JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **221,** 658–671 (1998)
ARTICLE NO. AY985933

The Behavior, on $\dot{B}_1^{0,1}$, of an Oscillatory Integral with Polynomial Phase Function

Philip J. Gloor*

Department of Mathematics, St. Olaf College, Northfield, Minnesota 55057 and

Received October 7, 1996

We study the convolution oscillatory singular integral operator $Tf = \text{p.v.}\,\Omega * f$, with $\Omega(x) = e^{iq(x)}K(x)$, where q is a real-valued polynomial of a real variable, of degree $d \geq 2$, and K is a Calderón–Zygmund-type kernel. We prove that this operator extends to an operator that maps the Besov space $\dot{B}_1^{0,1}$ into the Hardy-type space H_0^1.

1. INTRODUCTION

In recent years several results concerning the boundedness of oscillatory singular integral operators have appeared in the literature. The operator of interest here is a convolution operator of the form $Tf = \text{p.v.}\,\Omega * f$, with $\Omega(x) = e^{iq(x)}K(x)$, where q is a real-valued polynomial of a real variable and K is a Calderón–Zygmund-type kernel. Similar operators are studied in [1] and [3].

In [1] the phase function $q \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is real-valued and satisfies

$$|D^\alpha q(x)| \leq C_\alpha |x|^{b-|\alpha|} \tag{1.1}$$

for every multi-index α with $|\alpha| \leq M$ and $x \neq 0$, where M and b are positive integers, and

$$|\nabla q(x)| \geq C|x|^{b-1}. \tag{1.2}$$

*Current address is St. Olaf College

0022-247X/98 $25.00

In this same paper the kernel $K\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^1$ is said to be of type (k, m) if for every $x \neq 0$,

$$|D^\alpha K(x)| \leq C_\alpha |x|^{-k-|\alpha|}, \tag{1.3}$$

for every multi-index α with $|\alpha| \leq m$, where C_α is a constant that depends on α but not on x. The result in [1], which is of interest to us, is the following.

THEOREM 1.4. *For $k = 1, 2, 3, \ldots$ and $p_k = n/(n+k)$, let $m \geq kb + n$ and let K be a kernel of type $(n, k+1)$ near the origin and of type $(m, k+1)$ away from the origin. In addition, suppose K satisfies the cancellation condition* p.v. $\int_{|x| \leq \varepsilon} K(x)\,dx = 0$, *for some $\varepsilon > 0$. Then the convolution oscillatory singular integral operator $Tf =$* p.v. $\Omega * f$ *is bounded on H_0^p, for $p_k \leq p < 1$, provided $q \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfies* (1.1) *and* (1.2), *where $b > 0$, $b \neq 1$, and $M \geq k+1$.*

The space H_0^p of this theorem is the Hardy-type space defined in [5] and discussed below.

In [3] the phase function $q \in C^2(\mathbb{R}^n \setminus \{0\})$ is real-valued and satisfies conditions (1.1) and (1.2) with $M = 2$. Furthermore, K is a Calderón–Zygmund kernel. The result in [3], which is of interest to us, is the following.

THEOREM 1.5. *When $n = 1$, $Tf =$* p.v. $\Omega * f$ *extends to a bounded operator in the space $\dot{B}_1^{0,1}$ if and only if T is a bounded operator on $L^2(\mathbb{R}^1)$.*

The space $\dot{B}_1^{0,1}$ of this theorem is the Besov space defined in [3] and discussed below.

In this study we are interested in the convolution oscillatory singular integral operator

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^1} e^{iq(x-y)} K(x-y) f(y)\,dy, \tag{1.6}$$

where $q(x) = \sum_{l=0}^{d} \alpha_l x^l$ is a real-valued polynomial of degree $d \geq 2$ and K is a kernel of type $(1, 2)$ that satisfies a cancellation condition. The significant feature of this study is that the phase function in (1.6), the polynomial q, may not satisfy condition (1.2), a condition that is necessary in the proofs of Theorems 1.4 and 1.5 above. Our main result is the following.

THEOREM 1.7. *Let $q(x) = \sum_{l=0}^{d} \alpha_l x^l$ be a real-valued polynomial of degree $d \geq 2$, and suppose K is a kernel of type $(1, 2)$ that satisfies the cancellation condition* p.v. $\int_{|x| \leq \varepsilon} K(x)\,dx = 0$, *for some $\varepsilon > 0$. Then there* *exists a constant C such that*

$$\|Tf\|_{H_0^1} \le C\|f\|_{\dot{B}_1^{0,1}} \tag{1.8}$$

for every $f \in \dot{B}_1^{0,1}$, *where C is independent of f.*

The remainder of this paper is organized as follows. In the next section we give definitions of the Hardy-type space H_0^1 and the Besov space $\dot{B}_1^{0,1}$. In Section 3 we present the proof of Theorem 1.7, along with the statements of three necessary lemmas. We postpone the discussion of the proofs of these lemmas until Section 4.

Throughout this paper, the letter C will denote a constant, the value of which may change with each appearance.

2. DEFINITIONS AND NOTATION

In this section we present definitions of the Hardy-type space H_0^1 and the Besov space $\dot{B}_1^{0,1}$.

The Hardy-type space H_0^1 was studied first by Han [5] in the setting of $\mathbb{R}^1$ and later by Chen and Fan [1] in the setting of $\mathbb{R}^n$ for $n \ge 1$. In this study we are interested in the setting $\mathbb{R}^1$.

If $1 < q \le \infty$ and $s \ge 0$, then a measurable function $a: \mathbb{R}^1 \to \mathbb{C}$ is called a $(1, 1, q, s)$ atom centered at x_0 if there exists an interval $I(x_0, \rho) \subset \mathbb{R}^1$, with center x_0 and radius ρ, such that

$$\operatorname{supp}(a) \subset I(x_0, \rho), \tag{2.1}$$

$$\int_{\mathbb{R}^1} x^j a(x)\, dx = 0 \qquad \text{for every integer } 0 \le j \le s, \tag{2.2}$$

$$\|a\|_{L^q(\mathbb{R}^1)} \le \rho^{1/q-1}, \tag{2.3}$$

and

$$\|a'\|_{L^q(\mathbb{R}^1)} \le \rho^{1/q-2}, \tag{2.4}$$

where the derivative is in the distributional sense. For the same values of q and s as above, we define the atomic Hardy-type space, $H_0^{1,q,s}$, as the collection of all $f \in S'(\mathbb{R}^1)$, the tempered distributions in $\mathbb{R}^1$, that can be written as

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \tag{2.5}$$

where $\{\lambda_j\}_{j=1}^{\infty}$ satisfies $\sum_{j=1}^{\infty} |\lambda_j| < \infty$ and each a_j is a $(1, 1, q, s)$ atom. Convergence of (2.5) is in the distributional sense. If we define the norm of such an f as $\|f\|_{H_0^{1,q,s}} = \inf (\sum_{j=1}^{\infty} |\lambda_j|)$, where the infimum is taken over all sums (2.5), then $H_0^{1,q,s}$ becomes a Banach space.

Now suppose that $\varphi \in \mathcal{S}(\mathbb{R}^1)$, the Schwartz class of functions in $\mathbb{R}^1$, satisfies $\int_{\mathbb{R}^1} \varphi(x)\, dx = 0$. The generalized Lusin function S_φ^b is defined as

$$S_\varphi^b(f)(x) = \left(\int_{\Gamma(x)} |f * \varphi_t(y)|^{b+1} \frac{dy\, dt}{t^2} \right)^{1/(b+1)} \tag{2.6}$$

for $b \geq 0$ and $f \in \mathcal{S}'(\mathbb{R}^1)$, where $\Gamma(x) = \{(y, t) \in \mathbb{R}_+^2 : |x - y| < t\}$ and $\varphi_t(y) = t^{-1}\varphi(t^{-1}y)$ for $y \in \mathbb{R}^1$, and $t > 0$. If $s \geq 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^1)$ satisfies $\operatorname{supp}(\varphi) \subset \{x \in \mathbb{R}^1 : |x| < 1\}$, $\int_0^\infty |\hat{\varphi}(xt)|^2 (dt/t) \neq 0$ for every $x \neq 0$, and $\int_{\mathbb{R}^1} x^j \varphi(x)\, dx = 0$ for every integer $0 \leq j \leq s$, then we define the Hardy-type space H_0^1 to be the collection of all $f \in \mathcal{S}'(\mathbb{R}^1)$ that satisfy $\|f\|_{H_0^1} = \|S_\varphi^0 f\|_{L^1(\mathbb{R}^1)} < \infty$. The following theorem can be found in [5].

THEOREM 2.7. *Let* $1 < q \leq \infty$ *and* $s \geq 0$.

(i) *For any* $(1, 1, q, s)$ *atom* a, *there exists a constant* C, *independent of* a, *such that* $\|S_\varphi^0 a\|_{L^1(\mathbb{R}^1)} \leq C$.

(ii) $H_0^1 = H_0^{1,q,s}$ *and* $\|f\|_{H_0^{1,q,s}} \approx \|f\|_{H_0^1}$ *for every* $f \in H_0^1$.

We define the Besov space $\dot{B}_1^{0,1}$ as in [3]. Suppose $1 \leq q \leq \infty$. A measurable function $a: \mathbb{R}^1 \to \mathbb{C}$ is called a $(1, q)$ Besov atom if there exists an interval $I(x_0, \rho) \subset \mathbb{R}^1$ with center x_0 and radius ρ such that

$$\operatorname{supp}(a) \subset I(x_0, \rho), \tag{2.8}$$

$$\|a\|_{L^q(\mathbb{R}^1)} \leq \rho^{1/q - 1}, \tag{2.9}$$

$$\|a'\|_{L^q(\mathbb{R}^1)} \leq \rho^{1/q - 2}, \tag{2.10}$$

$$\|a''\|_{L^q(\mathbb{R}^1)} \leq \rho^{1/q - 3}, \tag{2.11}$$

and

$$\int_{\mathbb{R}^1} x^j a(x)\, dx = 0 \qquad \text{for } j = 0, 1. \tag{2.12}$$

The Besov space $\dot{B}_1^{0,1}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^1)$ that can be written as

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \tag{2.13}$$

where $\{\lambda_j\}_{j=1}^{\infty}$ satisfies $\sum_{j=1}^{\infty}|\lambda_j| < \infty$ and each a_j is a $(1,\infty)$ Besov atom. Convergence of (2.13) is in the distributional sense. If we define the norm of such an f as $\|f\|_{\dot{B}_1^{0,1}} = \inf(\sum_{j=1}^{\infty}|\lambda_j|)$, where the infimum is taken over all sums (2.13), then $\dot{B}_1^{0,1}$ becomes a Banach space.

3. PROOF OF THEOREM 1.7

To prove (1.8) it is enough to show that

$$\|Ta\|_{H_0^1} = \|S_\varphi^0(Ta)\|_{L^1(\mathbb{R}^1)} \le C \tag{3.1}$$

for every $(1,\infty)$ Besov atom, a, where C is independent of a. Indeed, suppose (3.1) holds and that $f \in \dot{B}_1^{0,1}$. Let $\varepsilon > 0$ be arbitrary and write $f = \sum_{j=1}^{\infty}\lambda_j a_j$, where each a_j is a $(1,\infty)$ Besov atom and $\sum_{j=1}^{\infty}|\lambda_j| < \|f\|_{\dot{B}_1^{0,1}} + \varepsilon$. It follows that

$$\begin{aligned}\|T(f)\|_{H_0^1} = \|S_\varphi^0(Tf)\|_{L^1(\mathbb{R}^1)} &= \left\| S_\varphi^0\left(T\left(\sum_{j=1}^{\infty}\lambda_j a_j\right)\right)\right\|_{L^1(\mathbb{R}^1)} \\ &\le \sum_{j=1}^{\infty}|\lambda_j|\,\|S_\varphi^0(Ta_j)\|_{L^1(\mathbb{R}^1)} \le C\sum_{j=1}^{\infty}|\lambda_j| \le C\big(\|f\|_{\dot{B}_1^{0,1}} + \varepsilon\big),\end{aligned}$$

and since $\varepsilon > 0$ was arbitrary, (1.8) follows.

Since T is a convolution operator, in proving (3.1) we may and do assume without loss of generality, that a is a $(1,\infty)$ Besov atom centered at the origin. We let $I = I(0,\rho)$ denote the interval, with radius ρ and center at the origin, associated with a as in (2.8).

The idea of the proof of this theorem is to break Ta into pieces and show that each piece is either a $(1,1,2,s)$ atom, for a particular value of $s > 0$, or is one term of a series that belongs to H_0^1. To this end we let $\psi \in C_0^\infty(\mathbb{R}^1)$ be such that $\operatorname{supp}(\psi) \subset \{1/2 \le |x| < 2\}$, $\psi \ge 0$, $\psi(1) = 1$, and $\sum_{j=-\infty}^{\infty}\psi(2^j x) = 1$ for every $x \ne 0$. For some fixed $N \ge 1$, to be determined in Lemma 3.15 below, we define

$$\eta(x) = 1 - \sum_{j=N+1}^{\infty}\psi_j(x),$$

where $\psi_j(x) = \psi(2^{-j-1}\rho^{-1}x)$. It follows that $\operatorname{supp}(\eta) \subset \{|x| \le 2^{N+2}\rho\}$ and $\eta(x) = 1$ for every $|x| \le 2^{N+1}\rho$. We let $\Omega(x) = e^{iq(x)}K(x)$ and write Ω as

$$\Omega(x) = \eta(x)\Omega(x) + \sum_{j=N+1}^{\infty}\psi_j(x)\Omega(x) = \Omega_0(x) + \sum_{j=N+1}^{\infty}\Omega_j(x)$$

so

$$Ta = (\Omega_0 * a) + \sum_{j=N+1}^{\infty} (\Omega_j * a). \tag{3.2}$$

Using arguments similar to those in [1], which are by now well known and therefore omitted, and the result in [6] concerning the L^2 boundedness of T, one can easily show that, up to a constant, $\Omega_0 * a$ is a $(1, 1, 2, s)$ atom, for $s = 1$, and hence by Theorem 2.7 there exists a constant C, independent of a, such that

$$\|\Omega_0 * a\|_{H_0^1} \leq C. \tag{3.3}$$

We now consider the terms $\Omega_j * a$ for $j \geq N + 1$ in (3.2). Because our phase function, q, is a polynomial, the arguments are more delicate than those for $\Omega_0 * a$.

Since $\operatorname{supp}(\psi_j) \subset \{2^j\rho \leq |x| < 2^{j+2}\rho\}$ and $\operatorname{supp}(a) \subset \{|x| \leq \rho\}$, it follows that

$$\operatorname{supp}(\Omega_j * a) \subset \{2^{j-1}\rho \leq |x| \leq 2^{j+3}\rho\}. \tag{3.4}$$

Using the cancellation properties of a, we see that for $l = 0, 1$,

$$\begin{aligned} &\int_{\mathbb{R}^1} x^l (\Omega_j * a)(x)\, dx \\ &\quad = \int_{\mathbb{R}^1} x^l \int_{\mathbb{R}^1} \Omega_j(y) a(x-y)\, dy\, dx \\ &\quad = \int_{\mathbb{R}^1} \Omega_j(y) \int_{\mathbb{R}^1} \sum_{i=0}^{l} \binom{l}{i} (x-y)^i y^{l-i} a(x-y)\, dx\, dy \\ &\quad = \sum_{i=0}^{l} \binom{l}{i} \int_{\mathbb{R}^1} y^{l-i} \Omega_j(y) \int_{\mathbb{R}^1} (x-y)^i a(x-y)\, dx\, dy = 0. \end{aligned} \tag{3.5}$$

In considering the series $\sum_{j=N+1}^{\infty} (\Omega_j * a)$ we will find it necessary to treat separately the $\Omega_j * a$ whose supports contain the zeros of q'. To this end we let

$$q(x) = \sum_{i=0}^{d} \alpha_i x^i = \alpha_d (x - \mu_1)(x - \mu_2) \cdots (x - \mu_d),$$

where μ_i, $1 \leq i \leq d$, are the roots of q. We note that

$$\frac{|q(x)|}{|x|^d} = |x|^{-d} \left| \sum_{i=0}^{d} \alpha_i x^i \right| \leq \sum_{i=0}^{d} |\alpha_i|$$

for every $|x| \geq 1$, so

$$|q(x)| \leq \left(\sum_{i=0}^{d} |\alpha_i| \right) |x|^d$$

for every $|x| \geq 1$. In fact, for $0 \leq l \leq d$,

$$|D^l q(x)| \leq C_l |x|^{d-l} \tag{3.6}$$

for every $|x| \geq 1$, where C_l depends on l and on the coefficients of q.

Along with the upper bound (3.6), we desire a lower bound on q'. To establish this bound we consider

$$q'(x) = \sum_{i=1}^{d} i\alpha_i x^{i-1} = d\alpha_d (x - \nu_1)(x - \nu_2) \cdots (x - \nu_{d-1}),$$

where ν_i, $1 \leq i \leq d-1$ are the roots of q'. Let $I_k = [-2^{k+1}\rho, -2^k\rho] \cup [2^k\rho, 2^{k+1}\rho]$ and suppose $\mathrm{Re}(\nu_i) \in I_k$.

Note that if $x \notin I_{k-1} \cup I_k \cup I_{k+1}$, then $|x - \nu_i| > |x|/2$. If we let

$$Z = \bigcup_{i=1}^{d-1} \{y \in I_{k-1} \cup I_k \cup I_{k+1} \colon \mathrm{Re}(\nu_i) \in I_k\} \cup \{0\},$$

then it follows that for any $x \notin Z$,

$$|q'(x)| = |d\alpha_d||x - \nu_1||x - \nu_2| \cdots |x - \nu_{d-1}| > \frac{d|\alpha_d|}{2^{d-1}} |x|^{d-1}. \tag{3.7}$$

This is the lower bound that we desire. Note that this bound is valid only on $\mathbb{R}^1 \setminus Z$.

If we define $B = \{j \in \mathbb{Z} \colon Z \cap \bigcup_{i=j-2}^{j+3} I_i \neq \emptyset\}$ and $G = \{j \in \mathbb{Z} \colon j \geq N + 1\} \setminus B$, then for any $j \in G$, the set of good indices, we have $Z \cap \{x \in \mathbb{R}^1 \colon 2^{j-2}\rho \leq |x| \leq 2^{j+4}\rho\} = \emptyset$. For those indices $j \in G$, if $x \in \mathrm{supp}(\Omega_j * a) \subset \{2^{j-1}\rho \leq |z| \leq 2^{j+3}\rho\}$ and $y \in \mathrm{supp}(a) \subset \{|z| \leq \rho\}$, then $2^{j-2}\rho \leq |x - y| \leq 2^{j+4}\rho$, since $j - 2 = (j-1) - 1 \geq N - 1 \geq 0$, and it follows that $x - y \notin Z$ and hence that

$$|q'(x - y)| > \frac{d|\alpha_d|}{2^{d-1}} |x - y|^{d-1} = C_d |x - y|^{d-1}.$$

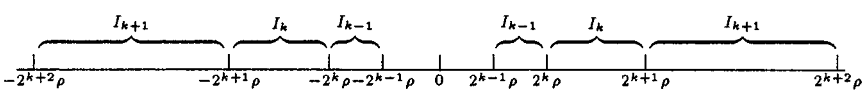

FIG. 1. The diadic intervals.

We now return to our series, which we break into two pieces:

$$\sum_{j=N+1}^{\infty} (\Omega_j * a) = \sum_{j \in G} (\Omega_j * a) + \sum_{\substack{j \in B \\ j \geq N+1}} (\Omega_j * a).$$

We concentrate first on the terms $\Omega_j * a$ for which $j \in B$. For each such term we obtain

$$\begin{aligned}|(\Omega_j * a)(x)| &= \left| \int_{|y| \leq \rho} e^{iq(x-y)} \psi(2^{-j-1} \rho^{-1}(x-y)) K(x-y) a(y)\, dy \right| \\ &\leq C \int \frac{|a(y)|}{|x-y|}\, dy \leq (2^{2-j} \rho^{-1})(2\rho) \|a\|_{L^\infty(\mathbb{R}^1)} \leq C 2^{-j} \rho^{-1}\end{aligned}$$

and it follows that

$$\|\Omega_j * a\|_{L^2(\mathbb{R}^1)} \leq C \rho^{-1/2}. \tag{3.8}$$

Using a similar argument, we obtain

$$\|(\Omega_j * a)'\|_{L^2(\mathbb{R}^1)} = \|\Omega_j * a'\|_{L^2(\mathbb{R}^1)} \leq C \rho^{-3/2}. \tag{3.9}$$

In light of (3.4), (3.5), (3.8), and (3.9), we see that, up to a constant, $\Omega_j * a$ is a $(1, 1, 2, s)$ atom, with $s = 1$, for each $j \in B$ for which $j \geq N + 1$. Since B contains at most $20(d-1)$ elements, it follows that

$$\left\| \sum_{\substack{j \in B \\ j \geq N+1}} (\Omega_j * a) \right\|_{H_0^1} \leq \sum_{j \in B} \|\Omega_j * a\|_{H_0^1} \leq C, \tag{3.10}$$

where C is a constant independent of the atom a.

In light of (3.3) and (3.10), we see that to finish this proof it will suffice to show that there exists a constant C, independent of a, such that

$$\sum_{j \in G} \|\Omega_j * a\|_{H_0^1} \leq C. \tag{3.11}$$

To prove the existence of such a constant we require three lemmas, the proofs of which are discussed in the final section. The first two of these lemmas are Lemmas 2.9 and 2.3 in [1], as adapted to our setting. The results of the third lemma are inequalities (3.4) and (3.5) in [3], as adapted to our setting.

LEMMA 3.12. *Suppose $2^{j-2}\rho \le 1$. Then there exist constants, C, such that*

$$\left|(\Omega_j * a)(x)\right| \le C\left(2^{j-2}\rho\right)^{-1} 2^{-j} \tag{3.13}$$

and

$$\left|(\Omega_j * a)'(x)\right| \le C\left(2^{j-2}\rho\right)^{-2} 2^{-j}, \tag{3.14}$$

where the constants depend on the coefficients of the polynomial q.

LEMMA 3.15. *Let s be a nonnegative integer less than or equal to 1, and suppose a is a measurable function, supported on the interval $I = (-\rho, \rho)$ and satisfying the conditions*

(i) $\int_{\mathbb{R}^1} x^l a(x)\,dx = 0$ *for every integer l such that* $0 \le l \le s$,

(ii) $\|a\|_{L^\infty(\mathbb{R}^1)} \le \rho^{-1}$,

(iii) $\|a'\|_{L^\infty(\mathbb{R}^1)} \le \rho^{-2}$, *where the derivative is in the distributional sense.*

Let

$$A_j(\rho, a) = \sup_{2^{j-1}\rho \le |x| \le 2^{j+3}\rho} \left|\int_{\mathbb{R}^1} e^{iq(x-y)} a(y) y^l\,dy\right|,$$

where $l \le s$ and q is a polynomial of degree $d \ge 2$. Then for each $j \in \mathbb{Z}$ for which $2^{j-2}\rho \ge 1$,

$$A_j(\rho, a) \le C\rho^{d+l} 2^{j(d-1)}, \tag{3.16}$$

where C is independent of a and j. Furthermore, there exists an $N \in \mathbb{N}$ such that for $j \ge N$, $2^{j-2}\rho \ge 1$, and for $j \notin B$,

$$A_j(\rho, a) \le C\left(2^j\rho\right)^{-d} \rho^l 2^j, \tag{3.17}$$

where N is independent of a, and C is independent of a and j.

LEMMA 3.18. *For $j \in G$ such that $2^{j-2}\rho \ge 1$, and any measurable function, a, supported in the interval $I = (-\rho, \rho)$ and satisfying the conditions of Lemma* 3.15,

$$\|\Omega_j * a\|_{L^\infty(\mathbb{R}^1)} \le C\left(2^j\rho\right)^{-1}\left(2^{-j} + \min\{\rho^d 2^{j(d-1)}, \rho^{-d} 2^{-j(d-1)}\}\right) \tag{3.19}$$

and

$$\|(\Omega_j * a)'\|_{L^\infty(\mathbb{R}^1)} \le C\left(2^j\rho\right)^{-1} \rho^{-1}\left(2^{-j} + \min\{\rho^d 2^{j(d-1)}, p^{-d} 2^{-j(d-1)}\}\right). \tag{3.20}$$

Returning to the proof of (3.11), we break each term of the series into several pieces as follows:

$$
\begin{aligned}
\|\Omega_j * a\|_{H_0^1} &= \int_{\mathbb{R}^1} \left| \int_0^\infty \int_{|x-y|<t} |(\varphi_t * \Omega_j * a)(y)|\, dy \frac{dt}{t^2} \right| dx \\
&= \int_{|x| \le 2^{j+5}\rho} \left| \int_0^{2\rho} \int_{|x-y|<t} |\cdots|\, dy \frac{dt}{t^2} \right| dx \\
&\quad + \int_{|x| \le 2^{j+5}\rho} \left| \int_{2\rho}^{2^{j+3}\rho} \int_{|x-y|<t} |\cdots|\, dy \frac{dt}{t^2} \right| dx \\
&\quad + \int_{|x| \le 2^{j+5}\rho} \left| \int_{2^{j+3}\rho}^{\infty} \int_{|x-y|<t} |\cdots|\, dy \frac{dt}{t^2} \right| dx \\
&\quad + \int_{|x| > 2^{j+5}\rho} \left| \int_{2^{j+3}\rho}^{\infty} \int_{|x-y|<t} |\cdots|\, dy \frac{dt}{t^2} \right| dx \\
&\quad + \int_{|x| > 2^{j+5}\rho} \left| \int_0^{2^{j+3}\rho} \int_{|x-y|<t} |\cdots|\, dy \frac{dt}{t^2} \right| dx \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

By the support condition (3.4) for $\Omega_j * a$, we obtain $I_5 = 0$. Using the cancellation property of φ we obtain for I_1 and I_3,

$$
\begin{aligned}
I_1 &= \int_{|x| \le 2^{j+5}\rho} \int_0^{2\rho} \int_{|x-y|<t} \left| \int_{\mathbb{R}^1} \varphi_t(y-z)\big((\Omega_j * a)(z) \right. \\
&\qquad\qquad \left. -(\Omega_j * a)(y)\big)\, dz \right| dy \frac{dt}{t^2}\, dx \\
&\le \|(\Omega_j * a)'\|_{L^\infty(\mathbb{R}^1)} C_\varphi 2^j \rho^2
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \int_{|x| \le 2^{j+5}\rho} \int_{2^{j+3}\rho}^{\infty} \int_{|x-y|<t} \left| \int_{\mathbb{R}^1} \left(\frac{1}{t} \varphi\left(\frac{y-z}{t} \right) \right. \right. \\
&\qquad\qquad \left. \left. - \frac{1}{t} \varphi\left(\frac{y}{t} \right) \right) (\Omega_j * a)(z)\, dz \right| dy \frac{dt}{t^2}\, dx \\
&\le \|\Omega_j * a\|_{L^\infty(\mathbb{R}^1)} 2^{j+3}\rho \|\varphi'\|_{L^1(\mathbb{R}^1)} \int_{|x| \le 2^{j+5}\rho} \int_{2^{j+3}\rho}^{\infty} \int_{|x-y|<t} dy \frac{dt}{t^3}\, dx \\
&= C_\varphi 2^j \rho \|\Omega_j * a\|_{L^\infty(\mathbb{R}^1)}.
\end{aligned}
$$

Using the cancellation property of $\Omega_j * a$ we bound I_4 as

$$\begin{aligned} I_4 \le & \int_{|x|>2^{j+5}\rho} \int_{2^{j+3}\rho}^{\infty} \int_{|x-y|<t} \int_{2^{j-1}\rho \le |z| \le 2^{j+3}\rho} \left| \varphi\left(\frac{y-z}{t}\right) - \varphi\left(\frac{y}{t}\right) \right| \\ & \times |(\Omega_j * a)(z)|\, dz\, dy \frac{dt}{t^3}\, dx \\ \le & \; (2^{j+3}\rho)^2 \|\varphi'\|_{L^\infty(\mathbb{R}^1)} \|\Omega_j * a\|_{L^\infty(\mathbb{R}^1)} \int_{|x|>2^{j+5}\rho} \int_{|x|/4}^{\infty} \int_{|x-y|<t} dy \frac{dt}{t^4}\, dx \\ \le & \; C(2^j\rho) \|\Omega_j * a\|_{L^\infty(\mathbb{R}^1)}. \end{aligned}$$

Finally, we bound I_2 as

$$\begin{aligned} I_2 \le & \|\Omega_j * a\|_{L^\infty(\mathbb{R}^1)} \int_{|x|\le 2^{j+5}\rho} \int_{2\rho}^{2^{j+3}\rho} \int_{|x-y|<t} \int_{\mathbb{R}^1} \left| \varphi\left(\frac{y-z}{t}\right) \right| \frac{dz}{t}\, dy \frac{dt}{t^2}\, dx \\ \le & \; C \|\varphi\|_{L^1(\mathbb{R}^1)} \|\Omega_j * a\|_{L^\infty(\mathbb{R}^1)} (2^j\rho) \ln(2^{j+2}) \\ = & \; C(2^j\rho) \ln(2^{j+2}) \|\Omega_j * a\|_{L^\infty(\mathbb{R}^1)}. \end{aligned}$$

Using the bounds for I_i, $1 \le i \le 5$, Lemma 3.12, and Lemma 3.18, we obtain

$$\begin{aligned} & \sum_{j\in G} \|\Omega_j * a\|_{H_0^1} \\ & \quad = \sum_{\substack{j\in G \\ 2^{j-2}\rho \ge 1}} \|\Omega_j * a\|_{H_0^1} + \sum_{\substack{j\in G \\ 2^{j-2}\rho \le 1}} \|\Omega_j * a\|_{H_0^1} \\ & \quad \le C \sum_{2^{j-2}\rho \ge 1} \Big(2^j\rho^2 \|(\Omega_j * a)'\|_{L^\infty(\mathbb{R}^1)} + 2^j\rho \|\Omega_j * a\|_{L^\infty(\mathbb{R}^1)} \\ & \qquad\qquad + (2^j\rho)\ln(2^{j+2}) \|\Omega_j * a\|_{L^\infty(\mathbb{R}^1)} \Big) \\ & \qquad + C \sum_{2^{j-2}\rho \le 1} \Big(2^j\rho^2 \|(\Omega_j * a)'\|_{L^\infty(\mathbb{R}^1)} + 2^j\rho \|\Omega_j * a\|_{L^\infty(\mathbb{R}^1)} \\ & \qquad\qquad + (2^j\rho)\ln(2^{j+2}) \|\Omega_j * a\|_{L^\infty(\mathbb{R}^1)} \Big) \\ & \quad \le C \sum_{j=2}^{\infty} \Big(2^{-j} + 2^{-j}\ln(2^{j+2}) + \min\{\rho^d 2^{j(d-1)}, \rho^{-d} 2^{-j(d-1)}\} \\ & \qquad\qquad + \ln(2^{j+2}) \min\{\rho^d 2^{j(d-1)}, \rho^{-d} 2^{-j(d-1)}\} \Big) \\ & \qquad + C \sum_{j=2}^{\infty} \left(2^{-2j} + 2^{-j} + 2^{-j}\ln(2^{j+2}) \right) \\ & \quad \le C_d, \end{aligned}$$

where C_d depends on d but not on ρ. This proves (3.11) and hence our theorem. ∎

4. PROOFS OF LEMMAS

In this section we discuss the proofs of Lemmas 3.12 and 3.15. The proof of Lemma 3.18 is quite similar to the proofs of statements (3.4) and (3.5) in [3] so we omit it.

Proof of Lemma 3.12. Recall that

$$(\Omega_j * a)(x) = \int_{|y|\le\rho} e^{iq(x-y)}\psi\big(2^{-j-1}\rho^{-1}(x-y)\big)K(x-y)a(y)\,dy$$

and $\mathrm{supp}(\Omega_j * a) \subset \{2^{j-1}\rho \le |x| \le 2^{j+3}\rho\}$. If we let

$$\tilde{K}(z) = e^{iq(z)}\psi\big(2^{-j-1}\rho^{-1}(z)\big)K(z),$$

then since K is a kernel of type $(1,2)$, we obtain

$$|D^l\tilde{K}(z)| \le C\big(2^{j-2}\rho\big)^{-1-l}$$

for $0 \le l \le 2$ and $2^{j-2}\rho \le |z| \le 2^{j+4}\rho$. Note that the constant, C, depends on the coefficients of the polynomial, q.

Using the cancellation condition of a,

$$\begin{aligned}|(\Omega_j * a)(x)| &= \left|\int_{|y|\le\rho}\big(\tilde{K}(x-y) - \tilde{K}(x)\big)a(y)\,dy\right| \\ &\le C\big(2^{j-2}\rho\big)^{-2}\rho^2\|a\|_{L^\infty(\mathbb{R}^1)} \le C\big(2^{j-2}\rho\big)^{-1}2^{-j},\end{aligned}$$

and similarly,

$$|(\Omega_j * a)'(x)| \le C\big(2^{j-2}\rho\big)^{-2}2^{-j},$$

where the constants in both expressions depend on the polynomial, q. ∎

Proof of Lemma 3.15. Using inequality (3.6) and the cancellation property of a, for any x satisfying $2^{j-1}\rho \le |x| \le 2^{j+3}\rho$, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^1} e^{iq(x-y)}a(y)y^l\,dy\right| &\le \int_{|y|\le\rho} |e^{iq(x-y)} - e^{iq(x)}|\,|a(y)|\,|y|^l\,dy \\ &= \int_{|y|\le\rho} |q'(x-\xi)|\,|y|^{l+1}|a(y)|\,dy \\ &\le C\int_{|y|\le\rho} |x-\xi|^{d-1}|y|^{l+1}|a(y)|\,dy \\ &\le C\rho^{d+1}2^{j(d-1)}\end{aligned}$$

from which (3.16) follows.

To prove (3.17) we fix $2^{j-1}\rho \le |x| \le 2^{j+3}\rho$ and consider $|y| \le \rho$. Note that $2^{j-2}\rho \le |x-y| \le 2^{j+4}\rho$ so for $j \notin B$ we have $x, x-y \notin Z$ and by (3.7) it follows that $|q'(x)| > C_d|x|^{d-1}$ and $|q'(x-y)| > C_d|x-y|^{d-1}$. Letting $q'(x) = q'(x-y) + yq''(x-\xi)$, where $|\xi| \le \rho$ we have

$$\begin{aligned}|q'(x-y)| &\ge |q'(x)| - \rho \sup_{|\xi|\le\rho} |q''(x-\xi)| \\ &\ge C|x|^{d-1} - \rho \sup_{|\xi|\le\rho} \left(C|x-\xi|^{d-2}\right) \\ &\ge C\left(2^{j-1}\rho\right)^{d-1} - \rho C\left(2^{j+4}\rho\right)^{d-2} \\ &= C\left(2^j\rho\right)^{d-1}(1 - C2^{-j}).\end{aligned}$$

So, there exists an $N \ge 1$ such that for every $j \ge N$,

$$|q'(x-y)| \ge C\left(2^j\rho\right)^{d-1}$$

for every x and y satisfying $2^{j-1}\rho \le |x| \le 2^{j+3}\rho$ and $|y| \le \rho$. Integrating by parts we obtain

$$\begin{aligned}\int_{\mathbb{R}^1} e^{iq(x-y)}a(y)y^l\,dy &= \int_{\mathbb{R}^1} \frac{d}{dy}\left(e^{iq(x-y)}\right)\frac{i}{q'(x-y)}a(y)y^l\,dy \\ &= -i\int_{|y|\le\rho} e^{iq(x-y)}\frac{d}{dy}\left(\frac{a(y)y^l}{q'(x-y)}\right)dy \\ &= -i\int_{|y|\le\rho} e^{iq(x-y)}\left[\frac{d}{dy}\left(\frac{1}{q'(x-y)}\right)a(y)y^l\right. \\ &\qquad\left. + \frac{a'(y)y^l}{q'(x-y)} + \frac{ly^{l-1}a(y)}{q'(x-y)}\right]dy.\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^1} e^{iq(x-y)} a(y) y^l \, dy\right| \\
&\quad \le \int_{|y|\le\rho} \Big(|q'(x-y)|^{-2} |q''(x-y)| \, |a(y)| \, |y|^l \\
&\qquad\qquad + |q'(x-y)|^{-1} \big(|a'(y)| \, |y|^l + l |a(y)| \, |y|^{l-1} \big) \Big) \, dy \\
&\quad \le \int_{|y|\le\rho} \Big(C|x-y|^{-2(d-1)} |x-y|^{d-2} \|a\|_{L^\infty(\mathbb{R}^1)} \rho^l \\
&\qquad\qquad + C|x-y|^{-d+1} \big(\|a'\|_{L^\infty(\mathbb{R}^1)} \rho^l + l \|a\|_{L^\infty(\mathbb{R}^1)} \rho^{l-1} \big) \Big) \, dy \\
&\quad \le C(2^j \rho)^{-d} 2^j \rho^l,
\end{aligned}$$

which proves (3.17). ∎

ACKNOWLEDGMENT

This study was part of the author's Ph.D. dissertation, which was written under the direction of Professor Alberto Torchinsky.

REFERENCES

1. L. K. Chen and D. Fan, Oscillatory kernels in certain Hardy-type spaces, *Studia Math*. **111** (1994), 195–206.
2. R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc*. **83** (1977), 99–157.
3. D. Fan, An oscillatory integral in $\dot{B}_1^{0,1}(\mathbb{R}^n)$, *J. Math. Anal. Appl*. **187** (1994), 986–1002.
4. P. J. Gloor, "Oscillatory Singular Integral Operators on Hardy Spaces," Ph.D. dissertation, Indiana University, Bloomington, IN, 1996.
5. Y. Han, "Certain Hardy-Type Spaces," Ph.D. dissertation, Washington University, St. Louis, MO, 1984.
6. F. Ricci and E. M. Stein, Harmonic analysis on nilpotent groups and singular integrals, I. Oscillatory integrals, *J. Funct. Anal*. **73** (1987), 179–194.