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Stability of an integro-differential equation☆

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ABSTRACT

In this work we study a scalar integro-differential equation and give some new conditions ensuring that the zero solution is asymptotically stable by means of the fixed-point theory. Our work extends and improves the results in the literature.

1. Introduction

In this work, we consider the following integro-differential equation

$$x'(t) = -\int_{t-r(t)}^{t} a(t,s)x(s)\mathrm{d}s \tag{1.1}$$

as well as the nonlinear analogue

$$x'(t) = -\int_{t-r(t)}^{t} a(t,s)g(x(s))\mathrm{d}s \tag{1.2}$$

for $t \geq 0$, where $r : [0,\infty) \to [0,\infty)$ is differentiable with $t - r(t) \to \infty$ as $t \to \infty$, $a : [-r_0,\infty) \times [-r_0,\infty) \to R$, where $r_0 = \max_{t\geq 0}\{t - r(t)\}$, $g : R \to R$ are continuous functions and $xg(x) > 0$ if $x \neq 0$ is sufficiently small.

Many authors (see [1–5] and the references quoted therein) have studied the stability of the above two equations and their special forms. Recently, Burton [1] considered the case that $r(t) = r$ is a constant. Becker and Burton [2] investigated (1.1) and (1.2) with the variable delay $r(t)$ under the assumption that the function $t - r(t)$ is strictly increasing. In the present work, however, we rewrite (1.1) and (1.2) in a fashion different from that in [1,2] and introduce a function $v(t)$ in constructing a a fixed-point mapping. Consequently, we eliminate the condition that $t-r(t)$ be strictly increasing and obtain less restrictive conditions for stability. For such a technique being used, our results extend and improve the results in [1,2]. We mainly use the fixed-point theory, which has been effectively employed to study the stability of functional differential equations with variable delays [6–9].

The rest of this paper is organized as follows. In Section 2 we consider the linear equation and in Section 3 we consider the nonlinear equation. In Section 4, we give some remarks and examples to illustrate that our results are stronger than that in [1,2].

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2. The linear equation

In order to find a new fixed-point mapping for (1.1), we write (1.1) as

$$x'(t) = B(t, t - r(t))(1 - r'(t))x(t - r(t)) + \frac{d}{dt}\int_{t-r(t)}^{t} B(t, s)x(s)ds, \tag{2.1}$$

where

$$B(t, s) := \int_t^s a(u, s)du, \quad \text{with } B(t, t - r(t)) = \int_t^{t-r(t)} a(u, t - r(t))du. \tag{2.2}$$

Lemma 2.1. *If $x(t)$ is a solution of (1.1) on an interval $[0, T)$ and satisfies the initial condition $x(t) = \psi(t)$ for $t \in [-r_0, 0]$, then $x(t)$ is a solution of the integral equation*

$$\begin{aligned} x(t) &= e^{-\int_0^t v(s)ds}\psi(0) - e^{-\int_0^t v(u)du}\int_{-r(0)}^{0}[v(u) + B(0, u)]\psi(u)du \\ &\quad + \int_{t-r(t)}^{t}[v(u) + B(t, u)]x(u)du - \int_0^t e^{-\int_s^t v(u)du}v(s)\int_{s-r(s)}^{s}[v(u) + B(s, u)]x(u)duds \\ &\quad + \int_0^t e^{-\int_s^t v(u)du}[v(s - r(s)) + B(s, s - r(s))](1 - r'(s))x(s - r(s))ds \end{aligned} \tag{2.3}$$

on $[0, T)$, where $v : [-r_0, \infty) \to R$ is an arbitrary continuous function. Conversely, if a continuous function $x(t)$ is equal to $\psi(t)$ for $t \in [-r_0, 0]$ and is a solution of (2.3) on an interval $[0, \tau)$, then $x(t)$ is a solution of (1.1) on $[0, \tau)$.

Proof. Multiplying both sides of (2.1) by the factor $e^{\int_0^t v(u)du}$ and integrating from 0 to any $t \in [0, T)$, we get

$$\begin{aligned} x(t) &= e^{-\int_0^t v(s)ds}\psi(0) + \int_0^t e^{-\int_s^t v(u)du}v(s)x(s)ds + \int_0^t e^{-\int_s^t v(u)du}\frac{d}{ds}\int_{s-r(s)}^{s} B(s, u)x(u)duds \\ &\quad + \int_0^t e^{-\int_s^t v(u)du}B(s, s - r(s))(1 - r'(s))x(s - r(s))ds. \end{aligned}$$

Performing an integration by parts, we have

$$\begin{aligned} x(t) &= e^{-\int_0^t v(s)ds}\psi(0) + \int_0^t e^{-\int_s^t v(u)du}\frac{d}{ds}\int_{s-r(s)}^{s}[v(u) + B(s, u)]x(u)duds \\ &\quad + \int_0^t e^{-\int_s^t v(u)du}[v(s - r(s)) + B(s, s - r(s))](1 - r'(s))x(s - r(s))ds \\ &= e^{-\int_0^t v(s)ds}\psi(0) + e^{-\int_s^t v(u)du}\int_{s-r(s)}^{s}[v(u) + B(s, u)]x(u)du\Big|_0^t \\ &\quad - \int_0^t e^{-\int_s^t v(u)du}v(s)\int_{s-r(s)}^{s}[v(u) + B(s, u)]x(u)duds \\ &\quad + \int_0^t e^{-\int_s^t v(u)du}[v(s - r(s)) + B(s, s - r(s))](1 - r'(s))x(s - r(s))ds, \end{aligned}$$

which leads to (2.3). Conversely, suppose that a continuous function $x(t)$ is equal to $\psi(t)$ on $[-r_0, 0]$ and satisfies (2.3) on an interval $[0, \tau)$. Then it is differentiable on $[0, \tau)$. Differentiating (2.3) with the aid of Leibniz's rule, we obtain (2.1). $\square$

Next, we will define a mapping directly from (2.3). By Lemma 2.1, a fixed point of that map will be a solution of (2.3) and (1.1). To obtain stability of the zero solution of (1.1), we need the mapping defined by (2.3) to map bounded functions into bounded functions. Let $(C, \|\cdot\|)$ be the set of real-valued bounded continuous functions on $[-r_0, \infty)$ with the supremum norm $\|\cdot\|$; that is, for $\phi \in C$,

$$\|\phi\| := \sup\{|\phi(t)| : t \in [-r_0, \infty)\}.$$

In other words, we carry out our investigations in the complete metric space (C, ρ), where ρ denotes the supremum (uniform) metric: for $\phi_1, \phi_2 \in C$,

$$\rho(\phi_1, \phi_2) = \|\phi_1 - \phi_2\|.$$

For a given continuous initial function $\psi : [-r_0, 0] \to R$, define the set $C_\psi \subset C$ by

$$C_\psi := \{\phi : [-r_0, \infty) \to R | \phi \in C, \phi(t) = \psi(t) \quad \text{for } t \in [-r_0, 0]\}.$$

Let $\|\cdot\|$ denote the supremum on $[-r_0, 0]$ or on $[-r_0, \infty)$. Finally, note that $(C_\psi, \|\cdot\|)$ is itself a complete metric space since C_ψ is a closed subset of C.

Lemma 2.2. *Let $v : [-r_0, \infty) \to R$ be a continuous function and P be a mapping on C_ψ as follows: for $\phi \in C_\psi$,*

$$(P\phi)(t) = \psi(t)$$

if $t \in [-r_0, 0]$, while

$$\begin{aligned}(P\phi)(t) = {} & \mathrm{e}^{-\int_0^t v(s)\mathrm{d}s}\psi(0) - \mathrm{e}^{-\int_0^t v(u)\mathrm{d}u}\int_{-r(0)}^0 [v(u) + B(0, u)]\psi(u)\mathrm{d}u + \int_{t-r(t)}^t [v(u) + B(t, u)]\phi(u)\mathrm{d}u \\ & - \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u} v(s)\int_{s-r(s)}^s [v(u) + B(s, u)]\phi(u)\mathrm{d}u\mathrm{d}s \\ & + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}[v(s - r(s)) + B(s, s - r(s))](1 - r'(s))\phi(s - r(s))\mathrm{d}s \end{aligned} \tag{2.4}$$

if $t > 0$. Suppose that there exist constants $k \ge 0$ and $\alpha > 0$ such that

$$-\int_0^t v(s)\mathrm{d}s \le k \tag{2.5}$$

and

$$\begin{aligned}&\int_{t-r(t)}^t |v(u) + B(t, u)|\mathrm{d}u + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}|v(s)|\int_{s-r(s)}^s |v(u) + B(s, u)|\mathrm{d}u\mathrm{d}s \\ &\quad + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}|v(s - r(s)) + B(s, s - r(s))||1 - r'(s)|\mathrm{d}s \le \alpha\end{aligned} \tag{2.6}$$

for $t \ge 0$, then $P : C_\psi \to C_\psi$.

Proof. For $\phi \in C_\psi$, $P\phi$ is continuous and agrees with ψ on $[-r_0, 0]$ by virtue of the definition of P. For $t > 0$, it follows from (2.5) and (2.6) that

$$|(P\phi)(t)| \le \mathrm{e}^k|\psi(0)| + \mathrm{e}^k\int_{-r(0)}^0 |v(u) + B(0, u)||\psi(u)|\mathrm{d}u + \alpha\|\phi\|.$$

Consequently,

$$\|P\phi\| \le \mathrm{e}^k\|\psi\|\left(1 + \int_{-r(0)}^0 |v(u) + B(0, u)|\mathrm{d}u\right) + \alpha\|\phi\| < \infty. \tag{2.7}$$

Thus, $P\phi \in C_\psi$. □

Definition 2.1. The zero solution of (1.1) is said to be stable at $t = 0$ if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\psi : [-r_0, 0] \to (-\delta, \delta)$ implies that $|x(t)| < \varepsilon$ for $t \ge -r_0$.

Theorem 2.1. *Suppose that there exist constants $k \ge 0$, $\alpha \in (0, 1)$ and a continuous function $v : [-r_0, \infty) \to R$ such that (2.5) and (2.6) hold for $t \ge 0$. Then for each continuous function $\psi : [-r_0, 0] \to R$, there is an unique continuous function $x : [-r_0, \infty) \to R$ with $x(t) = \psi(t)$ on $[-r_0, 0]$ that satisfies (1.1) on $[0, \infty)$. Moreover, $x(t)$ is bounded on $[-r_0, \infty)$. Furthermore, the zero solution of (1.1) is stable at $t = 0$. If, in addition,*

$$\int_0^t v(s)\mathrm{d}s \to \infty \tag{2.8}$$

as $t \to \infty$, then $x(t) \to 0$ as $t \to \infty$.

Proof. Consider the space C_ψ defined by the continuous initial function $\psi : [-r_0, 0] \to R$. For $\phi, \eta \in C_\psi$,

$$\begin{aligned}|(P\phi)(t) - (P\eta)(t)| &\le \int_{t-r(t)}^{t} |v(u) + B(t,u)||\phi(u) - \eta(u)|\mathrm{d}u \\ &\quad + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}|v(s)| \int_{s-r(s)}^{s} |v(u) + B(s,u)||\phi(u) - \eta(u)|\mathrm{d}u\mathrm{d}s \\ &\quad + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}|v(s - r(s)) + B(s, s - r(s))||1 - r'(s)||\phi(s - r(s)) - \eta(s - r(s))|\mathrm{d}s \\ &\le \left(\int_{t-r(t)}^{t} |v(u) + B(t,u)|\mathrm{d}u + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}|v(s)| \int_{s-r(s)}^{s} |v(u) + B(s,u)|\mathrm{d}u\mathrm{d}s \right. \\ &\quad \left. + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}|v(s - r(s)) + B(s, s - r(s))||1 - r'(s)|\mathrm{d}s\right)\|\phi - \eta\|. \end{aligned} \tag{2.9}$$

For $t > 0$. By the definition of P and (2.6), P is a contraction mapping with contraction constant α. By Banach's contraction mapping principle, P has a unique fixed point x in C_ψ which is a bounded continuous function. By Lemma 2.1, it is a solution of (1.1) on $[0, \infty)$. It follows that x is the only bounded continuous function satisfying (1.1) on $[0, \infty)$ and the initial condition. Similarly to the method in [2], we can show that (1.1) does not have any unbounded continuous solutions.

It is clear that the zero solution of (1.1) is stable. If $x(t)$ is a solution with the initial function ψ, by (2.7), we have

$$(1 - \alpha)\|x\| \le \mathrm{e}^k \|\psi\| \left(1 + \int_{-r(0)}^{0} |v(u) + B(0,u)|\mathrm{d}u\right).$$

Then for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x(t)| < \varepsilon$ for all $t \ge -r_0$ if $\|\psi\| < \delta$.

Next we prove that the solution of (1.1) tends to zero when (2.8) holds. First we define a subset of C_ψ as follows:

$$C_\psi^0 := \{\phi : [-r_0, \infty) \to R | \phi \in C, \phi(t) = \psi(t) \text{ for } t \in [-r_0, 0]\ \phi(t) \to 0 \text{ as } t \to \infty\}. \tag{2.10}$$

Since C_ψ^0 is a closed subset of C_ψ and (C_ψ, ρ) is complete, the metric space (C_ψ^0, ρ) is also complete. Now we show that $(P\phi)(t) \to 0$ as $t \to \infty$ when $\phi \in C_\psi^0$. By (2.4) and (2.6), we have

$$|(P\phi)(t)| \le \mathrm{e}^{-\int_0^t v(s)\mathrm{d}s}\left(|\psi(0)| + \int_{-r(0)}^{0} |v(u) + B(0,u)|\mathrm{d}u\right) + \alpha\|\phi\|_{[t-r(t),t]} + |I_4| + |I_5|$$

for $t > 0$, where I_4, I_5 denote the last two terms of (2.4), respectively. We can prove that each of the above terms tend to zero as $t \to \infty$. In fact, it is easy to see that the first term tends to 0 by (2.8) and the second term approaches zero as $t \to \infty$ since $t - r(t) \to \infty$. For each $\varepsilon > 0$, there exists a $T > 0$ such that

$$\|\phi\|_{[T-r(T),\infty)} < \varepsilon/2\alpha$$

since $t - r(t) \to \infty$ as $t \to \infty$. Thus, for $t \ge T$,

$$\begin{aligned}|I_4| &\le \int_0^T |v(s)\mathrm{e}^{-\int_s^T v(u)\mathrm{d}u} \int_{s-r(s)}^{s} |v(u) + B(s,u)|\mathrm{d}u\mathrm{d}s\|\phi\|\mathrm{e}^{-\int_T^t v(u)\mathrm{d}u} \\ &\quad + \int_T^t |v(s)\mathrm{e}^{-\int_s^t v(u)\mathrm{d}u} \int_{s-r(s)}^{s} |v(u) + B(s,u)|\mathrm{d}u\mathrm{d}s\|\phi\|_{[T-r(T),\infty)}.\end{aligned}$$

By (2.8), there exists a $\tau \ge T$ such that $\|\phi\|\mathrm{e}^{-\int_T^t v(u)\mathrm{d}u} < \varepsilon/2\alpha$ for $t > \tau$. Thus, for every $\varepsilon > 0$, there exists a $\tau > 0$ such that $t > \tau$ implies $I_4 < \varepsilon$; that is, $I_4 \to 0$ as $t \to \infty$. Similarly, we can show that I_5 tends to zero as $t \to \infty$. This yields $(P\phi)(t) \to 0$ as $t \to \infty$, and hence $P : C_\psi^0 \to C_\psi^0$. Therefore, P is a contraction on C_ψ^0 with a unique fixed point x. By Lemma 2.1, x is a solution of (1.1) on $[0, \infty)$. Hence, $x(t)$ is the only continuous solution of (1.1) agreeing with the initial function ψ. As $x \in C_\psi^0$, $x(t) \to 0$ as $t \to \infty$. □

3. The nonlinear equation

The nonlinear equation (1.2) is written as

$$x'(t) = B(t, t - r(t))(1 - r'(t))g(x(t - r(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\int_{t-r(t)}^{t} B(t,s)g(x(s))\mathrm{d}s. \tag{3.1}$$

Lemma 3.1. *Let $\psi : [-r_0, 0] \to R$ be a given continuous initial function. If $x(t)$ is a solution of (1.2) on an interval $[0, T)$ with $x(t) = \psi(t)$ for $t \in [-r_0, 0]$, then $x(t)$ is a solution of the integral equation*

$$\begin{aligned} x(t) = {} & \mathrm{e}^{-\int_0^t v(s)\mathrm{d}s}\psi(0) - \mathrm{e}^{-\int_0^t v(u)\mathrm{d}u}\int_{-r(0)}^{0}[v(u)+B(0,u)]g(\psi(u))\mathrm{d}u + \int_{t-r(t)}^{t}[v(u)+B(t,u)]g(x(u))\mathrm{d}u \\ & - \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}v(s)\int_{s-r(s)}^{s}[v(u)+B(s,u)]g(x(u))\mathrm{d}u\mathrm{d}s + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}[v(s-r(s)) \\ & + B(s,s-r(s))](1-r'(s))g(x(s-r(s)))\mathrm{d}s + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}v(s)[x(s)-g(x(s))]\mathrm{d}s. \end{aligned} \tag{3.2}$$

where $v : [-r_0, \infty) \to R$ is an arbitrary continuous function. Conversely, if a continuous function $x(t)$ is equal to $\psi(t)$ for $t \in [-r_0, 0]$ and is a solution of (3.2) on an interval $[0, \tau)$, then $x(t)$ is a solution of (1.2) on $[0, \tau)$.

Proof. Multiplying both sides of (3.1) by the factor $\mathrm{e}^{\int_0^t v(u)\mathrm{d}u}$ and integrating from 0 to any $t \in [0, T)$, we obtain

$$\begin{aligned} x(t) = {} & \mathrm{e}^{-\int_0^t v(s)\mathrm{d}s}\psi(0) + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}v(s)x(s)\mathrm{d}s + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}\frac{\mathrm{d}}{\mathrm{d}s}\int_{s-r(s)}^{s}B(s,u)x(u)\mathrm{d}u\mathrm{d}s \\ & + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}B(s,s-r(s))(1-r'(s))x(s-r(s))\mathrm{d}s \\ = {} & \mathrm{e}^{-\int_0^t v(s)\mathrm{d}s}\psi(0) + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}v(s)x(s)\mathrm{d}s + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}\frac{\mathrm{d}}{\mathrm{d}s}\int_{s-r(s)}^{s}[v(u)+B(s,u)]g(x(u))\mathrm{d}u\mathrm{d}s \\ & + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}B(s,s-r(s))(1-r'(s))g(x(s-r(s)))\mathrm{d}s - \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}\frac{\mathrm{d}}{\mathrm{d}s}\int_{s-r(s)}^{s}v(u)g(x(u))\mathrm{d}u\mathrm{d}s. \end{aligned}$$

Then an integration by parts yields (3.2). Conversely, suppose that a continuous function $x(t)$ is equal to $\psi(t)$ on $[-r_0, 0]$ and satisfies (3.2) on an interval $[0, \tau)$. Then it is differentiable on $[0, \tau)$. Differentiating (3.2) with the aid of Leibniz's rule, we obtain (3.1). □

Define

$$C_\psi^l := \{\phi : [-r_0, \infty) \to R \mid \phi \in C, \phi(t) = \psi(t) \text{ for } t \in [-r_0, 0],\ |\phi(t)| \le l\}, \tag{3.3}$$

where $\psi : [-r_0, 0] \to [-l, l]$ is a given continuous initial function.

Lemma 3.2. *Let $v : [-r_0, \infty) \to R$ be a continuous function and a mapping P be defined on C_ψ^l as follows: for $\phi \in C_\psi^l$,*

$$(P\phi)(t) = \psi(t) \quad \text{for } t \in [-r_0, 0];$$

while for $t > 0$

$$\begin{aligned} (P\phi)(t) = {} & \mathrm{e}^{-\int_0^t v(s)\mathrm{d}s}\psi(0) - \mathrm{e}^{-\int_0^t v(u)\mathrm{d}u}\int_{-r(0)}^{0}[v(u)+B(0,u)]g(\psi(u))\mathrm{d}u + \int_{t-r(t)}^{t}[v(u)+B(t,u)]g(\phi(u))\mathrm{d}u \\ & - \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}v(s)\int_{s-r(s)}^{s}[v(u)+B(s,u)]g(\phi(u))\mathrm{d}u\mathrm{d}s \\ & + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}[v(s-r(s))+B(s,s-r(s))](1-r'(s))g(\phi(s-r(s)))\mathrm{d}s \\ & + \int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}v(s)[\phi(s)-g(\phi(s))]\mathrm{d}s. \end{aligned} \tag{3.4}$$

Suppose that:

(i) *there exists a constant $l > 0$ such that g satisfies a Lipschitz condition on $[-l, l]$;*
(ii) *$v(t) \ge 0$ for $t \ge 0$;*
(iii) *there exists a continuous function q such that $|B(t, u)| \le q(u)$ for $t - r(t) \le u \le t$.*

Then there is a metric d for C_ψ^l such that:

(iv) *the metric space (C_ψ^l, d) is complete, and*
(v) *P has a contraction on (C_ψ^l, d) if P maps C_ψ^l into itself.*

Proof. By (i) we can choose a common Lipschitz constant L for $g(x)$ and $x-g(x)$ on $[-l,l]$. For $t\in[-r_0,\infty)$ and a constant $k>4$, define

$$h(t)=kL\int_0^t[v(u)+q(u)+\omega(u)]\mathrm{d}u,$$

where

$$\omega(u)=\begin{cases}0, & \text{if } u\in[-r_0,0]\\ |v(u-r(u))+B(u,u-r(u))(1-r'(u))|, & \text{if } u\in[0,\infty).\end{cases}$$

Now let $\mathcal{S}$ be the space of all continuous functions $\phi:[-r_0,\infty)\to R$ such that

$$|\phi|_h:=\sup\{|\phi(t)|\mathrm{e}^{-h(t)}:t\in[-r(0),\infty)\}<\infty.$$

It is clear that $(\mathcal{S},|\cdot|_h)$ is a Banach space. Thus $(\mathcal{S},d)$ is a complete metric space, where d denotes the induced metric: $d(\phi,\eta)=|\phi-\eta|_h$ for $\phi,\eta\in\mathcal{S}$. Since C_ψ^l is a closed subset of $\mathcal{S}$ with this metric, the metric space (C_ψ^l,d) is complete and the proof of (iv) is complete.

As for (v), since $P:C_\psi^l\to C_\psi^l$ and g satisfies a Lipschitz condition on $[-l,l]$, we can obtain, for $\phi,\eta\in C_\psi^l$

$$\begin{aligned}|(P\phi)(t)-(P\eta)(t)|\mathrm{e}^{-h(t)}\le&\int_{t-r(t)}^t|v(u)+B(t,u)|L|\phi(u)-\eta(u)|\mathrm{e}^{-h(t)+h(u)-h(u)}\mathrm{d}u\\&+\int_0^t\mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}v(s)\int_{s-r(s)}^s|v(u)+B(s,u)|L|\phi(u)-\eta(u)|\mathrm{e}^{-h(t)+h(u)-h(u)}\mathrm{d}u\mathrm{d}s\\&+\int_0^t\mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}\omega(s)L|\phi(s-r(s))-\eta(s-r(s))|\mathrm{e}^{-h(t)+h(s-r(s))-h(s-r(s))}\mathrm{d}s\\&+\int_0^t\mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}v(s)L|\phi(s)-\eta(s)|\mathrm{e}^{-h(t)+h(s)-h(s)}\mathrm{d}s\\\le&\int_{t-r(t)}^t\mathrm{e}^{-kL\int_u^t[v(\theta)+q(\theta)]\mathrm{d}\theta}|v(u)+B(t,u)|L|\phi(u)-\eta(u)|\mathrm{e}^{-h(u)}\mathrm{d}u\\&+\int_0^t\mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}v(s)\int_{s-r(s)}^s\mathrm{e}^{-kL\int_u^s[v(\theta)+q(\theta)]\mathrm{d}\theta}|v(u)+B(s,u)|L|\phi(u)-\eta(u)|\mathrm{e}^{-h(u)}\mathrm{d}u\mathrm{d}s\\&+\int_0^t\mathrm{e}^{-kL\int_s^t\omega(u)\mathrm{d}u}\omega(s)L|\phi(s-r(s))-\eta(s-r(s))|\mathrm{e}^{-h(s-r(s))}\mathrm{d}s\\&+\int_0^t\mathrm{e}^{-(kL+1)\int_s^t v(u)\mathrm{d}u}v(s)L|\phi(s)-\eta(s)|\mathrm{e}^{-h(s)}\mathrm{d}s.\end{aligned}$$

By (iii), we have

$$|v(u)+B(t,u)|\le v(u)+q(u)$$

for $t-r(t)\le u\le t$. Consequently,

$$|(P\phi)(t)-(P\eta)(t)|\mathrm{e}^{-h(t)}\le\left(\frac{1}{kL}+\frac{1}{kL}+\frac{1}{kL}+\frac{1}{kL+1}\right)L|\phi-\eta|_h\le\frac{4}{k}|\phi-\eta|_h \tag{3.5}$$

for all $t>0$. Thus $d(P\phi,P\eta)\le(4/k)d(\phi,\eta)$. Since $k>4$, we conclude that P is a contraction on (C_ψ^l,d). □

Theorem 3.1. *Suppose g, v and B satisfy conditions* (i)–(iii) *in Lemma 3.2 and further suppose that:*

(i) *g is odd and strictly increasing on $[-l,l]$;*
(ii) *$x-g(x)$ is non-decreasing on $[0,l]$;*
(iii) *there exists an $\alpha\in(0,1)$ such that, for $t\ge0$*

$$\begin{aligned}&\int_{t-r(t)}^t|v(u)+B(t,u)|\mathrm{d}u+\int_0^t\mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}|v(s)|\int_{s-r(s)}^s|v(u)+B(s,u)|\mathrm{d}u\mathrm{d}s\\&\quad+\int_0^t\mathrm{e}^{-\int_s^t v(u)\mathrm{d}u}|v(s-r(s))+B(s,s-r(s))||1-r'(s)|\mathrm{d}s\le\alpha.\end{aligned}$$

Then a $\delta\in(0,l)$ exists such that, for each continuous function $\psi:[-r_0,0]\to(-\delta,\delta)$, there is a unique continuous function $x:[-r_0,\infty)\to R$ with $x(t)=\psi(t)$ on $[-r_0,0]$, which is a solution of (1.2) on $[0,\infty)$. Moreover, $x(t)$ is bounded by l on $[-r_0,\infty)$. Furthermore, the zero solution of (1.2) is stable at $t=0$.

Proof. Since g is odd and satisfies the Lipschitz condition on $[-l, l]$, $g(0) = 0$ and g is (uniformly) continuous on $[-l, l]$. Thus we can choose a δ that satisfies the inequality

$$\delta + g(\delta)\int_{-r(0)}^{0} |v(u) + B(0, u)|\mathrm{d}u \le (1-\alpha)g(l). \tag{3.6}$$

Let $\psi : [-r_0, 0] \to (-\delta, \delta)$ be a continuous function. Note that (3.6) implies $\delta < l$ since $g(l) \le l$ by (ii). Thus, $|\psi(t)| < l$ for $-r_0 \le t \le 0$. Now we show that for such a ψ, $P : C_\psi^l \to C_\psi^l$. In fact, for an arbitrary $\phi \in C_\psi^l$, it follows from conditions (i) and (ii) that

$$\begin{aligned}|(P\phi)(t)| \le{} & \delta + g(\delta)\int_{-r(0)}^{0} |v(u) + B(0, u)|\mathrm{d}u + g(l)\int_{t-r(t)}^{t} |v(u) + B(t, u)|\mathrm{d}u \\ & + g(l)\int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u} v(s)\int_{s-r(s)}^{s} |v(u) + B(s, u)|\mathrm{d}u\mathrm{d}s \\ & + g(l)\int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u} |v(s-r(s)) + B(s, s-r(s))||1-r'(s)|\mathrm{d}s + (l - g(l))\int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u} v(s)\mathrm{d}s\end{aligned}$$

for $t > 0$. By (iii) and (3.6), this implies

$$\begin{aligned}|(P\phi)(t)| &\le \delta + g(\delta)\int_{-r(0)}^{0} |v(u) + B(0, u)|\mathrm{d}u + \alpha g(l) + l - g(l) \\ &\le (1-\alpha)g(l) + (\alpha - 1)g(l) + l = l.\end{aligned}$$

Hence, $|(P\phi)(t)| \le l$ for $t \in [-r_0, \infty)$ since $|(P\phi)(t)| = |\psi(t)| < l$ for $t \in [-r_0, 0]$. Therefore, $P\phi \in C_\psi^l$. By Lemma 3.2, P is a contraction on the complete metric space (C_ψ^l, d). Then P has a unique fixed point $x \in C_\psi^l$, which is a solution of (1.2) on $[0, \infty)$ by Lemma 3.1 and $|x(t)| \le l$ for all $t \ge -r_0$. Hence, $x(t)$ is the only continuous function satisfying (1.2) for $t \ge 0$ and with $x(t) = \psi(t)$ for $-r_0 \le t \le 0$.

To obtain stability at $t = 0$, let $\varepsilon > 0$ be given and choose $m > 0$ so that $m < \min\{\varepsilon, l\}$. Replacing l with m beginning with (3.6), we see there is a $\delta > 0$ such that $\|\psi\| < \delta$, which implies that the unique continuous solution x agreeing with ψ on $[-r_0, 0]$ satisfies $|x(t)| \le m < \varepsilon$ for all $t \ge -r_0$. □

Definition 3.1. The zero solution of (1.2) is asymptotically stable if it is stable at $t = 0$ and a δ exists such that for any continuous function $\psi : [-r_0, 0] \to (-\delta, \delta)$, the solution $x(t)$ with $x(t) = \psi(t)$ on $[-r_0, 0]$ tends to zero as $t \to \infty$.

The following theorem provides the asymptotic stability of Eq. (1.2). The proof is similar to that of Theorem 3.13 [2] and hence, we omit it.

Theorem 3.2. *Suppose that all of the conditions in Lemma 3.2 and Theorem 3.1 hold. Furthermore, suppose that g is continuously differentiable on $[-l, l]$ and $g'(0) \neq 0$. If $\int_0^t v(s)\mathrm{d}s \to \infty$ as $t \to \infty$, then the zero solution of (1.2) is asymptotically stable.*

4. Remarks and examples

Remark 4.1. The work of Becker and Burton in [2] requires that $t - r(t)$ be strictly increasing. However, in the present work, this condition is removed.

Remark 4.2. The conditions in [2], parallel to (2.6) and (2.8), are

$$\int_{t-r(t)}^{t} |G(t, u)|\mathrm{d}u + \int_0^t \mathrm{e}^{-\int_s^t G(u,u)\mathrm{d}u}|G(s, u)|\mathrm{d}u\mathrm{d}s \le \alpha \tag{4.1}$$

and

$$\int_0^t G(s, s)\mathrm{d}s \to \infty \quad \text{as } t \to \infty, \tag{4.2}$$

where $G(t, s) = \int_t^{f(s)} a(u, s)\mathrm{d}u$, $f(t)$ is the inverse of $t - r(t)$. Choosing $\mathrm{v}(\mathrm{s}) = \mathrm{G}(\mathrm{s}, \mathrm{s})$, Theorems 2.1 and 3.1 reduce to Theorem 3.3 and 3.10 of [2], respectively. So our results generalize and improve those of [1,2]. See Example 4.1.

Example 4.1. Consider the equation

$$x'(t) = -\int_{0.4635t}^{t} \frac{1}{s^2+1} x(s)\mathrm{d}s. \tag{4.3}$$

Following the notation in Remark 4.2, we have $f(t) = \frac{t}{0.4635}$, then

$$G(t,s) = \int_t^{s/0.4635} \frac{1}{s^2+1}\mathrm{d}u = \frac{s/0.4635 - t}{s^2+1}$$

for $t \geq 0$ and $0.4635t \leq s \leq t$. Consequently,

$$\lim_{t\geq 0} \left\{ \int_{0.4635t}^t |G(t,u)|\mathrm{d}u + \int_0^t \mathrm{e}^{-\int_s^t G(u,u)\mathrm{d}u} |G(s,s)| \int_{0.4635s}^s |G(s,u)|\mathrm{d}u\mathrm{d}s \right\} = 2\left(-\frac{\ln 0.4635 + 1}{0.4635} + 1\right) = 1.003,$$

Then there exists some $t_0 > 0$ such that

$$\int_{0.4635t}^t |G(t,u)|\mathrm{d}u + \int_0^t \mathrm{e}^{-\int_s^t G(u,u)\mathrm{d}u} |G(s,s)| \int_{0.4635s}^s |G(s,u)|\mathrm{d}u\mathrm{d}s > 1.002$$

for $t \geq t_0$. This implies that condition (4.1) does not hold. Thus, Theorem 3.3 in [2] cannot be applied to Eq. (4.3).

However, By (2.2),

$$B(t,s) = \int_t^s \frac{1}{s^2+1}\mathrm{d}u = \frac{s-t}{s^2+1}.$$

Choosing $v(t) = \frac{t}{t^2+1}$, clearly, condition (2.8) holds. Furthermore, we have

$$\begin{aligned}
\int_{t-r(t)}^t |v(u) + B(t,u)|\mathrm{d}u &= \int_{0.4635t}^t \left| \frac{2u-t}{u^2+1} \right| \mathrm{d}u \\
&= \int_{0.4635t}^{0.5t} \frac{t-2u}{u^2+1}\mathrm{d}u + \int_{0.5t}^t \frac{2u-t}{u^2+1}\mathrm{d}u \\
&= t(2\arctan 0.5t - \arctan t - \arctan 0.4635t) + \ln(t^2+1) \\
&\quad + \ln(0.4635^2 t^2 + 1) - 2\ln(0.25t^2+1) \\
&=: w(t).
\end{aligned}$$

Since the function $w(t)$ is increasing in $[0, \infty)$ and

$$\lim_{t\to\infty} w(t) = 1/0.4635 - 3 + 2\ln 2 + 2\ln 0.927 = 0.3992,$$

then

$$\int_{t-r(t)}^t |v(u) + B(t,u)|\mathrm{d}u < 0.3992,$$

$$\begin{aligned}
\int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u} |v(s-r(s)) + B(s,s-r(s))||1-r'(s)|\mathrm{d}s &= (1/0.4635 - 2)\int_0^t \mathrm{e}^{-\int_s^t \frac{u}{u^2+1}\mathrm{d}u} \frac{s}{s^2 + 1/0.4635^2}\mathrm{d}s \\
&< 1/0.4635 - 2 = 0.1575,
\end{aligned}$$

and

$$\int_0^t \mathrm{e}^{-\int_s^t v(u)\mathrm{d}u} |v(s)| \int_{s-r(s)}^s |v(u) + B(s,u)|\mathrm{d}u\mathrm{d}s < 0.3992.$$

Let $\alpha := 0.3992 + 0.1575 + 0.3992 = 0.9559 < 1$, then the zero solution of (4.3) is asymptotically stable by Theorem 2.1.

5. Conclusion

In this work, a scalar integro-differential equation has been studied. Some sufficient conditions to ensure that the zero solution is asymptotically stable have been established. These obtained results extend and improve the results in [1,2]. Moreover, an example is given to illustrate our results.

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