P \) if and only if

\[
c(P) + c(M) - c(R) = 4(b(M) - |Z(M)|).
\]

(3.8)

Indeed, applying (1.6) to \( R \) and \( M \) and using relations (3.6) and (3.7) we obtain

\[
A(P) - A(M) = \cdots = \frac{b(P)}{2} + i(P) - \frac{c(P)}{8} + \frac{c(M)}{8} + \frac{c(M)}{8} - \frac{c(R)}{2} - \frac{b(M)}{2} + \frac{|Z(M)|}{2}.
\]

This means that formula (1.6) is true for \( P \) iff

\[
\frac{c(P)}{8} + \frac{c(M)}{8} - \frac{c(R)}{8} - \frac{b(M)}{2} + \frac{|Z(M)|}{2} = 0,
\]

iff (3.8) holds. So, to complete the proof it remains to establish (3.8).
For convenience we define \( c(z, P) \) to be zero whenever \( z \notin \partial P \). Now traveling around the boundary of \( P \) we check what the contribution of each vertex \( z \) to the left side of (3.8) is. We divide all vertices of \( P \) into three groups. This leads us to consider the following three cases.

**Case 1.** \( z \in Z(M) \). Thus obviously \( c(z, P) = -c(z, M) \) and \( c(z, R) = 0 \). Hence for each such vertex \( z \) we have

\[
c(z, P) + c(z, M) - c(z, R) = 0. \tag{3.9}
\]

**Case 2.** \( z \in \partial R \setminus \partial M \). In this case (3.9) also holds since we have \( c(z, M) = 0 \) and \( c(z, P) = c(z, R) \).

**Case 3.** \( z \in \partial R \cap \partial M \). This is the case when \( z \) is a common vertex of \( P, R, \) and \( M \). The number of such vertices is \( b(M) - |Z(M)| \). Thus to prove (3.8) it is sufficient to show that in each such vertex \( z \) we have

\[
c(z, P) + c(z, M) - c(z, R) = 4. \]

This, in turn, will be true, if we show that for each tiling edge \( e \) at \( z \) the equality

\[
e(z, P) + e(z, M) - e(z, R) = 1
\]

holds. The last equality can be verified similarly as the equality (3.4) and details are omitted. Hence (3.8) is established and the proof is completed.

### 4. Areas of Figures in \( L(T) \)

In our introduction we have mentioned two formulas applicable to lattice figures in \( L(T) \). Formula (1.5) is a simple consequence of an application of formula (1.4) and therefore seems to be of a little less interest than formula (1.7). The latter has its own, strictly connected with the triangular tiling, purely combinatorial character.

**Theorem 2.** If \( P \in A(T) \), then the area of \( P \) is given by

\[
A(P) = b + 2i - c/6. \tag{1.7}
\]

**Proof.** The proof of Theorem 2 goes just in the same way as the proof of formula (1.6). At the beginning we notice that formula (1.7) coincides with formula (1.2) for \( A(T) \). Further we first show its validity for \( A_n(T) \)
and next for $\mathcal{R}(T)$. The only changes concern the boundary characteristic
and are the following: instead of (3.5) and (3.8) we use

$$c(P_1) + c(P_2) - 6 = c(P_1 \cup P_2)$$

and

$$c(P) + c(M) - c(R) = 6(b(M) - |Z(M)|),$$

respectively, which are proved similarly as (3.5) and (3.8) in Section 3.

Comparing formulas (1.4) with (1.6) and (1.5) with (1.7) we get relations
between the Euler characteristic and the boundary characteristic of regions
from $\mathcal{R}(S)$ and $\mathcal{R}(T)$. We formulate the relations, the existence of which
seems to be a bit surprising, in the corollary below.

**Corollary 1.** For any $P$ in $\mathcal{R}(S)$

$$c(P) = 8\chi(P) - 4\chi(\partial P),$$

and, if $P$ belongs to $\mathcal{R}(T)$, then

$$c(P) = 12\chi(P) - 6\chi(\partial P).$$

5. **Areas of Honeycombs**

Ding and Reay [3] have begun the study of finding an analog of Pick's
theorem for $L(H)$, where the area cannot be determined in terms of $b$ and
$i$ only (see Fig. 2). They have noted that the boundary characteristic
becomes a useful parameter in finding polygonal areas with vertices in
$L(H)$. Namely, they have shown that formula (1.3) is valid for polygons
whose boundary consists of diagonals and/or edges of hexagons in $L(H)$.
Broader classes of polygons satisfying formula (1.3) are given in [2] and in
a forthcoming paper by J. Reay and the author.

Instead of the class $\mathcal{P}(H)$ we will consider in this section only its sub-
class $\mathcal{P}^*(H)$ of polygons satisfying formula (1.3). Remaining classes from
Section 2 with an asterisk will denote respective unions (in the sense as in
Section 2) of elements from $\mathcal{P}^*(H)$.

We now come to the main theorem of this section which gives the area
of hexagonal lattice polygons from $\mathcal{R}^*(H)$ by means of formula (1.8). In
the case of $L(H)$ we "have to put up with" the appearance of the Euler
characteristic part in our formula because we do not know how it could be
replaced.
LEMMA 4. The area of any region $P$ in $\mathcal{B}_n^*(H)$ is given by

$$A(P) = \frac{b}{4} + \frac{i}{2} + \frac{c}{12} - 1 + \frac{1}{2} \chi(\partial P).$$

Note that this means that formula (1.8) is valid for $\mathcal{B}_n^*(H)$ because for each region $P \in \mathcal{B}_n^*(H)$ we have $\chi(P) = 1$.

Proof of Lemma 4. The proof goes through as the proof of Lemma 1. The only changes are caused by the boundary characteristic and by the appearance of the Euler characteristic in this case. It is enough to replace (3.5) by

$$c(P_1) + c(P_2) - 3 = c(P_1 \cup P_2)$$

and additionally use the relation

$$\chi(\partial P_1) + \chi(\partial P_2) - 1 = \chi(\partial(P_1 \cup P_2)),$$

which is simply verified. \[\square\]

THEOREM 3. If $P \in \mathcal{B}_*(H)$, then

$$A(P) = \frac{b}{4} + \frac{i}{2} + \frac{c}{12} - \chi(P) + \frac{1}{2} \chi(\partial P). \quad (1.8)$$

Proof. We proceed similarly as in the proof of Theorem 1. Lemma 4 implies that formula (1.8) is true for $\mathcal{B}_n^*(H)$. Suppose the formula is true for $\mathcal{B}_n^*(H)$, $t \geq 0$, and take a region $P$ from $\mathcal{B}_n^*(H)$ having $t + 1$ holes. If $M$ is one of the holes (for the same reason as in Section 3 we can assume that $M$ is a simple polygon), take a new polygon $R = P \cup M$. By the induction hypothesis we can apply formula (1.8) to $R$ and $M$ (strictly speaking, we apply (1.3) to $M$). Note that relations (3.6) and (3.7) remain true. We will also need the connections

$$\chi(R) = \chi(P) + 1 \quad (5.1)$$

and

$$\chi(\partial R) = \chi(\partial P) + b(M) - |Z(M)|. \quad (5.2)$$

Equation (5.1) easily follows from the fact that the Euler characteristic of a region with $t$ holes is equal to $1 - t$. To check (5.2) we note that because of (3.6) it is enough to observe that by adding $M$ to $P$, the number of edges of $P$ decreases by $b(M)$. 

Substituting relations (3.6), (3.7), (5.1), and (5.2) into formula (1.8) and using (3.1) we can reveal that formula (1.8) is valid for $P$ if and only if
\[ c(P) + c(M) - c(R) = 3(b(M) - |Z(M)|). \]
The last relation resembles (3.8) and is verified in a very similar way. This completes the proof.

Note that our method can be also applied to proving formulas (1.4) and (1.5). The virtue of this method is that it does not employ the notion of *primitive triangles* (ones whose only lattice points are their vertices) which in the case of $L(H)$ have different areas ($\frac{1}{6}$, $\frac{1}{3}$, or $\frac{1}{2}$).

Now we are going to obtain an alternative form for Pick’s theorem in $L(H)$. We remove the Euler characteristic from the formula for $A(P)$ by embedding the hexagonal lattice $L(H)$ in a triangular lattice $L_W(T)$, obtained by adding the auxiliary set $W$ of the centres of each hexagon. Following [3], we let:

\[ h = \text{the number of auxiliary points in int } P; \quad h = |\text{int } P \cap W|, \]
\[ m = \text{the number of auxiliary points on } \partial P; \quad m = |\partial P \cap W|, \]
\[ b' = b + m = |\partial P \cap L_W(T)|, \]
\[ r' = r + h = |\text{int } P \cap L_W(T)|. \]

We are ready to give another Pick’s theorem on the hexagonal tiling in which Euler characteristics do not appear.

**Theorem 4.** If $P \in \mathcal{R}^*(H)$, then
\[ A(P) = \frac{1}{8} (2b' - b) + \frac{1}{4} (2i' - i) - \frac{c}{24} = \frac{(b + 2m)}{8} + \frac{(i + 2h)}{4} - \frac{c}{24}. \]

**Proof.** Let $P$ be an element of $\mathcal{R}^*(H)$ then its area, in virtue of Theorem 3, is given by (1.8). On the other hand, treating $P$ as a polygon with vertices in $L_W(T)$ (here each triangle of the tiling has area $\frac{1}{6}$), we can apply an apparent modification of formula (1.5). Thus we obtain
\[ A(P) = \frac{1}{6} (b' + 2i' - 2\chi(P) + \chi(\partial P)). \quad (5.3) \]
Using Eq. (1.8) and (5.3) we can eliminate $\chi(P)$ and $\chi(\partial P)$. This leads to the desired result.

We conclude by indicating two restricted classes of nonsimple polygons in $\mathcal{R}^*(H)$, the area of which is a function of only the boundary charac-
teristic, and the number of lattice points in the interior and on the boundary. We give them in the following corollary which simply follows from Theorem 4.

**Corollary 2.** Let $P$ be a region in $\mathbb{R}^*(H)$; then

1. If $P$ misses the set $W$ of auxiliary points ($P \cap W = \emptyset$), then

   \[ A(P) = \frac{b}{8} + \frac{i}{4} - \frac{c}{24}. \]

2. If $P$ has no interior lattice points and, moreover, $P \cap W = \emptyset$, then

   \[ A(P) = \frac{b}{8} - \frac{c}{24}. \]

**References**