The stability of travelling fronts for general scalar viscous balance law

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Received 29 April 2003
Available online 20 January 2005
Submitted by M.D. Gunzburger

Abstract

This paper is concerned with the existence and stability of travelling front solutions for some general scalar viscous balance law. By shooting methods we prove the existence of some class of travelling fronts for any positive viscosity. Further by analytic semigroup theory and detailed spectral analysis, we show that the travelling fronts obtained are asymptotically stable in some appropriate exponentially weighted space. Especially for all sufficiently small viscosity, the travelling waves are proved to be uniformly exponentially stable in the same weighted space.

Keywords: Balance law; Travelling front; Existence; Phase plane method; Asymptotic stability; Spectral analysis

1. Introduction

This paper is concerned with the travelling front solutions of the viscous balance law

\[ u_t + f(u)_x = \varepsilon u_{xx} + g(u), \quad x \in \mathbb{R}, \tag{1.1} \]

here \( \varepsilon > 0 \) is called the viscosity parameter.
When \( f = 0 \), (1.1) is the standard reaction–diffusion equation, it is well known that the existence and stability of travelling waves for this type of equations has been widely investigated [1,11]. On the other hand, when \( g = 0 \), (1.1) becomes the standard viscous conservation law, to which a lot of important research work have been devoted in the past several decades, such as the admissibility of the shock waves of hyperbolic conservation law, the existence and stability of viscous shock profiles, etc. Especially by spectral analysis and semigroup methods [4,5,9,10,12], the asymptotic stability including exponential stability and algebraic stability of the viscous shock profiles has been extensively studied.

In recent years, much work [6–8] has been devoted to the following hyperbolic balance laws:

\[
 u_t + f(u)_x = g(u), \quad x \in \mathbb{R},
\]

where the reaction term \( g(u) \) describing some chemical reactions, combustion or other interactions [2] can dramatically change the long time behavior of solutions of (1.2), compared to hyperbolic conservation laws. For example, C. Mascia [6] showed that for the case \( f \) is convex there exist many types of travelling waves for (1.2), including the continuous monotone waves and many kinds of discontinuous waves (shock waves, sub-shocks, etc.), while for hyperbolic conservation laws the only travelling waves are shock waves.

More recently, when the convection term \( f \) is convex, by invariant manifold theory J. Härterich [2] showed that several types of waves of (1.2) obtained in [6] admit the viscous profiles of travelling waves of (1.1), and they are close in weighted space \( L^1_{\beta}(\mathbb{R}) \) when the viscosity \( \varepsilon \) is small enough. In particular, the author proved that for every continuous wave \( u_{0,c}(x - ct) \) connecting \( u_k \) to \( u_j \) of (1.2) in the following four cases:

\[
\begin{align*}
(i) & \quad j = k + 1, \quad u_k \in \mathcal{A}(g), \quad c > f'(u_{k+1}), \\
(ii) & \quad j = k - 1, \quad u_k \in \mathcal{A}(g), \quad c > f'(u_k), \\
(iii) & \quad j = k + 1, \quad u_k \in \mathcal{R}(g), \quad c < f'(u_k), \\
(iv) & \quad j = k - 1, \quad u_k \in \mathcal{R}(g), \quad c < f'(u_{k-1}).
\end{align*}
\]

when \( \varepsilon \) is small enough (1.1) possesses the smooth monotone waves \( u_{\varepsilon,c}(x - ct) \) which converge to \( u_{0,c}(x - ct) \) in the above mentioned weighted space as \( \varepsilon \downarrow 0 \). In fact, the methods and proof in [2] are still valid for the nonconvex convection cases, although the author did not mention that.

In fact, as explicitly shown in [7], the convexity hypothesis on \( f \) is too restrictive to understand the qualitative behavior of (1.2) in the general cases. Recently, the existence of travelling waves of hyperbolic balance law (1.2) for some nonconvex convection cases was investigated in [8], where the extremal speeds for the existence of continuous travelling waves were also obtained.

For the cases with convex convection terms and small compact initial perturbations, the asymptotic stability of continuous travelling waves for hyperbolic balance law (1.2) were obtained in [6]. As far as we know, until now there is no result on the stability of travelling waves for viscous balance law (1.1).

\[1 \text{ For } \mathcal{A}(g) \text{ and } \mathcal{R}(g), \text{ the reader may refer to the notations in (1.3).} \]
In this paper, we restrict our attention to the existence and stability of the above four types of travelling waves for (1.1), the existence and stability for the other kinds of traveling waves for (1.1) will be investigated in our another paper.

First, using shooting methods [1] we prove that in the general nonlinearity cases (no convexity condition on \( f \)), for any positive viscosity, there exist smooth monotone waves of (1.1) connecting adjacent zeroes of \( g \). As in [8,11], the estimates on the extremal speeds for (1.1) are also obtained.

Furthermore, based on our existence results, by semigroup theory and detailed spectral analysis we show that for every fixed viscosity \( \varepsilon \) the travelling fronts for general (1.1) are locally asymptotically stable in some suitable weighted spaces, and especially for all sufficiently small viscosity, we can choose the same weighted spaces in which the travelling fronts are uniformly exponentially stable, which further implies the stability of the corresponding waves of hyperbolic balance law (1.2) in some suitable exponential weighted space. These also extend the stability results on the travelling waves of (1.2) in [6] to the cases with general nonlinearity and with noncompact initial perturbations.

Finally, it should be remarked that the weights we imposed on are both necessary and sufficient for shifting the essential spectrum to the left to obtain exponential stability, so the stability results obtained in this paper are optimal in the sense of exponential stability.

This paper is planed as follows: The existence of travelling fronts of (1.1) for the more general cases is proved in Section 2, and the asymptotic stability of the travelling waves in the weighted spaces is obtained in Section 3.

1.1. Notations

The zeroes of \( g \) are denoted by \( u_k \), where \( k \in \{1, 2, \ldots, n\} \), and the set of all zeroes of \( g \) is called \( \mathcal{Z}(g) \). Depending on the sign of \( g' \) the zeroes of \( g \) are divided into two sets:

\[
\mathcal{A}(g) := \{ u_k \in \mathcal{Z}(g): g'(u_k) < 0 \}, \quad \mathcal{R}(g) := \{ u_k \in \mathcal{Z}(g): g'(u_k) > 0 \}. \tag{1.3}
\]

2. The existence of travelling fronts

In this and the following sections, we always assume the following two assumptions hold:

(I) \( g \in C^1(\mathbb{R}) \), and \( g \) has finitely many simple zeroes,

(II) \( f \in C^2(\mathbb{R}) \).

In this section, by shooting methods we shall prove the existence of continuous travelling fronts of (1.1) for every fixed \( \varepsilon > 0 \), here we also give the estimates on the maximal or minimum wave speed.

**Theorem 2.1.** Let (I) and (II) hold, then for any fixed \( \varepsilon > 0 \) there exist travelling fronts \( U_{\varepsilon,c}(x-ct) \) connecting \( u_k \) to \( u_j \) for (1.1), where \( u_k \), \( u_j \), and \( c \) are in one of the following cases:
Proof of Theorem 2.1. Without loss of generality, here we only give the proof for case (iii), the proof for the other cases is similar.

For case (iii), let $U_{e,c}(z)$ ($z = x - ct$) be the travelling wave of (1.1) connecting $u_k$ to $u_j$, then $U_{e,c}(z)$ satisfies

$$
\begin{align}
&\varepsilon U_{e,c}'' + (c - f'(U_{e,c}))U_{e,c}' + g(U_{e,c}) = 0, \\
&U_{e,c}(-\infty) = u_k, U_{e,c}(+\infty) = u_{k+1}. 
\end{align}
$$

(2.1)

Let $q = (U_{e,c} - u_k)/[u]$, with $[u] = u_{k+1} - u_k$, then (2.1) is equivalent to

$$
\begin{align}
&q' = p, \\
&\varepsilon p' = -g_1(q) + (f_1(q) - c)p, \\
&(q, p)(-\infty) = (0, 0), (q, p)(+\infty) = (1, 0),
\end{align}
$$

(2.2)

where $f_1(q) = f'(q[u] + u_k)$, $g_1(q) = g(q[u] + u_k)/[u]$.

Similar to [1,11], by linearizing the system (2.2) around $(1, 0)$ and $(0, 0)$ respectively, one can easily check that if

$$
c < f_1(0) - 2\sqrt{\varepsilon g_1'(0)},
$$

(2.3)

then $(0, 0)$ is an unstable node and $(1, 0)$ is a saddle point of (2.2), so there exists an unique trajectory $T_\varepsilon$ of (2.2) hitting $(1, 0)$ from the left and above as in Fig. 1.

For some $\mu > 0$ and small $\mu_0 > 0$, consider the lines $p = \mu q$ and $p = \mu_0 q$ as indicated in Fig. 1. Label the points $A$, $B$, $C$ and the radius $\varepsilon$ of sufficiently small neighborhood of $(1, 0)$ as shown in Fig. 1. Following the trajectory $T_\varepsilon$ backward into the triangle $OCA$, we know that it must either lead to the other critical point $(0, 0)$ in the closed triangle or else cross the boundary of the triangle.

By the standard phase plane analysis as in [11], it is easy to check that $T_\varepsilon$ once enters the triangle, it will neither leave the triangle from side $OC$ nor from $AB$. On the segment $OA$, $p = \mu q$,

$$
\varepsilon \frac{dp}{dq} \geq \min_{[0,1]} f_1(q) - c - \frac{1}{\mu} \sup_{[0,1]} \frac{g_1(q)}{q} = \tau - c - \frac{v}{\mu},
$$
\[ \nu = \sup_{[0, 1]} \left( g_1(q)/q \right) \geq g_1'(0), \quad \tau = \min_{[0, 1]} f_1(q) = \min_{[0, 1]} f'(u). \]

Thus, if \( \mu > 0 \) satisfies
\[ \tau - c - \mu^{-1} \nu \geq \varepsilon \mu, \tag{2.4} \]
then \( \Gamma_c \) will not leave the triangle through OA.

It is easy to check that if \( c \) satisfies
\[ c \leq \tau - 2\sqrt{\varepsilon \nu}, \tag{2.5} \]
then there exists \( \mu > 0 \) satisfying (2.4). Therefore for any fixed \( c \) satisfying (2.5), there exists an unique orbit of travelling front connecting \((0, 0)\) to \((1, 0)\). Further, by a similar argument as in [11] we can prove that there exists a maximal speed \( c^* \) satisfying
\[ \tau - 2\sqrt{\varepsilon \nu} \leq c^* \leq f_1(0) - 2\sqrt{\varepsilon g_1'(0)}, \]
such that for every \( c < c^* \), there exists a travelling front of (2.2). This completes the proof of Theorem 2.1 for case (iii).

In fact, from the above proof of Theorem 2.1, we notice that for any \( c < \tau - 2\sqrt{\varepsilon \nu} \), there exists \( \mu > 0 \) lying between the two distinct roots \((2\varepsilon)^{-1}(\tau - c \pm \sqrt{(\tau - c)^2 - 4\varepsilon \nu})\) such that the orbit \( \Gamma_c \) of travelling front satisfies
\[
\left. \frac{dp}{dq} \right|_{(0,0)} \leq \mu < (2\varepsilon)^{-1}(\tau - c + \sqrt{(\tau - c)^2 - 4\varepsilon \nu}) \\
\leq (2\varepsilon)^{-1} f_1(0) - c + \sqrt{(f_1(0) - c)^2 - 4\varepsilon g_1'(0)}) \]
then by (2.4), \( \Gamma_c \) must satisfy
Theorem 2.1 imply that some sub-shock profiles of (1.2) admit viscous profiles of travelling waves. Let \( u \) be a solution of (1.1). Then, there exists sufficiently small \( \epsilon > 0 \) such that for any \( 0 < \epsilon \leq \epsilon_0 \) there exist viscous profiles of travelling waves \( U_{0,\epsilon}(x - ct) \) connecting \( u_k \) and \( u_j \) of (1.2), where \( u_k, u_j, \) and \( c \) are in one of the following cases:

(i) \( j = k + 1, u_k \in A(g), c > \max_{[u_k, u_j]} f'(u) \),

(ii) \( j = k - 1, u_k \in A(g), c > \max_{[u_j, u_k]} f'(u) \),

(iii) \( j = k + 1, u_k \in R(g), c < \min_{[u_k, u_j]} f'(u) \),

(iv) \( j = k - 1, u_k \in R(g), c < \min_{[u_j, u_k]} f'(u) \).

Furthermore, the continuous dependence of \( \Gamma_c \) on \( c \) implies (2.6) are valid for any \( c < c^* \).

Thus additionally we have proved the following proposition.

**Proposition 2.1.** For the waves (iii) in Theorem 2.1, if \( c < c^* \), then the trajectory \( \Gamma_c \) must satisfy

\[
\frac{dp}{dq}(0,0) = (2\epsilon)^{-1}
\left(f'(u_k) - c - \sqrt{\left(f'(u_k) - c\right)^2 - 4\epsilon g'(u_k)}\right).
\]

**Remark 2.1.** (i) For the other waves (i), (ii), and (iv) in Theorem 2.1, similar results to Proposition 2.1 can be obtained.

(ii) Proposition 2.1 will be very useful in the proof of the stability theorem in the next section.

**Remark 2.2.** Considering the cases with sufficiently small viscosity and comparing our results with the results for inviscid case in [2,6,8], one can easily understand the following fact: There is a critical speed separating the continuous profiles from the discontinuous ones of (1.2), as was shown in [8], where the critical speed \( c_{\text{cri}} = \min_{[u_k, u_j]} f'(u) \). When \( \epsilon = 0 \), it is easy to check that in nonconvex convection cases \( f'(u_k) \) may not equal to \( \min_{[u_k, u_j]} f'(u) \), only for speeds \( c < c_{\text{cri}} \) there exist smooth monotone waves (iii) of (1.2), no smooth waves exist for the case \( c_{\text{cri}} < c < c^* \) with \( c^* \) satisfying \( \min_{[u_k, u_j]} f'(u) \leq c^* \leq f'(u_k) \). In fact, for every speed \( c \) between \( c_{\text{cri}} \) and \( c^* \), with \( c^* = f'(u^*) \) for some \( u^* \in (u_k, u_j) \), (1.2) possesses a sub-shock solution. The above phase plane analysis and Theorem 2.1 imply that some sub-shock profiles of (1.2) admit viscous profiles of travelling fronts of (1.1).

Combining the results in [2], we can immediately obtain

**Corollary 2.1.** Let (I) and (II) hold. For any wave \( U_{0,\epsilon}(x - ct) \) connecting \( u_k \) and \( u_j \) of (1.2), where \( u_k, u_j, \) and \( c \) are in one of the following cases:

(i) \( j = k + 1, u_k \in A(g), c > \max_{[u_k, u_j]} f'(u) \),

(ii) \( j = k - 1, u_k \in A(g), c > \max_{[u_j, u_k]} f'(u) \),

(iii) \( j = k + 1, u_k \in R(g), c < \min_{[u_k, u_j]} f'(u) \),

(iv) \( j = k - 1, u_k \in R(g), c < \min_{[u_j, u_k]} f'(u) \),

then there exists sufficiently small \( \epsilon_0 > 0 \) such that for any \( 0 < \epsilon \leq \epsilon_0 \) there exist viscous profiles of travelling fronts \( U_{\epsilon}(x - ct) \) connecting \( u_k \) to \( u_j \) of (1.1), which converge to \( U_{0,\epsilon}(x - ct) \) as \( \epsilon \downarrow 0 \) in \( L^1_\beta(\mathbb{R}) \), where \( 0 \leq \beta < \min\{\|g'(u_k)\|/|c - f'(u_k)|, \|g'(u_j)\|/|c - f'(u_j)|\} \), and \( L^1_\beta(\mathbb{R}) = \{u \in L^1(\mathbb{R}) \mid \int_\mathbb{R} (1 + \epsilon^{\beta/\delta}) |u(\xi)| \, d\xi < \infty \} \).
3. The asymptotic stability of travelling waves

In this section, we shall prove the asymptotic stability of travelling waves obtained in Section 2.

Consider the following initial value problem of (1.1):
\[
\begin{align*}
    u_t + f(u)_{x} &= \varepsilon u_{xx} + g(u), \quad x \in \mathbb{R}, \quad t > 0, \\
    u|_{t=0} &= u_0(x).
\end{align*}
\] (3.1)

Introducing new variable \( z = x - ct \), (3.1) can be written as
\[
\begin{align*}
    u_t - cu_z + f'(u)u_z &= \varepsilon u_{zz} + g(u), \\
    u|_{t=0} &= u_0(z).
\end{align*}
\] (3.2)

Define \( w(z,t) = u(z,t) - U_{\varepsilon,c}(z) \) with \( U_{\varepsilon,c}(z) \) the travelling front (i), (ii), (iii), or (iv) respectively in Theorem 2.1, then \( w(z,t) \) satisfies
\[
\begin{align*}
    w_t &= L_{\varepsilon,c}w + h(w,w_z) \\
    \text{with} \\
    L_{\varepsilon,c} &= \varepsilon \frac{\partial^2}{\partial z^2} + (c - f'(U_{\varepsilon,c})) \frac{\partial}{\partial z} + g'(U_{\varepsilon,c}) - f''(U_{\varepsilon,c})U'_{\varepsilon,c},
\end{align*}
\] (3.3)

and \( \|h(w,w_z)\|_{L^2} = O(\|w\|_{H^1}^2) \) as \( \|w\|_{H^1} \) is sufficiently small.

Now we state the main stability results in this paper:

**Theorem 3.1.** Let \( U_{\varepsilon,c}(z) \) be the travelling waves (i) or (ii) ((iii) or (iv) respectively) in Theorem 2.1, and define weight function \( V_\alpha(z) = 1 + e^{\alpha z} \) (\( V_\alpha(z) = 1 + e^{-\alpha z} \) respectively).

Given \( \varepsilon > 0 \), for any fixed \( \alpha > 0 \) satisfying
\[
\alpha_1(\varepsilon) < \alpha < \alpha_2(\varepsilon)
\]

with
\[
\alpha_1(\varepsilon) = \frac{2g'(u_j)}{|f'(u_j) - c| + \sqrt{(f'(u_j) - c)^2 - 4\varepsilon g'(u_j)}}
\]

and
\[
\alpha_2(\varepsilon) = \frac{2g'(u_k)}{|f'(u_k) - c| - \sqrt{(f'(u_k) - c)^2 - 4\varepsilon g'(u_k)}}
\]

then there exists small \( \delta > 0 \) such that if
\[
\|V_\alpha(u_0(z) - U_{\varepsilon,c}(z))\|_{H^1} \leq \delta,
\]
then
\[ \| V_\alpha(z)(u(t, z) - U_{\varepsilon,c}(z)) \|_{H^1(\mathbb{R})} \leq C e^{-\sigma t}, \]
with \( \sigma > 0 \) and \( C \) depends on \( c, \varepsilon, \) and \( \alpha. \)

Note that, in sufficiently small viscosity case, we can choose the uniform bound for the weight parameter \( \alpha \) both from bottom and above such that for any \( \alpha \) between them an uniform decay rate for all positive small enough \( \varepsilon \) can be obtained.

**Theorem 3.2.** Let \( U_{\varepsilon,c}(z) \) be the travelling waves (i) or (ii) ((iii) or (iv)) respectively of (1.1) in Corollary 2.1, for \( 0 < \varepsilon \leq \varepsilon_0 \) with \( \varepsilon_0 > 0 \) small enough. Define weight function
\[ V_\alpha(z) = 1 + e^{\alpha z}, \]
for any fixed \( \alpha > 0 \) satisfying
\[ \alpha_1(\varepsilon_0) < \alpha < \alpha_2(\varepsilon_0) \quad (3.7) \]
with \( \alpha_1(\varepsilon_0) \) and \( \alpha_2(\varepsilon_0) \) defined in (3.5) and (3.6) respectively. There exists small \( \delta > 0 \) such that if
\[ \| V_\alpha(z)(u_0(z) - U_{\varepsilon,c}(z)) \|_{H^1(\mathbb{R})} \leq \delta, \]
then
\[ \| V_\alpha(z)(u(t, z) - U_{\varepsilon,c}(z)) \|_{H^1(\mathbb{R})} \leq C e^{-\sigma t}, \]
with \( \sigma > 0 \) and \( C \) depends only on \( c, \varepsilon_0, \) and \( \alpha. \)

As in Section 2, without losing generality, in the following we only give the proof of Theorem 3.1 for case (iii) in Theorem 2.1 and the proof of Theorem 3.2 for case (iii) in Corollary 2.1, the proof for other cases is similar.

### 3.1. The distribution of essential spectra of \( A_{\varepsilon,\alpha} \)

Denote weighted space
\[ X_\alpha = \{ w(z) \mid V_\alpha(z)w(z) \in X \} \quad \text{with weight function } V_\alpha(z) = 1 + e^{-\alpha z}, \]
for some \( \alpha > 0, \)

and the norm \( \| w \|_{X_\alpha} = \| V_\alpha(z)w(z) \|_X. \)

Obviously,
\[ L^2_{\alpha}(\mathbb{R}) = \{ w(z) \mid V_\alpha(z)w(z) \in L^2(\mathbb{R}) \}, \]
\[ H^k_{\alpha}(\mathbb{R}) = \{ w(z) \mid V_\alpha(z)w(z) \in H^k(\mathbb{R}) \}. \]

For any fixed \( \varepsilon > 0 \) and \( c < c^*, \) where \( c^* \) is given in Theorem 2.1 (iii), i.e.
\[ \min_{[u_k, u_j]} f'(u) - 2 \sqrt{e} \sup_{(u_k, u_j)} (g(u)/(u - u_k)) \leq c^* \leq f'(u_k) - 2 \sqrt{e} g'(u_k), \]
define an operator
\[ A_{\varepsilon,\alpha} : H^2_\alpha(\mathbb{R}) \to L^2_\alpha(\mathbb{R}), \quad A_{\varepsilon,\alpha}w = L_{\varepsilon,c}w, \quad w \in H^2_\alpha(\mathbb{R}), \]
thus

\[ A_{\varepsilon,\alpha}w = \varepsilon w_{zz} + a(z)w_z + b(z)w, \]

with

\[ a(z) = c - f'(U_{\varepsilon,c}(z)), \quad b(z) = g'(U_{\varepsilon,c}(z)) - f''(U_{\varepsilon,c}(z))U'_{\varepsilon,c}(z). \]  

(3.8)

Define the map

\[ \psi : X \to X, \quad \psi w = \frac{w}{V_{\alpha}}, \quad w \in X, \]

and further, we define an operator

\[ \hat{A}_{\varepsilon,\alpha} : H^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \hat{A}_{\varepsilon,\alpha} = \psi^{-1}A_{\alpha}\psi, \]

i.e. for \( w \in H^2(\mathbb{R}) \),

\[
\hat{A}_{\varepsilon,\alpha}w = \varepsilon w_{zz} + \left(2\varepsilon V_{\alpha} \left(\frac{1}{V_{\alpha}}\right) + c - f'(U_{\varepsilon,c})\right)w_z
\]
\[ + \left(\varepsilon V_{\alpha} \left(\frac{1}{V_{\alpha}}\right)'' + (c - f''(U_{\varepsilon,c}))V_{\alpha} \left(\frac{1}{V_{\alpha}}\right)'\right)w
\]
\[ + g'(U_{\varepsilon,c}) - f''(U_{\varepsilon,c})U'_{\varepsilon,c}w. \]  

(3.9)

Obviously, \( A_{\varepsilon,\alpha} : H^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is equivalent to \( \hat{A}_{\varepsilon,\alpha} : H^2(\mathbb{R}) \to L^2(\mathbb{R}) \), with

\[ \sigma_p(A_{\varepsilon,\alpha}) = \sigma_p(\hat{A}_{\varepsilon,\alpha}), \quad \rho(A_{\varepsilon,\alpha}) = \rho(\hat{A}_{\varepsilon,\alpha}), \]  

(3.10)

and

\[ \| (\lambda I - A_{\varepsilon,\alpha})^{-1} \|_{L^2_{\alpha} \to L^2_{\alpha}} = \| (\lambda I - \hat{A}_{\varepsilon,\alpha})^{-1} \|_{L^2 \to L^2}. \]  

(3.11)

Obviously (3.9)–(3.11) imply \( A_{\varepsilon,\alpha} \) generates an analytic semigroup on \( L^2_{\alpha} \).

In the following, we consider the essential spectra of \( A_{\varepsilon,\alpha} \).

By (3.10), obviously

\[ \sigma_{\text{ess}}(A_{\varepsilon,\alpha}) = \sigma_{\text{ess}}(\hat{A}_{\varepsilon,\alpha}). \]  

(3.12)

By (3.9), we can define

\[
\hat{A}_{\varepsilon,\alpha}^{-\infty}w = \varepsilon w_{zz} + (2\varepsilon\alpha + c - f'(u_k))w_z
\]
\[ + (\varepsilon\alpha^2 + \alpha(c - f'(u_k)) + g'(u_k))w, \]  

(3.13)

\[
\hat{A}_{\varepsilon,\alpha}^{+\infty}w = \varepsilon w_{zz} + (c - f'(u_j))w_z + g'(u_j)w. \]  

(3.14)

Define

\[ S_{\alpha}^- = \{ \lambda \in \mathbb{C} \mid \lambda = \varepsilon(\tau)^2 + (2\varepsilon\alpha + c - f'(u_k))i\tau \]
\[ + \varepsilon\alpha^2 + \alpha(c - f'(u_k)) + g'(u_k) \text{for some } \tau \in \mathbb{R} \}, \]  

(3.15)

\[ S_{\alpha}^+ = \{ \lambda \in \mathbb{C} \mid \lambda = \varepsilon(\tau)^2 + (c - f'(u_j))i\tau + g'(u_j) \text{for some } \tau \in \mathbb{R} \}. \]  

(3.16)
For travelling wave (iii) with \( g'(u_k) > 0, g'(u_j) < 0 \), if
\[
\varepsilon \alpha^2 + \alpha (c - f'(u_k)) + g'(u_k) < 0,
\]
then
\[
\sup \{ \text{Re} \{ \sigma_{\text{ess}}(A_{\varepsilon, \alpha}) \} \} \leq \max \{ \varepsilon \alpha^2 + \alpha (c - f'(u_k)) + g'(u_k), g'(u_j) \} < 0.
\]
(3.17)

By [3] and (3.12)–(3.18), we have shown that

**Lemma 3.1.** For each fixed \( \varepsilon > 0, c < c^* \), and \( \alpha > 0 \) satisfying
\[
\alpha_1(\varepsilon) < \alpha < \alpha_2(\varepsilon),
\]
with \( \alpha_1(\varepsilon) \) and \( \alpha_2(\varepsilon) \) defined in (3.5') and (3.6') respectively, there exists some constant \( C_{\alpha} > 0 \) such that
\[
\sup \{ \text{Re} \{ \sigma_{\text{ess}}(A_{\varepsilon, \alpha}) \} \} \leq -C_{\alpha} < 0.
\]
(3.19)

**Remark 3.1.** From (3.19) it is easy to see that for any \( 0 < \varepsilon \leq \varepsilon_0 \) (with \( \varepsilon_0 > 0 \) small enough), we can choose the uniform lower bound \( \alpha_1(\varepsilon_0) \) and upper bound \( \alpha_2(\varepsilon_0) \) such that for any weight parameter \( \alpha \) lying between them all the essential spectrum of the linearized operator \( A_{\varepsilon, \alpha} \) have negative real parts. Furthermore, for each fixed \( c < \min_{[u_k, u_{k+1}]} f'(u) \), it is easy to check that there exists constant \( C_{\varepsilon_0, \alpha} > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_0 \),
\[
\sup \{ \text{Re} \{ \sigma_{\text{ess}}(A_{\varepsilon, \alpha}) \} \} \leq -C_{\varepsilon_0, \alpha} < 0.
\]
(3.20)

### 3.2. The distribution of eigenvalues of \( A_{\varepsilon, \alpha} \)

In this section, we always assume \( \alpha > 0 \) satisfies (3.19).

First, we consider the asymptotic behavior of eigenfunction \( w_\lambda \) of \( A_{\varepsilon, \alpha} \)
\[
A_{\varepsilon, \alpha} w_\lambda = \lambda w_\lambda, \quad w_\lambda \in H_\alpha^2(\mathbb{R}).
\]
(3.21)

By standard asymptotic analysis, it is easy to prove that
\[
w_\lambda(z) \sim \exp \left( 2\varepsilon^{-1} (f'(u_k) - c) + \frac{4\varepsilon (\lambda - g'(u_k))}{c_2} \right) z, \quad \text{as } z \to -\infty.
\]
(3.22)

Let \( \lambda = \text{Re} \lambda + i \text{ Im} \lambda \) and \( \sqrt{(f'(u_k) - c)^2 + 4\varepsilon (\lambda - g'(u_k))} = c_1 + ic_2 \), with \( c_1 \geq 0 \), and \( c_2 \in \mathbb{R} \), then
\[
\begin{cases}
(f'(u_k) - c)^2 + 4\varepsilon (\text{Re} \lambda - g'(u_k)) = c_1^2 - c_2^2, \\
2\varepsilon \text{ Im} \lambda = c_1 c_2.
\end{cases}
\]
(3.23)

For any fixed \( \varepsilon > 0 \) and \( \alpha > 0 \) satisfying (3.19), if \( \text{Re} \lambda \geq -\sigma_1, \sigma_1 > 0 \) small enough, then
\[
c_1^2 \geq (f'(u_k) - c)^2 - 4\varepsilon (\sigma_1 + g'(u_k)),
\]
thus...
(2\varepsilon)^{-1}(f'(u_k) - c + c_1) \geq \frac{2g'(u_k)}{|f'(u_k) - c| - \sqrt{(f'(u_k) - c)^2 - 4\varepsilon(\sigma_1 + g'(u_k))}} > \alpha,
(2\varepsilon)^{-1}(f'(u_k) - c - c_1) \leq \frac{2g'(u_k)}{|f'(u_k) - c| + \sqrt{(f'(u_k) - c)^2 - 4\varepsilon(\sigma_1 + g'(u_k))}} < \alpha,

which implies

\begin{equation}
\hat{w}_\lambda(z) \sim \exp\left(2\varepsilon^{-1}(f'(u_k) - c + \text{Re} \sqrt{(f'(u_k) - c)^2 + 4\varepsilon(\lambda - g'(u_k))})z\right),
\end{equation}
as \ z \to -\infty.

A similar but simpler analysis tells that for any fixed \varepsilon > 0, if \lambda is an eigenvalue of \bar{A}, with \text{Re} \lambda \geq -\sigma_2, \sigma_2 > 0 small enough, then the corresponding eigenfunction \hat{w}_\lambda(z) must satisfy

\begin{equation}
\hat{w}_\lambda(z) \sim \exp\left(2\varepsilon^{-1}(f'(u_j) - c - \text{Re} \sqrt{(f'(u_j) - c)^2 + 4\varepsilon(\lambda - g'(u_j))})z\right),
\end{equation}
as \ z \to +\infty.

Thus we obtain

**Lemma 3.2.** For \varepsilon > 0, c < c^*, \alpha > 0 satisfying (3.19) and \text{Re} \lambda \geq -\sigma_0, with \sigma_0 > 0 small enough, if \lambda is an eigenvalue of \bar{A}, then the eigenfunction \hat{w}_\lambda(z) \in H^2(\mathbb{R}) must decay to zero exponentially as \ z \to \pm \infty, satisfying (3.23) and (3.24).

Let

\begin{equation}
\hat{w}_\lambda(z) = \exp(2\varepsilon^{-1} \int_0^z (c - f'(U_{c,e}(s))) \, ds) \, w_\lambda(z)
\end{equation}

for \text{Re} \lambda \geq -\sigma_0, the eigenvalue problem (3.20) can be written as

\begin{equation}
\hat{L} \hat{w}_\lambda = (c \hat{w}_\lambda')' - \left(\frac{a^2(z)}{4\varepsilon} + \frac{a'(z)}{2} - b(z)\right) \hat{w}_\lambda = \lambda \hat{w}_\lambda,
\end{equation}

with \hat{L} : H^2(\mathbb{R}) \to L^2(\mathbb{R}), and a(z) and b(z) defined in (3.8).

By Lemma 3.2, (3.25)–(3.26), it is easy to check that

**Lemma 3.3.** For \varepsilon > 0, c < c^*, \alpha > 0 satisfying (3.19) and \text{Re} \lambda \geq -\sigma_0, with \sigma_0 > 0 small enough, there exists a nonzero function \hat{w}_\lambda \in H^2(\mathbb{R}) satisfying \bar{A}(\varepsilon, \alpha) \hat{w}_\lambda = \lambda \hat{w}_\lambda if and only if there exists a nonzero function \hat{w}_\lambda \in H^2(\mathbb{R}) such that \hat{L} \hat{w}_\lambda = \lambda \hat{w}_\lambda.

Obviously, operator \hat{L} is a self-adjoint operator on \ H^2(\mathbb{R}), then Lemma 3.3 implies that under the condition of Lemma 3.3, for \text{Re} \lambda \geq -\sigma_0, all the eigenvalues of \bar{A} are real. In fact, in the following we can further show that all the eigenvalues of \bar{A} are negative.

**Lemma 3.4.** For any \varepsilon > 0, c < c^*, \alpha > 0 satisfying (3.19) and \sigma_0 > 0 small enough, the eigenvalues of \bar{A} in \Omega = \{\lambda \in \mathbb{C}, \text{Re} \lambda \geq -\sigma_0\} are real and

\begin{equation}
\sup\{\text{Re} \{\sigma_p(\bar{A})\}\} < 0.
\end{equation}
Proof. By contradiction, assume $\lambda \geq 0$ is an eigenvalue of (3.20) with eigenfunction $w_\lambda \in H^2_\alpha(\mathbb{R})$. Let

$$w_\lambda(z) = U_0(z)u_\lambda(z)$$  
(3.27)

with $U_0(z) = U'(z)$, then $u_\lambda(z)$ satisfies

$$\varepsilon u''_\lambda + \left(\frac{2\varepsilon U'_0}{U_0} + a(z)\right)u'_\lambda = \lambda u_\lambda.$$  
(3.28)

Let $p_\lambda = u'_\lambda$, then (3.28) becomes

$$\varepsilon p''_\lambda + \left(\frac{2\varepsilon U'_0}{U_0} + a(z)\right)p_\lambda = \lambda u_\lambda.$$  
(3.29)

thus

$$\left\{\exp\left\{-\int_0^z \left(\frac{2\varepsilon U'_0}{U_0} + a(s)\right)ds\right\} p_\lambda(z)\right\}' = \lambda u_\lambda \varepsilon \exp\left\{-\int_0^z \left(\frac{2\varepsilon U'_0}{U_0} + a(s)\right)ds\right\}. \quad (3.30)$$

First, we shall deduce a contradiction for the case $\lambda = 0$. Let $\lambda = 0$, then from (3.30) we have

$$p_0(z) = C \exp\left\{-\int_0^z \left(\frac{2\varepsilon U'_0}{U_0} + a(s)\right)ds\right\}, \quad (3.31)$$

with $C$ some constant.

Note that $U_0(z) > 0$, and satisfies $A_\varepsilon U_0 = 0$. By Proposition 2.1 we have

$$U_0(z) \sim \exp\left\{-2(\varepsilon)^{-\frac{1}{2}} \left(a_\pm + \sqrt{a_\pm^2 - 4\varepsilon b_\pm}\right)z\right\}, \quad \text{as } z \to \pm \infty \quad (3.32)$$

with

$$a_- = c - f'(u_k), \quad b_- = g'(u_k), \quad a_+ = c - f'(u_j), \quad b_+ = g'(u_j),$$

thus

$$\left(\frac{2\varepsilon U'_0}{U_0} + a(z)\right) \to (2\varepsilon)^{-\frac{1}{2}} \sqrt{a_\pm^2 - 4\varepsilon b_\pm} > 0, \quad \text{as } z \to +\infty. \quad (3.33)$$

By (3.24), (3.27), and (3.32), it is easy to check that $p_0(z) = u'_0(z)$ must be bounded as $z \to +\infty$. (3.31) and (3.33) implies that $p_0(z)$ is bounded at $+\infty$ iff $p_0(z) \equiv 0$ for $z \in \mathbb{R}$, which further means $w_0(z) = C_0 U_0(z)$ with $C_0$ some constant.

Comparing (3.23) with (3.32), it is easy to see that $U_0(z)$ is not an eigenfunction of $A_{\varepsilon, \alpha}$ with respect to $\lambda = 0$, thus $w_0(z) \equiv 0, z \in \mathbb{R}$, which contradicts $\lambda = 0$ is an eigenvalue of $A_{\varepsilon, \alpha}$.

In the following, we consider that case $\lambda > 0$.

Let $\lambda_1 > 0$ be the first eigenvalue of $\hat{L}$. By Sturm–Liouville theorem, the eigenfunction $\hat{w}_{\lambda_1}$ of $\hat{L}$ corresponding to $\lambda_1$ does not change sign. Let $\hat{w}_{\lambda_1}(z) > 0$, Lemma 3.4 implies there exists eigenfunction $w_{\lambda_1} \in H^2_\alpha(\mathbb{R})$ corresponding to the first eigenvalue $\lambda_1$ of $A_{\varepsilon, \alpha}$, with $w_{\lambda_1} > 0$. 


From (3.23), (3.24), (3.27), and (3.32), \( u_{\lambda_1}(z) \) satisfies
\[
\left| u'_{\lambda_1}(z) \right|, \left| u_{\lambda_1}(z) \right| \leq C \exp \left\{ (2\epsilon)^{-1} \left( +\sqrt{a_{\pm}^2 + 4\epsilon(\lambda_1 - b_{\pm}) + \sqrt{a_{\pm}^2 - 4\epsilon b_{\pm}}} \right) z \right\},
\]
as \( z \to \pm \infty \).
(3.34)

From (3.29), it is easy to see \( p_{\lambda_1}(z) \neq 0, z \in \mathbb{R} \). By (3.34) and the fact \( \lambda_1 > 0, w_{\lambda_1} > 0 \), there exists \( z_0 \in \mathbb{R} \) such that \( p_{\lambda_1}(z_0) > 0 \). Without losing generality, let \( p_{\lambda_1}(0) > 0 \), then (3.30) implies
\[
\exp \left\{ \int_0^z \left( \frac{2U_0'}{U_0} + \frac{a(s)}{\epsilon} \right) ds \right\} p_{\lambda_1}(z) > p_{\lambda_1}(0), \quad z > 0,
\]
i.e.
\[
p_{\lambda_1}(z) > p_{\lambda_1}(0) \exp \left\{ \int_0^z \left( -\frac{2U_0'}{U_0} + \frac{a(s)}{\epsilon} \right) ds \right\}, \quad z > 0.
\]
Again, by (3.33), we have \( p_{\lambda_1}(z) \to +\infty \) (as \( z \to +\infty \)), which contradicts (3.34). So the first eigenvalue \( \lambda_1 \) of \( A_{\epsilon,\alpha} \) must be negative, which completes the proof of Lemma 3.4.

By Lemmas 3.1 and 3.4, we obtain

**Lemma 3.5.** For each fixed \( \epsilon > 0, c < c^*, \) and \( \alpha > 0 \) satisfying (3.19),
\[
\sup \left\{ \Re \sigma(A_{\epsilon,\alpha}) \right\} < 0.
\]

Applying the standard stability theory to the viscous equation (1.1), we complete the proof of Theorem 3.1 for case (iii) in Theorem 2.1.

By (3.8) and (3.26) it is easy to prove that

**Proposition 3.1.** For each fixed \( c < \min_{[u_k, u_{k+1}]} f'(u) \), there exists small \( \epsilon_0 > 0 \), and constant \( \sigma > 0 \) such that for any \( 0 < \epsilon < \epsilon_0 \),
\[
\sup \left\{ \Re \sigma_p(\hat{L}) \right\} \leq -\sigma < 0.
\]

Thus by Lemma 3.3, Proposition 3.1, and Remark 3.1, we have

**Lemma 3.6.** For each fixed \( c < \min_{[u_k, u_{k+1}]} f'(u), \) \( \alpha > 0 \) satisfying (3.7), and \( \epsilon_0 \) small enough, there exists constant \( \sigma_{\epsilon_0,\alpha} > 0 \) such that for any \( 0 < \epsilon < \epsilon_0 \),
\[
\sup \left\{ \Re \sigma(A_{\epsilon,\alpha}) \right\} \leq -\sigma_{\epsilon_0,\alpha} < 0.
\]

By Lemma 3.6 and the standard semigroup theory, we complete the proof of Theorem 3.2 for case (iii) in Corollary 2.1.

The proof for other cases in Theorems 3.1 and 3.2 can be similarly obtained.
References