A manifold $X$ has an almost-complex structure if there is a vector 1-form $J$ over $X$, with the property $J \cdot J = -I$. Almost-complex structures are a generalization of complex structures.

The theory of scalar and vector forms as developed in Part I admits certain refinements in the case of almost-complex structures. Scalar forms are bi-graded rather than just graded (the bi-degree is usually called the type), and vector forms can be classified even more finely. The derivations on differential forms admit now a finer classification; there are the types $i'_* \cdot i'_*, d'_* \cdot d'_*$; and each of these are bi-graded. The decompositions of the commutators become more involved than before, especially when the torsion $T = \frac{1}{2}[J, J]$ of the structure does not vanish.

The identities for scalar and vector forms derived in Part I can now be decomposed into parts of the various types, in which also the new exterior derivatives of vector forms play an important part. Establishing these new identities is the main purpose of this paper. Some of them have already been used in the case of complex manifolds in order to develop a cohomology theory with various cohomology operations.

A change in the notation of Part I has proven more practical, especially in lengthy computations: for $\omega \cdot L$ and $M \cdot L$ we will now often simply write $\omega L$ and $ML$, or $\omega \cdot L$ and $M \cdot L$. To eliminate a number of brackets we further introduce as a convention that $\alpha L$ and $ML$ are one unit with respect to other operations, hence $d\omega L = d(\omega L)$, $\mathcal{D}''ML = \mathcal{D}''(ML)$, but the operators $d, \mathcal{D}'$, etc. will have precedence over the dot; hence $d\omega \cdot L = (d\omega)L$; $\mathcal{D}''M \cdot L = (\mathcal{D}''M)L$; and also $\omega L \cdot M = (\omega L)M$, while

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1) Part I appeared in this Journal [Proceedings A.59, Indagationes 18, No. 3, pp. 338–359 (1956)], and contains § 1–6. Bibliographical references are in square brackets; those numbered below 15 were first given in Part I.

2) This has been published independently of this paper in [15] and has, in [16], been applied to stability of complex structures. Further applications to the study of certain problems in almost-complex manifolds are being investigated.
The implication that vectors satisfy fiber formal and (7.1) forms. Fields, as derivation by mappings of almost-complex w·LM are vector usual and formula complex by almost-product w(LM), etc.

The terminology used in the following sections is typically that for almost-complex structures. The reader will notice, however, that from formula (7.2) on everything (except examples) applies to any (real or complex) almost-product structure.

§ 7. Types of scalar and vector forms. Throughout this paper it will be necessary to deal with complex scalar and vector forms.

Methodically, the simplest approach is to replace in Part I the ring  \( F \) of real-valued \( C^\infty \) functions by the ring \( \bar{F} \) of complex-valued \( C^\infty \) functions, and the field \( R \) of real numbers by the field \( C \) of complex numbers. A complex tangent vector field \( u \) is a mapping \( u: \bar{F} \to \bar{F} \) satisfying the usual rules.

A complex vector field \( u \) is real if \( u(f) \in F \) for all \( f \in F \). A complex vector field \( u \) acting on a real function \( f \in F \) gives \( u(f)=f'+if'' \). The mappings \( u': F \to \bar{F} \); \( u'': F \to F \) by \( u'(f)=f' \); \( u''(f)=f'' \) are easily seen to be derivations on \( F \). This shows that \( u=u'+iu'' \), where \( u' \) and \( u'' \) are real. Conversely, let \( u: F \to F \) be a real vector field, as defined in Part I. Then \( u \) can be extended to a derivation \( u: \bar{F} \to \bar{F} \) by \( u(f+ig)=u(f)+iu(g) \). It follows now easily:

Proposition (7.1). Every real vector field can be extended to a derivation \( u: \bar{F} \to \bar{F} \); and every complex vector field \( u \) is representable as \( u=u'+iu'' \); where \( u' \) and \( u'' \) are real. If \( u' \) and \( u'' \) are real vector fields, then \( u=u'+iu'' \) is a complex vector field.

In a similar way one defines complex scalar forms and complex vector forms. The vector spaces, bundles, and modules so obtained are denoted by \( \bar{T}_x(X), \bar{T}(X), \bar{T}^{*(x)}(X), \bar{\phi}_q, \bar{V}_x, \bar{\psi}_t, \bar{\phi} \), etc. One easily shows:

Proposition (7.2). The complexifications of the vector spaces, fiber bundles, modules of tangent vector fields, scalar and vector forms satisfy

\[
\begin{align*}
\bar{T}_x(X) &= T_x(X) \otimes_R C; \quad \bar{\psi}_0 = \psi_0 \otimes_R C; \\
\bar{T}^{*(x)}(X) &= T^{*(x)}(X) \otimes_R C; \quad \bar{\phi}_q = \phi_q \otimes_R C; \\
\bar{V}_x &= V_x \otimes_R C; \quad \bar{\psi}_t = \psi_t \otimes_R C;
\end{align*}
\]

and the operations \( \wedge, \wedge, \partial, [\cdot, \cdot] \) on the complexified objects obtained by formal extension from the real ones satisfy the formal identities stated in Part I §§ 2, 3, 4, 5, 6; in particular, the Theorems I, II; and (6.15).

Let now \( J \) be an almost-complex structure over \( X \); i.e., \( J \) is a real vector 1-form over \( X \), satisfying \( JJ = -I \). The existence of such a \( J \) implies certain topological restrictions for the manifold \( X \); in particular,
$X$ has to be even-dimensional. The form $J$ represents an endomorphism of $\mathcal{T}(X)$ which carries $T(X)$ onto itself. If the manifold is a complex one, then the complex structure determines a field $J$ as follows. Let $z^1, \ldots, z^n$ be complex coordinates, and $z^a = x^a + iy^a$. Then $x^1, \ldots, x^n, y^1, \ldots, y^n$ are real coordinates, and every tangent vector of $\mathcal{T}(X)$ is a linear combination of

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}$$

with complex coefficients. For real vectors the coefficients are real. Using

$$\frac{\partial}{\partial z^a} = \frac{1}{2} \left( \frac{\partial}{\partial x^a} - i \frac{\partial}{\partial y^a} \right), \quad \frac{\partial}{\partial \bar{z}^a} = \frac{1}{2} \left( \frac{\partial}{\partial x^a} + i \frac{\partial}{\partial y^a} \right)$$

as a new basis, every vector $v \in \mathcal{T}_z(X)$ can be written as

$$v = \sum_a \left( v^a \frac{\partial}{\partial z^a} + \bar{v}^a \frac{\partial}{\partial \bar{z}^a} \right),$$

with complex $v^a$, $\bar{v}^a$; and the real vectors are characterized by $\bar{v}^a = v^a$. The mapping $v \to Jv$ is now defined by its action on the basis:

$$J \frac{\partial}{\partial z^a} = i \frac{\partial}{\partial x^a}, \quad J \frac{\partial}{\partial \bar{z}^a} = -i \frac{\partial}{\partial x^a}.$$

It obviously carries real vectors into real vectors and satisfies $JJ = -I$. For a proof that $J$ is determined independent of the choice of the $z^a$, and that, conversely, an arbitrary almost-complex structure determines at most one complex structure, we refer to [13].

From $J$ one derives the following two complex vector 1-forms:

$$P = \frac{1}{2}(I - iJ), \quad Q = \frac{1}{2}(I + iJ)$$

which are projection operators because $PP = P$, $QQ = Q$, and they are complementary because $P + Q = I$, $PQ = QP = 0$. The operators $P$, $Q$ determine a direct sum decomposition of $\mathcal{T}_z(X)$:

$$\mathcal{T}_z(X) = T'_z(X) + T''_z(X), \quad T'_z(X) = P_z \mathcal{T}_z(X), \quad T''_z(X) = Q_z \mathcal{T}_z(X).$$

$T'_z(X)$ is the eigenspace of $J_z$ belonging to the eigenvalue $i$, and $T''_z(X)$ belongs to $-i$. Since in a complex manifold

$$J \frac{\partial}{\partial z^a} = i \frac{\partial}{\partial x^a}, \quad J \frac{\partial}{\partial \bar{z}^a} = -i \frac{\partial}{\partial x^a},$$

the $\frac{\partial}{\partial z^a}, \ldots, \frac{\partial}{\partial \bar{z}^a}$ form a basis for $T'_z(X)$ and the

$$\frac{\partial}{\partial x^a}, \ldots, \frac{\partial}{\partial y^a}$$

are a basis for $T''_z(X)$.

Remark 1. It is not unusual to call $T'_z(X)$ the tangent space at $x$. The consistent component set-up automatically leads there, although then always remains the problem what $T''_z(X)$ is. We prefer to distinguish between $T_z(X)$, $\mathcal{T}_z(X)$, $T'_z(X)$, $T''_z(X)$ even though e.g. $P_z$ maps $T_z(X)$ in a 1:1 way onto $T'_z(X)$. 

Remark 2. If in a complex manifold $z_U^i$ and $z_U^j$ are complex coordinates in open sets $U$ and $V$, and if $U \cap V \neq 0$, then

$$\frac{\partial}{\partial z_U^i} = \sum_{\sigma} \frac{\partial}{\partial z_U^\sigma} \frac{\partial}{\partial z_U^i}, \quad \text{with } \frac{\partial}{\partial z_U^i} \text{ holomorphic.}$$

This shows that $T''(X)$ is a complex vector bundle over $X$ [17]. Similarly, $T''(X)$ is a complex vector bundle with respect to the "conjugate complex structure" given by coordinates $\overline{z_U^i}$ (or by $-J$ instead of $J$).

The *types* of differential forms are determined by the decomposition (7.3) of $\overline{T}(X)$. Let $\omega$ be a $q$-form, either scalar or vector-valued. Then the operator $\prod_{r,s}$ is defined by $\prod_{r,s} \omega = 0$ if $r+s \neq q$, and for $r+s = q$ by

$$(7.4) \quad (\prod_{r,s} \omega) (u_1, \ldots, u_q) = \frac{1}{r!s!} \sum_\alpha |\alpha| \omega(Pu_{\alpha_1}, \ldots, Pu_{\alpha_r}, Qu_{\alpha_{r+1}}, \ldots, Qu_{\alpha_{r+s}}).$$

One easily shows $\prod_{r,s} \prod_{r+s=q} = \delta_{r,r} \delta_{s,s} \prod_{r,s}$ and $\sum_{r+s=q} \prod_{r,s} = \text{the identity on } \overline{T}^{*q}(X)$ resp. $\overline{V}^q(X)$, which means that the $\prod_{r,s}$ are projection operators and give a direct sum decomposition of $\overline{T}^{*q}(X)$ and $\overline{V}_q(X)$ resp. $\overline{V}^q(X)$ and $\overline{V}_q$. $\omega$ is called of *type* $(r, s)$ if $\prod_{r,s} \omega = \omega$.

For vector forms an even finer decomposition is possible. Define $\prod'$ and $\prod''$ by $\prod' L = PL$, $\prod'' L = QL$, then it is easy to see that $\prod' \prod'' = \prod$, and the same for $\prod'$. If $\prod' L = L$, $L$ is called $T''(X)$-valued, and if $\prod'' L = L$, $L$ is $T''(X)$-valued. Write $\prod' = \prod' \prod$ and $\prod'' = \prod'' \prod$, then $\sum_{r+s=q} (\prod' + \prod'') = \text{the identity on } \overline{V}^q(X)$. $L$ is called of *type* $(r, s)'$, or $(r, s)''$ if $\prod' L = L$ or $\prod'' L = L$ respectively.

The bundles $T^{*q}(r,s)(X)$, $\overline{V}^{(r,s)}(X)$, $V^{(r,s)}(X)$, $V^{(r,s)'}(X)$, and the modules $\Phi_{r,s}$, $\Psi_{r,s}$, $\Psi'$, $\Psi''$, $\Psi'$, $\Psi''$, $\Psi_{r,s}$, $\Psi_{r,s}$, $\Psi_{r,s}$ have now obvious meanings, e.g. $V^{(r,s)'}(X) = \prod_{r,s} \overline{V}^{r+s}(X)$, and $\Psi_{r,s} = \prod_{r,s} \overline{V}^{r+s}.$

Lemma (7.3). A differential form $\omega$ of degree $q$ (scalar or vector-valued) is of type $(r, s)$, $r+s = q$, if either one of the two equivalent conditions holds:

$$(7.5) \quad \omega P = r \omega, \quad \omega Q = s \omega.$$ If $\omega$ is a vector form, then $\omega$ is type $(r, s)'$, $(r, s)''$ respectively if in addition $Q \omega = 0, P \omega = 0$ respectively.

The proof is obvious from (7.4) and (2.10).

Lemma (7.4). The operators $\lambda$, $\pi$ behave as follows with respect to types:

$$(7.6) \begin{cases} \Phi_{r,s} \lambda \Phi_{r+s+u} \subset \Phi_{r+t,s+u} & \Phi_{r,s} \lambda \Psi_{r+s+u} \subset \Psi_{r+t,s+u} \\ \Phi_{r,s} \pi \Psi_{r+s+u} \subset \Phi_{r-t+1,s+u} & \Phi_{r,s} \pi \Phi_{r+s+u-1} \subset \Phi_{r-t+1,s+u-1} \\ \Psi_{r,s} \pi \Psi_{r+s+u} \subset \Psi_{r-t+1,s+u} & \Psi_{r,s} \pi \Phi_{r+s+u-1} \subset \Phi_{r-t+1,s+u-1} \end{cases}$$
where \( v \) denotes either one or two primes. If in the terms with \( x \Psi'_{l,u} \), \( r \) is zero, then \( \ldots x \Psi'_{l,u} = 0 \), and for the terms with \( x \Psi''_{l,u} \) the same happens if \( s = 0 \).

Proof. The first line of relations follows from (7.5) and (2.11a) with \( M = P \). The proofs of the following two lines are similar. We take the first one of these as an example. The proof requires the use of (7.5) and (2.12) with \( N = P \).

\[ (7.7) \quad qL \cdot P = q \cdot LP + qP \cdot L - q \cdot PL = tqL + rpL - qL = (t + r - 1)qL, \quad \text{QED.} \]

If \( r = 0 \), \( \omega(u_1, \ldots, u_q) \) vanishes if one of the \( u \)'s is \( T'(X) \)-valued. The evaluation of \( \omega L \) requires \( L \) to be substituted in \( \omega \), hence if \( L \) is \( T'(X) \)-valued, \( \omega L \) must vanish.

Lemma (7.5). If \( \omega \) is of type \((1, q-1)\), then \( \omega M \cdot N = \omega \cdot MN \) whenever \( M \) and \( N \) are \( T'(X) \)-valued.

Proof. We show that in this case (2.14) vanishes. \( \omega(u_1, \ldots, u_q) \) vanishes whenever two of the \( u \)'s are \( T'(X) \)-valued. But \( M \) and \( N \) are \( T'(X) \)-valued, hence \( \omega(N(...), M(...), ...) \) vanishes, and this proves the statement.

§ 8. Types of derivations. A derivation \( D \) on \( \Phi \) is called of type \((r, s)\) if \( D(\Phi_{l,u}) \subset \Phi_{l+r+u,s} \). If \( D \) is a derivation on \( \Phi \) of degree \( r + s \), then \( D_{r,s} \) is defined by \( D_{r,s} \omega = \prod_{r+t,s+u} D_{t,u} \omega \) for \( \omega \in \Phi_{l,u} \). It is easy to see that \( D_{r,s} \) is also a derivation.

Every derivation is determined by its action on \( F, \Phi_{1,0} \) and \( \Phi_{0,1} \). In fact, a derivation is already determined if for all locally defined \( C^\infty \) functions \( f \) its action on \( f \), \( \partial f \), \( \partial^s f \), is known, where \( \partial f \) and \( \partial^s f \) are defined as \( df \cdot P \) and \( df \cdot Q \) respectively (cf. (8.6)).

A derivation of type \( i_\bullet \) is of the form \( i_L \). If \( L \in \Psi' \) (\( L \in \Psi'' \)) it is said to be of type \( i'_L \) (type \( i''_L \)). Derivations \( D \) of type \( i'_L \) are characterized by \( D(\bar{\Phi}) = \{0\}, D(\Phi_{0,1}) = \{0\} \). It is often convenient to write \( i'_L \) for \( i_L \) in this case. Derivations of type \( i''_L \) are treated similarly.

The derivations of type \( d_L \) are commutators: \( d_L = [i_L, d] \). A decomposition of \( d_L \) is therefore preceded by one for \( d \).

Let \( \omega \in \Phi_{p,q} \), then \( \omega P = p\omega \). Applying (5.10), with \( L = M = P \), gives, using \( PP = P \):

\[ (8.1) \quad [P, \omega]P = (p + 1)[P, \omega] = -\omega[P, P]. \]

Writing out \([P, \omega]\) (cf. (4.11)), and defining the torsion \( T = -\frac{1}{2}[P, P] \) we find thus:

\[ (8.2) \quad (d\omega \cdot P)P - (2p + 1)d\omega \cdot P + p(p + 1)d\omega = 2\omega T, \]

or, using \( d\omega = \sum_{r,s} \prod_{r,s} d\omega \):

\[ \left\{ \begin{array}{l} \sum_r (r^2 - r(2p + 1) + p(p + 1)) \prod_{r, p + q - r + 1} d\omega = 0 \\
\sum_r (r - p)(r - p - 1) \prod_{r, p + q - r + 1} d\omega = 2\omega T \end{array} \right. \]

(8.3)
Here, $T$ is real, and consists of two parts of different types:

$$T = \frac{1}{8}[J,J], \quad T' = PT, \quad T'' = QT.$$  
(8.4)

$T'$ is of type $(0,2)'$, and $T''$ is of type $(2,0)'$, as follows from (6.15); in our present notation: $T(u,v) = -Q[Pu,Pv] - P[Qu,Qv]$; because this implies

$$T''(u,v) = QT(u,v) = -Q[Pu,Pv] = T'(Pu,Pv);$$

similarly $T'(u,v) = T'(Qu,Qv)$. Hence since $T = (P+Q)T = T' + T''$ we have $\omega T = \omega T' + \omega T''$, and $\omega T' \in \Phi_{p-1,q+2}$, $\omega T'' \in \Phi_{p+2,q-1}$. It follows thus that $d\omega$ consists of four parts, of types $(p,q+1)$, $(p+1,q)$, $(p-1,q+2)$, $(p+2,q-1)$ respectively; the latter two being algebraically determined by $\omega$ and $T$, and the first two leading to the definitions for $d'$ and $d''$. Thus follows the well-known result 3):

**Lemma (8.1).** If $\omega \in \Phi_{p,q}$ then $d\omega$ has the following decomposition,

$$\begin{align*}
\omega &= \sum_{p-1,q+1} d\omega = \frac{\partial}{\partial \omega}, \\
\omega &= \sum_{p,q+1} d\omega = \frac{\partial}{\partial \omega}, \\
\omega &= \sum_{p+1,q} d\omega = \frac{\partial}{\partial \omega}, \\
\omega &= \sum_{p,q+1} d\omega = \frac{\partial}{\partial \omega},
\end{align*}$$

(8.6)

where $T'$ and $T''$ are the parts of the torsion $T$, defined by (8.4).

Each of these four operations is a derivation, as was remarked in the beginning of this section, and $d$ is thus decomposed:

$$d = d' + d'' + i_{T'} + i_{T''}.$$  
(8.7)

For the operation $d_\gamma: \omega \rightarrow [P,\omega]$ the decomposition (8.6) gives $d_\gamma \omega = -d' \omega - i_{T'} \omega + 2i_{T''} \omega$. $d_\gamma$ satisfies $d_\gamma f = d' f = \prod_{p=0}^{l} d f$; and further being a derivation of type $d_\gamma$, satisfies $d d_\gamma = -d_\gamma d$; which shows that it is the same as $\gamma$ in [9] and [14].

The general operation $d_L$, $L \in \Psi_{r,s}'$, is decomposed as follows:

$$d_L = [i_L,d] = [i_L,d'] + [i_L,d''] + [i_L,i_{T'}] + [i_L,i_{T''}].$$

(8.8)  

The fourth term is (cf. (5.6a)) $i_{T'La} + (-1)^{l}i_{T''a}$ and the third term equals $(-1)^{l}i_{Tb}$ because $T' L$ vanishes. The first term is denoted by $d'_{L}$. The second term is, like the third and fourth, of type $i_\Phi$. This follows from the fact that $[i_L,d''] f = 0, f \in \overline{F}$:

$$[i_L,d''] f = d' f \cdot L + (-1)^{l}d''(f \cdot L) = d''(f \cdot L) = 0,$$

(8.9)

because $d' f \in \Phi_{a,1}$ and $L = PL$. Hence there exists a unique vector $(r+s+1)$-form, $\Psi'' L$ such that

$$[i_L,d'] = (-1)^{l}i_{\Psi'' L}.$$  
(8.10)

3) This result and some others of our lemmas have been known to many authors in one form or another. Attempts to establish priorities here would be rather pointless, it seems. However, a few of our formulas have appeared elsewhere only in LeGrand [18].
Since \([i_L, d^*] \omega \in \Phi_{p+r-1,a+r+1} \) for \( \omega \in \Phi_{p,0} \) it follows that \( \mathcal{D}^n L \) has a part of type \((r, s+1)\) and a part of type \((r-1, s+2)\). The latter part must vanish, however, since for all \( \varphi \in \Phi_{0,1} \)

\[
\begin{align*}
0 &= d^* \varphi \cdot L + (-1)^{r+s} d^* \varphi L = [i_L, d^*] \varphi = \\
&= (-1)^q i_{\mathcal{D}^n L} \varphi = (-1)^q \varphi \times \prod_{r-1,s+2} \mathcal{D}^n L.
\end{align*}
\]

This gives the following decomposition of \( d_L \) for \( L \in \Psi' \):

\[
d_L = d'_L + (-1)^{q+1} d_{L^\prime} + (-1)^{q+1} d_{T^\prime} + i_{T^\prime L}.
\]

Similarly, for \( M \in \Psi'' \), \( \mathcal{D}'' M \) is defined by \([i_M, d^*] = (-1)^{a} i_{\mathcal{D}'' M} \).

Using the types of the terms in (8.12), and Lemma (7.4), we find

**Lemma (8.2).** If \( L \in \Psi'_r, s \) and \( \omega \in \Phi_{p,q,0} \) then \( d_L \) takes the form (8.12), and \( d_L \omega \) has the following decomposition

\[
\prod_{p+t} d_L \omega = d'_L \omega \quad \prod_{p+s} d_L \omega = (-1)^q i_{\mathcal{D}^n L} \omega
\]

\[
\prod_{p+t} d_L \omega = (-1)^q i_{L^\prime} \omega \quad \prod_{p+s} d_L \omega = (-1)^q i_{T^\prime L} + i_{T^\prime L} \omega
\]

in which \( d'_L = [i_L, d^*] \), while \( \mathcal{D}^n L \) is defined by (8.10).

The derivations \( d'_L \) are said to be of type \( d_1^* \), and similarly for type \( d_2^* \). \( \mathcal{D}^t, \mathcal{D}^u \) denote the modules of derivations of types \( d_1^* \), \( d_2^* \) respectively. It is to be remembered that \( \mathcal{D}^t \) and \( \mathcal{D}^u \) are not subspaces of \( \mathcal{D}^d \) defined in § 4. But we do have \( \mathcal{D} = \mathcal{D}^t + \mathcal{D}^u + \mathcal{D}^t + \mathcal{D}^u \) and \( \mathcal{D} = \mathcal{D}^t + \mathcal{D}^u \).

It will turn out useful to denote \( d_L \omega \) also by \( \{L, \omega\}_1 \), and \( d_L \omega \) by \( \{L, \omega\}_2 \), just as we wrote \([L, \omega]\) for \( d_L \omega \).

The following results are obtained in exactly the same way as the corresponding ones (5.3,4).

**Proposition (8.3).** The derivations on \( \tilde{\Phi} \) form a module over \( \tilde{\Phi} \), and we have, for \( \omega \in \Phi_{p,0}, L \in \Psi'_r \):

\[
\omega i'_L = i'_{\omega L}, \quad \omega d'_L = d'_{\omega L} + (-1)^q \omega i'_{d_{\omega L}};
\]

the latter being equivalent to

\[
\{\omega \wedge L, \pi\}' = \omega \wedge \{L, \pi\}' + (-1)^q \omega \wedge \pi L.
\]

**Proposition (8.4).** The following identities hold:

\[
\begin{align*}
\text{a)} & \quad d' d' = -d', & d' d' = -d', \\
\text{b)} & \quad d' d' + d' d' = -d_{T^t + T^u} = -d_{T^t} \\
\text{c)} & \quad \mathcal{D}^t T' = 0, & \mathcal{D}' T' = 0, \\
\text{d)} & \quad \mathcal{D}^t P = 0, & \mathcal{D}' Q = 0.
\end{align*}
\]

\[^4\] The double meaning of the symbol \( \mathcal{D} \) (operator and module) might be confusing. However, the modules will not occur again in this paper.
Proof. The identities (8.16, a, b, c) follow by writing out $d^2 = 0$ and using (8.6). (8.16d) follows from $\omega \mathcal{D}^* P = -d^* \omega P + d^* \omega \cdot P = (r - r)d^* \omega = 0$, where $\omega$ is of type $(r, s)$.

Proposition (8.5). The following product rules hold, if $L \in \mathcal{V}'$, $M \in \mathcal{V}'$:

\[
\begin{align*}
\text{a)} & \quad d^* \omega L = (-1)^{l-1} d^* \omega \cdot L + \omega \mathcal{D}^* L, \\
\text{b)} & \quad \mathcal{D}^* (\pi \wedge L) = d^* \pi \wedge L + (-1)^{p} \pi \wedge \mathcal{D}^* L, \\
\text{c)} & \quad \mathcal{D}^* LM = L \mathcal{D}^* M + (-1)^{m-1} \mathcal{D}^* L \cdot M.
\end{align*}
\]

Proof. (8.17a) is obtained by re-writing (8.10). For (8.17b) we use (8.10) and (2.11b), taking $L \in \mathcal{V}'$ and $\pi \in \mathcal{V}':$

\[
\begin{align*}
\omega \mathcal{D}^* (\pi \wedge L) &= d^* \omega (\pi \wedge L) + (-1)^{l+p} d^* \omega \cdot (\pi \wedge L) = \\
&= d^* \pi \wedge \omega L + (-1)^{p} \pi \wedge d^* \omega L + (-1)^{l+p} \pi \wedge (d^* \omega \cdot L) = \\
&= d^* \pi \wedge \omega L + (-1)^{p} \pi \wedge \mathcal{D}^* L = \omega (d^* \pi \wedge L) + (-1)^{p} \omega (\pi \wedge \mathcal{D}^* L).
\end{align*}
\]

For the proof of (8.17c) we assume for simplicity that $\omega$ is of type $(1, 0)$. Then $d^* \omega$ is of type $(1, 1)$ and by Lemma (7.5), $\omega L \cdot M = \omega \cdot LM$, and $(d^* \omega \cdot L)M = d^* \omega \cdot LM$. We find thus

\[
\begin{align*}
\omega \mathcal{D}^* LM &= d^* (\omega \cdot LM) + (-1)^{l+m+1} d^* \omega \cdot LM = d^* (\omega L \cdot M) + \\
&\quad + (-1)^{m} d^* \omega L \cdot M + (-1)^{m-1} d^* \omega L \cdot M + (-1)^{l+m-1} (d^* \omega \cdot L)M = \\
&= \omega L \cdot \mathcal{D}^* M + (-1)^{m-1} \omega \mathcal{D}^* L \cdot M = \omega (L \mathcal{D}^* M + (-1)^{m-1} \mathcal{D}^* L \cdot M).
\end{align*}
\]

(To be continued)