Symplectic automorphisms of prime order on K3 surfaces

Alice Garbagnati a, Alessandra Sarti b,∗,1

a Dipartimento di Matematica, Università di Milano, via Saldini 50, I-20133 Milano, Italy
b Institut für Mathematik, Universität Mainz, Staudingerweg 9, 55099 Mainz, Germany

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Abstract

We study algebraic K3 surfaces (defined over the complex number field) with a symplectic automorphism of prime order. In particular we consider the action of the automorphism on the second cohomology with integer coefficients (by a result of Nikulin this action is independent on the choice of the K3 surface). With the help of elliptic fibrations we determine the invariant sublattice and its perpendicular complement, and show that the latter coincides with the Coxeter–Todd lattice in the case of automorphism of order three.

Keywords: K3 surfaces; Automorphisms; Moduli

0. Introduction

In the paper [Ni1] Nikulin studies finite abelian groups \( G \) acting symplectically (i.e. \( G|_{\text{H}^2,0(X,\mathbb{C})} = \text{id}|_{\text{H}^2,0(X,\mathbb{C})} \)) on K3 surfaces (defined over \( \mathbb{C} \)). One of his main result is that the action induced by \( G \) on the cohomology group \( \text{H}^2(X,\mathbb{Z}) \) is unique up to isometry. In [Ni1] all abelian finite groups of automorphisms of a K3 surface acting symplectically are classified. Later Mukai in [Mu] extends the study to the non-abelian case. Here we consider only abelian groups of prime order \( p \) which, by Nikulin, are isomorphic to \( \mathbb{Z}/p\mathbb{Z} \) for \( p = 2, 3, 5, 7 \).

∗ Corresponding author.

E-mail addresses: garba@mat.unimi.it (A. Garbagnati), sarti@mathematik.uni-mainz.de (A. Sarti).

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In the case of \( p = 2 \) the group is generated by an involution, which is called by Morrison in [Mo, Definition 5.1] Nikulin involution. This was very much studied in the last years, in particular because of its relation with the Shioda–Inose structure (cf. e.g. [CD,GL,vGT,L,Mo]). In [Mo] Morrison proves that the isometry induced by a Nikulin involution \( \iota \) on the lattice \( \Lambda_{K3} \approx U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1) \), which is isometric to \( H^2(X, \mathbb{Z}) \), switches the two copies of \( E_8(-1) \) and acts as the identity on the sublattice \( U \oplus U \oplus U \). As a consequence one sees that \( (H^2(X, \mathbb{Z})^*)^\perp \) is the lattice \( E_8(-2) \). This implies that the Picard number \( \rho \) of an algebraic K3 surface admitting a Nikulin involution is at least nine. In [vGS] van Geemen and Sarti show that if \( \rho \geq 9 \) and \( E_8(-2) \subset NS(X) \) then the algebraic K3 surface \( X \) admits a Nikulin involution and they classify completely these K3 surfaces. Moreover they discuss many examples and in particular those surfaces admitting an elliptic fibration with a section of order two. This section operates by translation on the fibers and defines a Nikulin involution on the K3 surface.

The aim of this paper is to identify the action of a symplectic automorphism \( \sigma_p \) of the remaining possible prime orders \( p = 3, 5, 7 \) on the K3 lattice \( \Lambda_{K3} \) and to describe such algebraic K3 surfaces with minimal possible Picard number. Thanks to Nikulin’s result [Ni1, Theorem 4.7], to find the action on \( \Lambda_{K3} \), it suffices to identify the action in one special case. For this purpose it seemed to be convenient to study algebraic K3 surfaces with an elliptic fibration with a section of order three, five, respectively seven. Then the translation by this section is a symplectic automorphism of the surface of the same order. A concrete analysis leads us to the main result of the paper which is the description of the lattices \( H^2(X, \mathbb{Z})^\sigma_p \) and \( \Omega_p = (H^2(X, \mathbb{Z})^\sigma_p)^\perp \) given in Theorem 4.1. The proof of the main theorem consists in Propositions 4.2, 4.4, 4.6. We describe the lattice \( \Omega_p \) also as \( \mathbb{Z}[\omega_p] \)-lattices, where \( \omega_p \) is a primitive \( p \) root of the unity. This kind of lattices are studied e.g. in [Ba], [BS] and [E]. In particular in the case \( p = 3 \) the lattice \( \Omega_3 \) is the Coxeter–Todd lattice with the form multiplied by \(-2\), \( K_{12}(-2) \), which is described in [CT] and in [CS].

The elliptic surfaces we used to find the lattices \( \Omega_p \) do not have the minimal possible Picard number. We prove in Proposition 5.1 that for K3 surfaces, \( X \), with minimal Picard number and symplectic automorphism, if \( L \) is a class in \( NS(X) \) which is invariant for the automorphisms, with \( L^2 = 2d > 0 \), then either \( NS(X) = \mathbb{Z}L \oplus \Omega_p \) or the latter is a sublattice of index \( p \) in \( NS(X) \). Using this result and the one of Proposition 5.2 we describe the coarse moduli space of the algebraic K3 surfaces admitting a symplectic automorphism of prime order.

The structure of the paper is the following: in Section 1 we compute the number of moduli of algebraic K3 surfaces admitting a symplectic automorphism of order \( p \) and their minimal Picard number. In Section 2 we give the definition of \( \mathbb{Z}[\omega_p] \)-lattice and we associate to it a module with a bilinear form, which in some cases is a \( \mathbb{Z} \)-lattice (we use this construction in Section 4 to describe the lattices \( \Omega_p \) as \( \mathbb{Z}[\omega_p] \)-lattices). In Section 3 we recall some results about elliptic fibrations and elliptic K3 surfaces (see e.g. [Mi1,Mi2,Shim,Shio] for more on elliptic K3 surfaces). In particular we introduce the three elliptic fibrations which we use in Section 4 and give also their Weierstrass form. In Section 4 we state and proof the main result, Theorem 4.1: we identify the lattices \( \Omega_p \) and we describe them as \( \mathbb{Z}[\omega_p] \)-lattices. In Section 5 we describe the Néron–Severi group of K3 surfaces admitting a symplectic automorphism and having minimal Picard number (Proposition 5.1). In Section 5 we describe the coarse moduli space of the algebraic K3 surfaces admitting a symplectic automorphism and the Néron–Severi group of those having minimal Picard number.
1. Preliminary results

**Definition 1.1.** A symplectic automorphism $\sigma_p$ of order $p$ on a K3 surface $X$ is an automorphism such that:

1. the group $G$ generated by $\sigma_p$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$,
2. $\sigma_p^*(\delta) = \delta$, for all $\delta$ in $H^{2,0}(X)$.

We recall that by [Ni1] an automorphism on a K3 surface is symplectic if and only if it acts as the identity on the transcendental lattice $T_X$. In local coordinates at a fixed point $\sigma_p$ has the form $\text{diag}(\omega_p, \omega_p^{-1})$ where $\omega_p$ is a primitive $p$-root of unity. By a result of Nikulin the only possible values for $p$ are 2, 3, 5, 7 see [Ni1, Theorem 4.5] and [Ni1, §5]. The automorphism $\sigma_3$ has six fixed points on $X$, $\sigma_5$ has four fixed points and $\sigma_7$ has three fixed points. The automorphism $\sigma_p$ induces a $\sigma_p^*$ isometry on $H^2(X, \mathbb{C}) \cong \Lambda_{K3}$. Nikulin proved [Ni1, Theorem 4.7] that if $\sigma_p$ is symplectic, then the action of $\sigma_p^*$ is unique up to isometry of $\Lambda_{K3}$.

Let $\omega_p$ be a primitive $p$-root of the unity. The vector space $H^2(X, \mathbb{C})$ can be decomposed in eigenspaces of the eigenvalues 1 and $\omega_p^j$:

$$H^2(X, \mathbb{C}) = H^2(X, \mathbb{C})^{\sigma_p^*} \oplus \bigoplus_{i=1, \ldots, p-1} H^2(X, \mathbb{C})^{\omega_p^j}.$$ 

We observe that the non-rational eigenvalues $\omega_p^j$ have all the same multiplicity. So we put:

$$a_p := \text{multiplicity of the eigenvalue 1}, \quad b_p := \text{multiplicity of the eigenvalues } \omega_p^j.$$ 

In the following we find $a_p$ and $b_p$ by using the Lefschetz fixed point formula:

$$\mu_p = \sum_{r} (-1)^r \text{trace}(\sigma_p^*|H^r(X, \mathbb{C})) \quad (1)$$

where $\mu_p$ denotes the number of fixed points. For K3 surfaces we obtain

$$\mu_p = 1 + 0 + \text{trace}(\sigma_p^*|H^2(X, \mathbb{C})) + 0 + 1.$$ 

**Proposition 1.1.** Let $X$, $\sigma_p$, $a_p$, $b_p$ be as above. $p = 3, 5, 7$. Let $\rho_p$ be the Picard number of $X$, and let $m_p$ be the dimension of the moduli space of the algebraic K3 surfaces admitting a symplectic automorphism of order $p$. Then

$$a_3 = 10, \quad b_3 = 6, \quad \rho_3 \geq 13, \quad m_3 \leq 7,$$

$$a_5 = 6, \quad b_5 = 4, \quad \rho_5 \geq 17, \quad m_5 \leq 3,$$

$$a_7 = 4, \quad b_7 = 3, \quad \rho_7 \geq 19, \quad m_7 \leq 1.$$ 

**Proof.** The proof is similar in all the cases, here we give the details only in the case $p = 5$.

A symplectic automorphism of order five on a K3 surface has exactly four fixed points. Applying the Lefschetz fixed points formula (1), we have $a_5 + b_5(\omega_5 + \omega_5^2 + \omega_5^3 + \omega_5^4) = 2$. Since $\omega_p^{p-1} = -(\sum_{i=0}^{p-2} \omega_p^i)$, the equation becomes $a_5 - b_5 = 2$. 

Since \( \dim H^2(X, \mathbb{C}) = 22 \), \( a_5 \) and \( b_5 \) have to satisfy:

\[
\begin{align*}
\begin{cases}
a_5 - b_5 = 2, \\
a_5 + 4b_5 = 22.
\end{cases}
\end{align*}
\]

We have

\[
dim H^2(X, \mathbb{C})^{\omega_5^*} = 6 = a_5 \quad \text{and} \quad dim H^2(X, \mathbb{C})_{\omega_5} = dim H^2(X, \mathbb{C})_{\omega_2^3} = dim H^2(X, \mathbb{C})_{\omega_3^2} = 4 = b_5.
\]

Since \( T_X \otimes \mathbb{C} \subset H^2(X, \mathbb{C})^{\omega_5^*} \), \( (H^2(X, \mathbb{C})^{\omega_5^*})^\perp = H^2(X, \mathbb{C})_{\omega_5} \oplus H^2(X, \mathbb{C})_{\omega_2^3} \oplus H^2(X, \mathbb{C})_{\omega_3^2} \oplus H^2(X, \mathbb{C})_{\omega_5^*} \subset \text{NS}(X) \otimes \mathbb{C} \). We consider only algebraic K3 surfaces and so we have an ample class \( h \) on \( X \), by taking \( h + \omega_5^* h + \omega_2^* h + \omega_3^* h + \omega_5^* h \) we get a \( \omega_5 \)-invariant class, hence in \( H^2(X, \mathbb{C})^{\omega_5^*} \). From here it follows that \( \rho_p = \text{rank NS}(X) \geq 16 + 1 = 17 \), whence \( \text{rank } T_X \leq 22 - 17 = 5 \). The number of moduli is at most \( 20 - 17 = 3 \). \( \square \)

**Remark.** In [Ni1, §10] Nikulin computes \( \text{rank}(H^2(X, \mathbb{Z})^{\omega_p^*})^\perp = (p - 1)b_p \) and \( \text{rank}(H^2(X, \mathbb{Z})^{\omega_p^*}) = a_p \). In [Ni1, Lemma 4.2] he also proves that there are no classes with self intersection \(-2\) in the lattices \( (H^2(X, \mathbb{Z})^{\omega_p^*})^\perp \); we describe these lattices in Sections 4.1, 4.4, 4.6 and we find again the result of Nikulin.

### 2. The \( \mathbb{Z}[\omega] \)-lattices

In Sections 4.2, 4.5, 4.7 our purpose is to describe \( (H^2(X, \mathbb{Z})^{\omega_p^*})^\perp \) as \( \mathbb{Z}[\omega_p] \)-lattice. We recall now some useful results on these lattices.

**Definition 2.1.** Let \( p \) be an odd prime and \( \omega := \omega_p \) be a primitive \( p \)-root of the unity. A \( \mathbb{Z}[\omega] \)-lattice is a free \( \mathbb{Z}[\omega] \)-module with an hermitian form (with values in \( \mathbb{Z}[\omega] \)). Its rank is its rank as \( \mathbb{Z}[\omega] \)-module.

Let \( \{L, h_L\} \) be a \( \mathbb{Z}[\omega] \)-lattice of rank \( n \). The \( \mathbb{Z}[\omega] \)-module \( L \) is also a \( \mathbb{Z} \)-module of rank \( (p - 1)n \). In fact if \( \{e_i, i = 1, \ldots, n\} \) is a basis of \( L \) as \( \mathbb{Z}[\omega] \)-module, \( \omega^j e_i, i = 1, \ldots, n, j = 0, \ldots, p - 2 \), is a basis for \( L \) as \( \mathbb{Z} \)-module (recall that \( \omega^{p-1} = -(\omega^{p-2} + \omega^{p-3} + \cdots + 1) \)). The \( \mathbb{Z} \)-module \( L \) will be called \( L_{\mathbb{Z}} \).

Let \( \Gamma_p := \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \) be the group of the automorphisms of \( \mathbb{Q}(\omega) \) which fix \( \mathbb{Q} \). We recall that the group \( \Gamma_p \) has order \( p - 1 \) and its elements are automorphisms \( \rho_i \) such that \( \rho_i(1) = 1, \rho_i(\omega) = \omega^j \) where \( i = 1, \ldots, p - 1 \). We define a bilinear form on \( L_{\mathbb{Z}} \)

\[
b_L(\alpha, \beta) = -\frac{1}{p} \sum_{\rho \in \Gamma_p} \rho(h_L(\alpha, \beta)).
\]

Note that \( b_L \) takes values in \( \frac{1}{p} \mathbb{Z}[\omega] \), so in general \( \{L_{\mathbb{Z}}, b_L\} \) is not a \( \mathbb{Z} \)-lattice. We call it the **associated** module (respectively lattice) of the \( \mathbb{Z}[\omega] \)-lattice \( L \).
Remark. By the definition of the bilinear form is clear that

\[ b_L(\alpha, \beta) = -\frac{1}{p} \text{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}(h_L(\alpha, \beta)). \]

For a precise definition of the Trace see [E, p. 128].

2.1. The \( \mathbb{Z} \)-lattice \( F_p \)

We consider a K3 surface admitting an elliptic fibration. Let \( p \) be an odd prime number. Let \( I_p \) be a semistable fiber of a minimal elliptic fibration, i.e. (cf. Section 3) \( I_p \) is a fiber which is a reducible curve, whose irreducible components are the edges of a \( p \)-polygon, as described in [Mi1, Table I.4.1], we denote the \( p \)-irreducible components by \( C_i \), \( i = 0, \ldots, p-1 \), then

\[ C_i \cdot C_j = \begin{cases} -2 & \text{if } i \equiv j \mod p, \\ 1 & \text{if } |i - j| \equiv 1 \mod p, \\ 0 & \text{otherwise}. \end{cases} \]

We consider now the free \( \mathbb{Z} \)-module \( F_p \) with basis the elements of the form \( C_i - C_{i+1} \), \( i = 1, \ldots, p-1 \) and with bilinear form \( b_{F_p} \) which is the restriction of the intersection form to the basis \( C_i - C_{i+1} \), then \( \{F_p, b_{F_p}\} \) is a \( \mathbb{Z} \)-lattice.

2.2. The \( \mathbb{Z}[\omega_p] \)-lattice \( G_p \)

Let \( G_p \) be the \( \mathbb{Z}[\omega] \)-lattice \( G_p := (1 - \omega)^2 \mathbb{Z}[\omega] \), with the standard hermitian form: \( h(\alpha, \beta) = \alpha \bar{\beta} \). A basis for the \( \mathbb{Z} \)-module \( G_{p, \mathbb{Z}} \) is \( (1 - \omega)^2 \omega^i \), \( i = 0, \ldots, p-2 \).

On \( \mathbb{Z}[\omega] \) we consider the bilinear form \( b_L \) defined in (3), with values in \( \frac{1}{p} \mathbb{Z} \),

\[ b(\alpha, \beta) = -\frac{1}{p} \sum_{\rho \in F_p} \rho(\alpha \bar{\beta}), \quad \alpha, \beta \in \mathbb{Z}[\omega], \]

then we have

Lemma 2.1. The bilinear form \( b \) restricted to \( G_p \) (denoted by \( b_G \)) has values in \( \mathbb{Z} \) and coincides with the intersection form on \( F_p \) by using the map \( F_p \rightarrow G_p \) defined by \( C_i - C_{i+1} \mapsto \omega^i (1 - \omega)^2 \), \( i = 1, \ldots, p-1 \), \( C_p = C_0 \).

Proof. An easy computation shows that we have for \( p > 3 \):

\[ b_G(\omega^k (1 - \omega)^2, \omega^h (1 - \omega)^2) = \begin{cases} -6 & \text{if } k \equiv h \mod p, \\ 4 & \text{if } |k - h| \equiv 1 \mod p, \\ -1 & \text{if } |k - h| \equiv 2 \mod p, \\ 0 & \text{otherwise}, \end{cases} \]

and for \( p = 3 \):

\[ b_G(\omega^k (1 - \omega)^2, \omega^h (1 - \omega)^2) = \begin{cases} -6 & \text{if } k \equiv h \mod p, \\ 3 & \text{if } |k - h| \equiv 1 \mod p. \end{cases} \]
The intersection form on $F_p$ is easy to compute (cf. Section 2.1) and this computation proves that the map $F_p \to G_p$ defined in the lemma is an isometry. □

In Section 4 we apply the results of this section and we find a $\mathbb{Z}[\omega]$-lattice $\{L_p, h_{L_p}\}$ such that

- $\{L_p, h_{L_p}\}$ contains $G_p$ as sublattice;
- $\{L_p, \mathbb{Z}, b_{L_p}\}$ is a $\mathbb{Z}$-lattice;
- the $\mathbb{Z}$-lattice $\{L_p, \mathbb{Z}, b_{L_p}\}$ is isometric to the $\mathbb{Z}$-lattice $(H^2(X, \mathbb{Z})^{\sigma_p})^\perp$ for $p = 3, 5, 7$.

3. Some general facts on elliptic fibrations

In the next section we give explicit examples of K3 surfaces admitting a symplectic automorphism $\sigma_p$ by using elliptic fibrations. Here we recall some general results about these fibrations.

Let $X$ be an elliptic K3 surface, this means that we have a morphism $f : X \to \mathbb{P}^1$ such that the generic fiber is a (smooth) elliptic curve. We assume moreover that we have a section $s : \mathbb{P}^1 \to X$. The sections of $X$ generate the Mordell–Weil group $MW_f$ of $X$ and we take $s$ as zero section. This group acts on $X$ by translation (on each fiber), hence it leaves the two form invariant. We assume that the singular fibers of the fibration are all of type $I_{m_j}$, $m_j \in \mathbb{N}$. Let $F_j$ be a fiber of type $I_{m_j}$, we denote by $C_j^{(i)}$ the irreducible components of the fibers meeting the zero section. After choosing an orientation, we denote the other irreducible components of the fibers by $C_j^{(i)}$, $i = 1, \ldots, m_j - 1$. In the sequel we always consider $m_j$ a prime number, and the notation $C_j^{(i)}$ means $i \in \mathbb{Z}/m_j\mathbb{Z}$. For each section $r$ we define the number $k := k_j(r)$ by

$$r \cdot C_j^{(k)} = 1 \quad \text{and} \quad r \cdot C_j^{(i)} = 0 \quad \text{if} \quad i = 0, \ldots, m_j - 1, \; i \neq k.$$ 

If the section $r$ is a torsion section and $h$ is the number of reducible fibers of type $I_{m_j}$, then by [Mi2, Proposition 3.1] we have

$$\sum_{j=1}^h k_j(r) \left(1 - \frac{k_j(r)}{m_j}\right) = 4. \quad (4)$$

Moreover we recall the Shioda–Tate formula (cf. [Shio, Corollary 5.3] or [Mi1, p. 70])

$$\text{rank}(NS(X)) = 2 + \sum_{j=1}^h (m_j - 1) + \text{rank}(MW_f). \quad (5)$$

The rank$(MW_f)$ is the number of generators of the free part. If there are no sections of infinite order then rank$(MW_f) = 0$. Assume that $X$ has $h$ fibers of type $I_m$, $m \in \mathbb{N}$, $m > 1$, and the remaining singular fibers are of type $I_1$, which are rational curves with one node. Let $U \oplus (A_{m-1})^h$ denote the lattice generated by the zero section, the generic fiber and by the components of the reducible fibers not meeting $s$. If there are no sections of infinite order then it has finite index in $NS(X)$ equal to $n$, the order of the torsion part of the group $MW_f$. Using this remark we find that
\[
|\det(\text{NS}(X))| = \frac{\det(A_{m-1})^h}{n^2} = \frac{m^h}{n^2}.
\]

3.1. Elliptic fibrations with a symplectic automorphism

Now we describe three particular elliptic fibrations which admit a symplectic automorphism \(\sigma_3, \sigma_5\) or \(\sigma_7\). Assume that we have a section of prime order \(p = 3, 5, 7\). By [Shim, No. 560, 2346, 3256] there exist elliptic fibrations with one of the following configurations of components of singular fibers \(I_p\) not meeting \(s\) such that all the singular fibers of the fibrations are semistable (i.e. they are all of type \(I_n\) for a certain \(n \in \mathbb{N}\)) and the order of the torsion subgroup of the Mordell–Weil group \(o(\text{MW}_f) = p\):

\[
\begin{align*}
  p = 3: & \quad 6A_2, \quad o(\text{MW}_f) = 3, \\
  p = 5: & \quad 4A_4, \quad o(\text{MW}_f) = 5, \\
  p = 7: & \quad 3A_6, \quad o(\text{MW}_f) = 7.
\end{align*}
\]

We can assume that the remaining singular fibers are of type \(I_1\). Since the sum of the Euler characteristic of the fibers must add up to 24, these are six, four, respectively three fibers. Observe that each section of finite order induces a symplectic automorphism of the same order which corresponds to a translation by the section on each fiber, we denote it by \(\sigma_p\). The nodes of the \(I_1\) fibers are then the fixed points of these automorphisms, whence \(\sigma_p\) permutes the \(p\) components of the \(I_p\) fibers. For these fibrations we have rank \(\text{NS}(X) = 14, 18, 20\) and dimensions of the moduli spaces six, two and zero, which is one less than the maximal possible dimension of the moduli space we have given in Proposition 1.1.

3.1.1. Weierstrass forms

We compute the Weierstrass form for the elliptic fibration described in (7). When \(X\) is a K3 surface then this form is

\[
y^2 = x^3 + A(t)x + B(t), \quad t \in \mathbb{P}^1,
\]

or in homogeneous coordinates

\[
x_3x_2^2 = x_1^3 + A(t)x_1x_3^2 + B(t)x_3^3
\]

where \(A(t)\) and \(B(t)\) are polynomials of degrees eight and twelve respectively, \(x_3 = 0\) is the line at infinity and also the tangent to the inflectional point \((0 : 1 : 0)\).

Fibration with a section of order 3. In this case the point of order three must be an inflectional point (cf. [C, Ex. 5, p. 38]), we want to determine \(A(t)\) and \(B(t)\) in Eq. (8). We start by imposing to a general line \(y = l(t)x + m(t)\) to be an inflectional tangent so the equation of the elliptic fibration is

\[
y^2 = x^3 + A(t)x + B(t), \quad t \in \mathbb{P}^1, \quad \text{with}
\]

\[
A(t) = \frac{6l(t)m(t) + l(t)^4}{3}, \quad B(t) = \frac{27m(t)^2 - l(t)^6}{3^3}.
\]
Since $A(t)$ and $B(t)$ are of degrees eight and twelve, we have $\deg l(t) = 2$ and $\deg m(t) = 6$. The section of order three is

$$t \mapsto \left( \frac{l(t)^2}{3}, \frac{l(t)^3}{3} + m(t) \right).$$

The discriminant $\Delta = 4A^3 + 27B^2$ of the fibration is

$$\Delta = \frac{(5l(t)^3 + 27m(t))(l(t)^3 + 3m(t))^3}{27}$$

hence in general it vanishes to the order three on six values of $t$ and to the order one on other six values. Since $A$ and $B$ in general do not vanish on these values, this equation parametrizes an elliptic fibration with six fibers $I_3$ (so we have six curves $A_2$ not meeting the zero section) and six fibers $I_1$ (cf. [Mi1, Table IV.3.1, p. 41]).

**Fibration with a section of order 5.** In the same way we can compute the Weierstrass form of the elliptic fibration described in (7) with a section of order five.

In [BM] a geometrical condition for the existence of a point of order five on an elliptic curve is given. For fixed $t$ let the cubic curve be in the form (9) then take two arbitrary distinct lines through $O$ which meet the cubic in two other distinct points each. Call 1, 4 the points on the first line and 2, 3 the points on the second line, then 1 (or any of the other point) has order five if:

- the tangent through 1 meets the cubic in 3,
- the tangent through 4 meets the cubic in 2,
- the tangent through 3 meets the cubic in 4,
- the tangent through 2 meets the cubic in 1.

These conditions give the Weierstrass form:

$$y^2 = x^3 + A(t)x + B(t), \quad t \in \mathbb{P}^1,$$

with

$$A(t) = \frac{(-b(t)^4 + b(t)^2a(t)^2 - a(t)^4 - 3a(t)b(t)^3 + 3a(t)^3b(t))}{3},$$

$$B(t) = \frac{(b(t)^2 + a(t)^2)(19b(t)^4 - 34b(t)^2a(t)^2 + 19a(t)^4 + 18a(t)b(t)^3 - 18a(t)^3b(t))}{108}$$

where $\deg a(t) = 2$, $\deg b(t) = 2$. The section of order five is

$$t \mapsto \left( (2b(t)^2 - a(t)^2)^2 : 3(a(t) + b(t))(a(t) - b(t))^2 : 6 \right)$$

and the discriminant is

$$\Delta = \frac{1}{16} \frac{(b(t)^2 - a(t)^2)^5}{(11(b(t)^2 - a(t)^2) + 4a(t)b(t))}.$$

By a careful analysis of the zeros of the discriminant we can see that the fibration has four fibers $I_5$ and four fibers $I_1$ (cf. [Mi1, Table IV.3.1, p. 41]).
Fibration with a section of order 7. To find the Weierstrass form we use also in this case the results of [BM]. We explain briefly the idea to find a set of points of order seven on an elliptic curve. One takes points 0, 3, 4 and 1, 2, 4 on two lines in the plane. Then the intersections of a line through 3 different from the lines {3, 2}, {3, 4}, {3, 1} with the lines {1, 0} and {2, 0} give two new points $-1$ and $-2$. By using the conditions that the tangent through 1 goes through $-2$ and the tangent through 2 goes through 3 one can determine a cubic having a point of order seven which is e.g. 1. By using these conditions one can find the equation, but since the computations are quite involved, we recall the Weierstrass form given in [T, p. 195]

$$y^2 + (1 + t - r^2)xy + (r^2 - r^3)y = x^3 + (r^2 - r^3)x^2.$$  

By a direct check one sees that the point of order seven is $(0(t), 0(t))$. This elliptic fibration has three fibers $I_7$ and three fibers $I_1$.

4. Elliptic K3 surfaces with an automorphism of prime order

In this section we prove the main theorem:

**Theorem 4.1.** For any K3 surface $X$ with a symplectic automorphism $\sigma_p$ of order $p = 2, 3, 5, 7$ the action on $H^2(X, \mathbb{Z})$ decomposes in the following way:

$$p = 2: \quad H^2(X, \mathbb{Z})^{\sigma_2} = E_8(-2) \oplus U \oplus U \oplus U; \quad \left( H^2(X, \mathbb{Z})^{\sigma_2} \right)_{\perp} = E_8(-2),$$

$$p = 3: \quad H^2(X, \mathbb{Z})^{\sigma_3} = U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2; \quad \left( H^2(X, \mathbb{Z})^{\sigma_3} \right)_{\perp} = \left\{ (x_1, \ldots, x_6) \in (\mathbb{Z}[\omega_3])^{\oplus 6}: x_i \equiv x_j \mod (1 - \omega_3), \quad \sum_{i=1}^{6} x_i \equiv 0 \mod (1 - \omega_3)^2 \right\} = K_1(-2)$$

with hermitian form $h(\alpha, \beta) = \sum_{i=1}^{6} (\alpha_i \bar{\beta}_i)$.

$$p = 5: \quad H^2(X, \mathbb{Z})^{\sigma_5} = U \oplus U(5) \oplus U(5), \quad \left( H^2(X, \mathbb{Z})^{\sigma_5} \right)_{\perp} = \left\{ (x_1, \ldots, x_4) \in (\mathbb{Z}[\omega_5])^{\oplus 4}: x_1 \equiv x_2 \equiv 2x_3 \equiv 2x_4 \mod (1 - \omega_5), \quad (3 - \omega_5)(x_1 + x_2) + x_3 + x_4 \equiv 0 \mod (1 - \omega_5)^2 \right\}$$

with hermitian form $h(\alpha, \beta) = \sum_{i=1}^{4} \alpha_i \bar{\beta}_i + \sum_{j=3}^{4} f \alpha_j \bar{\beta}_j$ where $f = 1 - (\omega_5^2 + \omega_5^3)$.

$$p = 7: \quad H^2(X, \mathbb{Z})^{\sigma_7} = U(7) \oplus \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}, \quad \left( H^2(X, \mathbb{Z})^{\sigma_7} \right)_{\perp} = \left\{ (x_1, x_2, x_3) \in (\mathbb{Z}[\omega_7])^{\oplus 3}: x_1 \equiv x_2 \equiv 6x_3 \mod (1 - \omega_7), \quad (1 + 5\omega_7)x_1 + 3x_2 + 2x_3 \equiv 0 \mod (1 - \omega_7)^2 \right\}.$$


with hermitian form \( h(\alpha, \beta) = \alpha_1 \overline{\beta}_1 + f_1 \alpha_2 f_1 \overline{\beta}_2 + f_2 \alpha_3 f_2 \overline{\beta}_3 \) where \( f_1 = 3 + 2(\omega_7 + \omega_6) + (\omega_2^7 + \omega_5^7) \) and \( f_2 = 2 + (\omega_7 + \omega_6^7) \).

In the case \( p = 3 \), \( K_{12}(-2) \) denotes the Coxeter–Todd lattice with the bilinear form multiplied by \(-2\).

This theorem gives a complete description of the invariant sublattice \( H^2(X, \mathbb{Z})^{\sigma_p} \) and its orthogonal complement in \( H^2(X, \mathbb{Z}) \) for the symplectic automorphisms \( \sigma_p \) of all possible prime order \( p = 2, 3, 5, 7 \) acting on a K3 surface. The results about the order two automorphism is proven by Morrison in [Mo, Theorem 5.7].

We describe the lattices of the theorem and their hermitian forms in the sections from 4.1 to 4.7. The proof is the following: we identify the action of \( \sigma_p^* \) on \( H^2(X, \mathbb{Z}) \) in the case of \( X \) an elliptic K3 surface, this is done in several propositions in these sections, then we apply [Ni1, Theorem 4.7] which assure the uniqueness of this action.

### 4.1. A section of order three

Let \( X \) be a K3 surface with an elliptic fibration which admits a section of order three described in (7) of Section 3. We recall that \( X \) has six reducible fibers of type \( I_3 \) and six singular irreducible fibers of type \( I_1 \). In the preceding section we have seen that the rank of the Néron–Severi group is 14. We determine now \( NS(X) \) and \( T_X \).

Let \( t_1 \) denote the section of order three and \( t_2 = t_1 + t_1 \). Let \( \sigma_3 \) be the automorphism of \( X \) which corresponds to the translation by \( t_1 \). It leaves each fiber invariant and \( \sigma_3^*(s) = t_1 \), \( \sigma_3^*(t_1) = t_2 \), \( \sigma_3^*(t_2) = s \). Denoted by \( C^{(i)}_0, C^{(i)}_1, C^{(i)}_2 \) the components of the \( i \)th reducible fiber \( (i = 1, \ldots, 6) \), we can assume that \( C^{(i)}_1 \cdot t_1 = C^{(i)}_2 \cdot t_2 = C^{(i)}_0 \cdot s = 1 \).

**Proposition 4.1.** A \( \mathbb{Z} \)-basis for the lattice \( NS(X) \) is given by

\[
\{ s, t_1, t_2, F, C^{(1)}_1, C^{(1)}_2, C^{(2)}_1, C^{(2)}_2, C^{(3)}_1, C^{(3)}_2, C^{(4)}_1, C^{(4)}_2, C^{(5)}_1, C^{(5)}_2 \}.
\]

Let \( U \oplus A^6_2 \) be the lattice generated by the section, the fiber and the irreducible components of the six fibers \( I_3 \) which do not intersect the zero section \( s \). It has index three in the Néron–Severi group of \( X, NS(X) \). The lattice \( NS(X) \) has discriminant \(-3^4\) and its discriminant form is

\[
\mathbb{Z}_3 \left( \frac{2}{3} \right) \oplus \mathbb{Z}_3 \left( \frac{2}{3} \right) \oplus \mathbb{Z}_3 \left( \frac{2}{3} \right) \oplus \mathbb{Z}_3 \left( -\frac{2}{3} \right).
\]

The transcendental lattice \( T_X \) is

\[
T_X = U \oplus U(3) \oplus A_2 \oplus A_2
\]

and has a unique primitive embedding in the lattice \( \Lambda_{K3} \).

**Proof.** It is clear that a \( \mathbb{Q} \)-basis for \( NS(X) \) is given by \( s, F, C^{(i)}_1, C^{(i)}_2, i = 1, \ldots, 6 \). This basis generates the lattice \( U \oplus A^6_2 \). It has discriminant \( d(U \oplus A^6_2) = -3^6 \). We denote by
\[ c_i = 2C_1^{(i)} + C_2^{(i)}, \quad C = \sum c_i, \]
\[ d_i = C_1^{(i)} + 2C_2^{(i)}, \quad D = \sum d_i. \]

Since we know that \( t_1 \in \text{NS}(X) \) we can write
\[ t_1 = \alpha s + \beta F + \sum \gamma_i C_1^{(i)} + \sum \delta_i C_2^{(i)}, \quad \alpha, \beta, \gamma_i, \delta_i \in \mathbb{Q}. \]

Then by using the fact that \( t_1 \cdot s = t_1 \cdot C_2^{(i)} = 0 \) and \( t_1 \cdot C_1^{(i)} = t_1 \cdot F = 1 \) one obtains that \( \alpha = 1, \beta = 2 \) and \( \gamma_1 = -2/3, \delta_1 = -1/3 \) hence \( \frac{1}{3}C \in \text{NS}(X) \). A similar computation with \( t_2 \) shows that \( \frac{1}{3}D \in \text{NS}(X) \). So one obtains that
\[ t_1 = s + 2F - \frac{1}{3}C \in \text{NS}(X), \]
\[ t_2 = s + 2F - \frac{1}{3}D \in \text{NS}(X), \tag{10} \]
and so
\[ 3(t_2 - t_1) = \sum_{i=1}^{6} (C_1^{(i)} - C_2^{(i)}) = C - D. \]

We consider now the \( \mathbb{Q} \)-basis for the Néron–Severi group
\[ s, t_1, t_2, F, C_1^{(1)}, C_2^{(1)}, C_1^{(2)}, C_2^{(2)}, C_1^{(3)}, C_2^{(3)}, C_1^{(4)}, C_2^{(4)}, C_1^{(5)}, C_2^{(5)}. \]

By computing the matrix of the intersection form respect to this basis one finds that the determinant is \(-3^4\). By the Shioda–Tate formula we have \( |\text{det}(\text{NS}(X))| = 3^4 \). Hence this is a \( \mathbb{Z} \)-basis for the Néron–Severi group. We add to the classes which generate \( U \oplus A_2^6 \) the classes \( t_1 \) and \( \sigma^4_3(t_1) = t_2 \) given in the formula (10). Since \( d(U \oplus A_2^6) = 3^6 \) and \( d(\text{NS}(X)) = -3^4 \) the index of \( U \oplus A_2^6 \) in \( \text{NS}(X) \) is 3. Observe that this is also a consequence of a general result given at the end of Section 3.

The classes
\[ v_i = \frac{C_1^{(i)} - C_2^{(i)} - (C_1^{(5)} - C_2^{(5)})}{3}, \quad i = 1, \ldots, 4, \]
generate the discriminant group, which is \( \text{NS}(X)^\vee/\text{NS}(X) \cong (\mathbb{Z}/3\mathbb{Z})^\oplus 4 \).

These classes are not orthogonal to each other with respect to the bilinear form, so we take
\[ w_1 = v_1 - v_2, \quad w_2 = v_3 - v_4, \quad w_3 = v_1 + v_2 + v_3 + v_4, \quad w_4 = v_1 + v_2 - (v_3 + v_4) \]
which form an orthogonal basis with respect to the bilinear form with values in \( \mathbb{Q}/\mathbb{Z} \). And it is easy to compute that \( w_1^2 = w_2^2 = w_3^2 = 2/3, \ w_4^2 = -2/3 \). The discriminant form of the lattice \( \text{NS}(X) \) is then
\[
\mathbb{Z}_3\left(\frac{2}{3}\right) \oplus \mathbb{Z}_3\left(\frac{2}{3}\right) \oplus \mathbb{Z}_3\left(\frac{2}{3}\right) \oplus \mathbb{Z}_3\left(-\frac{2}{3}\right). \tag{11}
\]

The transcendental lattice \( T_X \) orthogonal to \( NS(X) \) has rank eight. Since \( NS(X) \) has signature (1, 13), the transcendental lattice has signature (2, 6). The discriminant form of the transcendental lattice is the opposite of the discriminant form of the Néron–Severi lattice. So the transcendental lattice has signature (2, 6), discriminant 34, discriminant group \( T^\vee_X / T_X \cong \left( \mathbb{Z}/3\mathbb{Z} \right)^{\oplus 4} \) and discriminant form \( \mathbb{Z}_3\left(-\frac{2}{3}\right) \oplus \mathbb{Z}_3\left(-\frac{2}{3}\right) \oplus \mathbb{Z}_3\left(\frac{2}{3}\right) \). By [Ni2, Corollary 1.13.5] we have \( T = U \oplus T' \) where \( T' \) has rank six, signature (1, 5) and \( T' \) has discriminant form as before. These data identify \( T' \) uniquely [Ni2, Corollary 1.13.3]. Hence it is isomorphic to \( U(3) \oplus A_2 \oplus A_2 \) with generators for the discriminant form

\[
\frac{e - f}{3}, \quad \frac{e + f}{3}, \quad \frac{A - B}{3}, \quad \frac{A' - B'}{3},
\]

where \( e, f, A, B, A', B' \) are the usual bases of the lattices.

The transcendental lattice

\[
T_X = U \oplus U(3) \oplus A_2 \oplus A_2
\]

has a unique embedding in the lattice \( \Lambda_{K3} \) by [Ni2, Theorem 1.14.4] or [Mo, Corollary 2.10].

4.1.1. The invariant lattice and its orthogonal complement

**Proposition 4.2.** The invariant sublattice of the Néron–Severi group is isometric to \( U(3) \) and it is generated by the classes \( F \) and \( s + t_1 + t_2 \).

The invariant sublattice \( H^2(X, \mathbb{Z})^\sigma_3 \) is isometric to \( U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2 \).

Its orthogonal complement \( \Omega_3 := (H^2(X, \mathbb{Z})^\sigma_3)^\perp \) is the negative definite twelve dimensional lattice \( \{ \mathbb{Z}^{12}, M \} \) where \( M \) is the bilinear form

\[
\begin{pmatrix}
-4 & 2 & -3 & -2 & 0 & -2 & 0 & -2 & 0 & -2 & 0 & -2 \\
2 & -4 & 3 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
-3 & 3 & -18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -9 \\
-2 & 1 & 0 & -6 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & -4 & 3 & 2 & 0 & 0 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 & 3 & -6 & -3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -3 & -4 & 3 & 2 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 & 0 & 0 & 3 & -6 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -3 & -4 & 3 & 2 & 0 \\
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -6 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & -3 & -4 & 3 \\
-2 & 1 & -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -6
\end{pmatrix}
\]

and it is equal to the lattice \( (NS(X)^\sigma_3)^\perp \).

The lattice \( \Omega_3 \) admits a unique primitive embedding in the lattice \( \Lambda_{K3} \).

The discriminant of \( \Omega_3 \) is \( 3^6 \) and its discriminant form is \( \left( \mathbb{Z}_3\left(\frac{2}{3}\right) \right)^{\oplus 6} \).

The isometry \( \sigma_3^* \) acts on the discriminant group \( \Omega_3^\vee / \Omega_3 \) as the identity.

**Proof.** It is clear that the isometry \( \sigma_3^* \) fixes the classes \( F \) and \( s + t_1 + t_2 \). These generate a lattice \( U(3) \) (with basis \( F \) and \( F + s + t_1 + t_2 \)).
The invariant sublattice $H^2(X, \mathbb{Z})^{\sigma^*}$ contains $T_X$ and the invariant sublattice of the Néron–Severi group. So $(H^2(X, \mathbb{Z})^{\sigma^*})^\perp = (NS(X)^{\sigma^*})^\perp$, this lattice has signature $(0, 12)$ and by [Ni1, p. 133] the discriminant group is $(\mathbb{Z}/3\mathbb{Z})^{\oplus 6}$. Hence by [Ni2, Theorem 1.14.4] there is a unique primitive embedding of $(H^2(X, \mathbb{Z})^{\sigma^*})^\perp$ in the $K3$-lattice. By using the orthogonality conditions one finds the following basis of $\Omega_3 = (NS(X)^{\sigma^*})^\perp$:

$$b_1 = t_2 - t_1, \quad b_2 = s - t_2, \quad b_3 = F - 3c_2^{(5)}, \quad b_{2(i+1)} = C_i^{(i)} - C_2^{(i)}, \quad i = 1, \ldots, 5,$$

$$b_{2j+3} = C_j^{(j)} - C_2^{(j+1)}, \quad j = 1, \ldots, 4.$$

An easy computation shows that the Gram matrix of this basis is exactly the matrix $M$ which indeed has determinant 36.

Since $H^2(X, \mathbb{Z})^{\sigma^*} \supseteq T_X \oplus NS(X)^{\sigma^*} = U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2$ and these lattices have the same rank, to prove that the inclusion is an equality we compare their discriminants. The lattice $(H^2(X, \mathbb{Z})^{\sigma^*})^\perp$ has determinant 36. So the lattice $(H^2(X, \mathbb{Z})^{\sigma^*})$ has determinant $-3^6$ (because these are primitive sublattices of $H^2(X, \mathbb{Z})$). The lattice $U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2$ has determinant exactly $-3^6$, so

$$H^2(X, \mathbb{Z})^{\sigma^*} = U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2.$$

Since $NS(X)^{\vee}/NS(X) \subset \Omega_3^{\vee}/\Omega_3$ the generators of the discriminant form of the lattice $\Omega_3$ are classes $w_1, \ldots, w_6$ with $w_1, \ldots, w_4$ the classes which generate the discriminant form of $NS(X)$ (cf. the proof of Proposition 4.1) and

$$w_5 = \frac{1}{3}(b_1 + 2b_2) = \frac{1}{3}(2s - t_1 - t_2),$$

$$w_6 = \frac{1}{3}(b_1 + 2b_2 - 2b_3) = \frac{1}{3}(2s - t_1 - t_2 - 2F + 6C_2^{(5)}).$$

These six classes are orthogonal, with respect to the bilinear form taking values in $\mathbb{Q}/\mathbb{Z}$, and generate the discriminant form. Their squares are $w_1^2 = w_2^2 = w_3^2 = w_5^2 \equiv \frac{2}{3} \mod 2\mathbb{Z}$, $w_4^2 = w_6^2 \equiv -\frac{2}{3} \mod 2\mathbb{Z}$. By replacing $w_4$, $w_6$ by $w_4 - w_6$, $w_4 + w_6$ we obtain the discriminant form $(\mathbb{Z}_{3^2})^{\oplus 6}$.

By computing the image of $w_i$, $i = 1, \ldots, 6$, under $\sigma_3^*$ one finds that $\sigma_3^*(w_i) - w_i \in \Omega_3$. For example: $\sigma_3^*(w_5) - w_5 = \frac{1}{3}(2t_1 - t_2 - s) = \frac{1}{3}(2s - t_1 - t_2) = t_1 - s$ which is an element of $\Omega_3$ (in fact it is orthogonal to $F$ and to $s + t_1 + t_2$). Hence the action of $\sigma_3^*$ is trivial on $\Omega_3^{\vee}/\Omega_3$ as claimed. \hfill $\square$

In the next two subsections we apply the results of Section 2 about the $\mathbb{Z}[\omega]$-lattices to describe the lattice $\{\Omega_3, M\}$ and to prove that $\Omega_3$ is isomorphic to the lattice $K_{12}(-2)$, where $K_{12}$ is the Coxeter–Todd lattice (cf. e.g. [CT,CS] for a description of this lattice).

4.2. The lattice $\Omega_3$

Let $\omega_3$ be a primitive third root of the unity. In this section we prove the following result (we use the same notations of Section 2):
Theorem 4.2. The lattice $\Omega_3$ is isometric to the $\mathbb{Z}$-lattice associated to the $\mathbb{Z}[\omega_3]$-lattice 
$\{L_3, h_{L_3}\}$ where

$$L_3 = \left\{ (x_1, \ldots, x_6) \in \left( \mathbb{Z}[\omega_3] \right)^{\oplus 6} : x_i \equiv x_j \mod (1 - \omega_3), \sum_{i=1}^6 x_i \equiv 0 \mod (1 - \omega_3)^2 \right\}$$

and $h_{L_3}$ is the restriction of the standard hermitian form on $\mathbb{Z}[\omega_3]^{\oplus 6}$.

Proof. Let $F = F_3^6$ be the $\mathbb{Z}$-sublattice of $NS(X)$ generated by

$$C_i^{(j)} - C_i^{(j+1)}, \quad i = 0, 1, 2, \quad j = 1, \ldots, 6,$$

with bilinear form induced by the intersection form on $NS(X)$.

Let $G = G_3^6$ denote the $\mathbb{Z}[\omega_3]$-lattice $\left( 1 - \omega_3 \right)^2 \mathbb{Z}[\omega_3]^{\oplus 6}$ with the standard hermitian form. This is a sublattice of $\mathbb{Z}[\omega_3]^{\oplus 6}$. Applying to each component of $G$ Lemma 2.1 we know that $\{G, b_G\}$ is a $\mathbb{Z}$-lattice isometric to the lattice $F$. The explicit isometry is given by

$$C_i^{(1)} - C_i^{(1)} \mapsto (1 - \omega_3)^2 \left( \omega_3^{i-1}, 0, 0, 0, 0, 0 \right),$$
$$C_i^{(2)} - C_i^{(2)} \mapsto (1 - \omega_3)^2 \left( 0, \omega_3^{i-1}, 0, 0, 0, 0 \right),$$
$$\vdots$$
$$C_i^{(6)} - C_i^{(6)} \mapsto (1 - \omega_3)^2 \left( 0, 0, 0, 0, 0, \omega_3^{i-1} \right).$$

The multiplication by $\omega_3$ of an element $(1 - \omega_3)^2 e_j$ (where $e_j$ is the canonical basis) corresponds to a translation by $t_1$ on a singular fiber, which sends the curve $C_i^{(j)}$ to the curve $C_i^{(j+1)}$. Hence we have a commutative diagram:

$$\begin{array}{ccc}
F & \longrightarrow & G \\
\sigma_3^* \downarrow & & \downarrow \omega_3 \\
F & \longrightarrow & G.
\end{array}$$

The elements $C_i^{(j)} - C_k^{(j)}$, $i, k = 0, 1, 2, \quad j = 1, \ldots, 6$, are all contained in the lattice $\Omega_3 = (NS(X)^{\sigma_3^*})^\perp$, but they do not generate this lattice. A set of generators for $\Omega_3$ is

$$s - t_1, \quad t_1 - t_2, \quad C_i^{(j)} - C_h^{(k)}, \quad i, h = 0, 1, 2, \quad j, k = 1, \ldots, 6.$$ 

From the formula (10) we obtain that

$$s - t_1 = \sum_{j=1}^6 \left[ \frac{1}{3} (C_1^{(j)} - C_2^{(j)}) + \frac{1}{3} \sigma_3^a (C_1^{(j)} - C_2^{(j)}) \right].$$
After the identification of $F$ with $G_Z$ we have

$$s - t_1 = (1 - \omega_3)^2 \left( \frac{1}{3} (1 + \omega_3) \right) (1, 1, 1, 1, 1, 1) = (1, 1, 1, 1, 1, 1).$$

The divisor $t_1 - t_2$, which is the image of $s - t_1$ under the action of $\sigma_3^*$, corresponds to the vector $(\omega_3, \omega_3, \omega_3, \omega_3, \omega_3, \omega_3)$. Similarly one can see that the element $C_1^{(1)} - C_1^{(2)}$ corresponds to the vector $(1 - \omega_3)(1, -1, 0, 0, 0, 0, 0)$ and more in general $C_i^{(j)} - C_i^{(k)}$ with $j \not= k$ corresponds to the vector $(1 - \omega_3)(\omega_3^{i-1} e_j - \omega_3^{i-1} e_k)$ where $e_i$ is the standard basis. The lattice $L_3$ generated by the vectors of $G_Z$ and by

$$\omega_3^i (1, 1, 1, 1, 1, 1), \quad (1 - \omega_3)\omega_3^{i-1} (-e_j + \omega_3 e_k), \quad i = 0, 1, 2, \quad j, k = 1, \ldots, 6,$$

is thus isometric to $\Omega_3$.

In conclusion a basis for $\Omega_3$ is

$$l_1 = -\omega_3 (1, 1, 1, 1, 1, 1), \quad l_2 = -\omega_3^2 (1, 1, 1, 1, 1, 1) = (1 + \omega_3)(1, 1, 1, 1, 1, 1),$$

$$l_3 = (1 - \omega_3)^2 (0, 0, 0, 0, 1 - \omega_3, 0), \quad l_4 = (1 - \omega_3)^2 (0, 1, 0, 0, 0, 0),$$

$$l_5 = (1 - \omega_3)(1, -1, 0, 0, 0, 0), \quad l_6 = (1 - \omega_3)^2 (0, 1, 0, 0, 0, 0),$$

$$l_7 = (1 - \omega_3)(0, 1, -1, 0, 0, 0), \quad l_8 = (1 - \omega_3)^2 (0, 0, 1, 0, 0, 0),$$

$$l_9 = (1 - \omega_3)(0, 0, 1, -1, 0, 0), \quad l_{10} = (1 - \omega_3)^2 (0, 0, 0, 1, 0, 0),$$

$$l_{11} = (1 - \omega_3)(0, 0, 0, 1, -1, 0), \quad l_{12} = (1 - \omega_3)^2 (0, 0, 0, 0, 1, 0).$$

The identification between $\Omega_3$ and $L_3$ is given by the map $b_i \mapsto l_i$.

After this identification the intersection form on $\Omega_3$ is exactly the form $b_{|L_3}$ on $L_3$.

The basis $l_i$ of $L_3$ satisfies the condition given in the theorem, and so

$$L_3 \subseteq \left\{ (x_1, \ldots, x_6) \in (\mathbb{Z}[\omega_3])^\oplus_6 : x_i \equiv x_j \ mod \ (1 - \omega_3), \sum_{i=1}^{6} x_i \equiv 0 \ mod \ (1 - \omega_3)^2 \right\}.$$

Since the vectors $(1 - \omega_3)^2 e_j, (1 - \omega_3)(e_i - e_j)$ and $(1, 1, 1, 1, 1, 1)$ generate the $\mathbb{Z}[\omega_3]$-lattice \{(x_1, \ldots, x_6) \in (\mathbb{Z}[\omega_3])^\oplus_6 : x_i \equiv x_j \ mod \ (1 - \omega_3), \sum_{i=1}^{6} x_i \equiv 0 \ mod \ (1 - \omega_3)^2 \} and since they are all vectors contained in $L_3$, the equality holds. $\Box$

4.3. The Coxeter–Todd lattice $K_{12}$

**Theorem 4.3.** The lattice $\Omega_3$ is isometric to the lattice $K_{12}(-2)$.

**Proof.** The lattice $K_{12}$ is described by Coxeter and Todd in [CT] and by Conway and Sloane in [CS]. The lattice $K_{12}$ is the twelve dimensional $\mathbb{Z}$-module associated to a six dimensional $\mathbb{Z}[\omega_3]$-lattice $A_6^{\omega_3}$. 
The \( \mathbb{Z}[\omega_3] \)-lattice \( A_6^{\omega_3} \) is described in [CS] in four different ways. We recall one of them denoted by \( A^{(3)} \) in [CS, Definition 2.3], which is convenient for us. Let \( \theta = \omega_3 - \bar{\omega}_3 \), then \( \Lambda_6^{\omega_3} = \{ (x_1, \ldots, x_6) : x_i \in \mathbb{Z}[\omega_3], x_i \equiv x_j \mod \theta, \sum_{i=1}^6 x_i \equiv 0 \mod 3 \} \) with hermitian form \( \frac{1}{3} \bar{x} \bar{y} \). We observe that \( \theta = \omega_3(1 - \omega_3) \). The element \( \omega_3 \) is a unit in \( \mathbb{Z}[\omega_3] \) so the congruence modulo \( \theta \) is the same as the congruence modulo \( 1 - \omega_3 \). Observing that \( -3 = \theta^2 \) it is then clear that the \( \mathbb{Z}[\omega_3] \)-module \( \Lambda_6^{\omega_3} \) is the \( \mathbb{Z}[\omega_3] \)-module \( L_3 \). The \( \mathbb{Z} \)-modules \( K_{12} \) and \( L_{3,\mathbb{Z}} \) are isomorphic since they are the twelve dimensional \( \mathbb{Z} \)-lattices associated to the same \( \mathbb{Z}[\omega_3] \)-lattice. The bilinear form on the \( \mathbb{Z} \)-module \( K_{12} \) is given by \( b_{K_{12}}(x, y) = \frac{1}{3} \bar{x} \bar{y} = \frac{1}{6} \text{Tr}(x \bar{y}) \) and the bilinear form on \( L_{3,\mathbb{Z}} \) is given by \( b_{L_3}(x, y) = -\frac{1}{3} \text{Tr}(x \bar{y}) \).

So the \( \mathbb{Z} \)-lattice \( \{ L_{3,\mathbb{Z}}, b_{L_3} \} \) is isometric to \( K_{12}(-2) \).

**Remark.** (1) In [CT] Coxeter and Todd give an explicit basis of the \( \mathbb{Z} \)-lattice \( K_{12} \). By a direct computation one can find the change of basis between the basis described in [CT] and the basis \( \{ b_i \} \) given in the proof of Proposition 4.2.

(2) The lattice \( \Omega_3 \) does not contain vectors of norm \(-2\) (cf. [Ni1, Lemma 4.2]), but has 756 vectors of norm \(-4\), 4032 of norm \(-6\) and 20 412 of norm \(-8\). Since these properties define the lattice \( K_{12}(-2) \) (cf. [CS, Theorem 1]), this is another way to prove the equality between \( \Omega_3 \) and \( K_{12}(-2) \).

(3) The lattice \( K_{12}(-2) \) is generated by vectors of norm \(-4\), [PP, Section 3].

### 4.4. Section of order five

Let \( X \) be a K3 surface with an elliptic fibration which admits a section of order five as described in Section 3. We recall that \( X \) has four reducible fibers of type \( I_5 \) and four singular irreducible fibers of type \( I_1 \). We have seen that the rank of the Néron–Severi group is 18. We determine now \( NS(X) \) and \( T_X \).

We label the fibers and their components as described in Section 3. Let \( t_1 \) denote the section of order five which meets the first singular fiber in \( C_1^{(1)} \). By the formula (4) of Section 3 up to permutation of the fibers only the following situations are possible:

\[
t_1 \cdot C_1^{(1)} = t_1 \cdot C_1^{(2)} = t_1 \cdot C_2^{(3)} = t_1 \cdot C_2^{(4)} = 1 \quad \text{and} \quad t_1 \cdot C_i^{(j)} = 0 \quad \text{otherwise};
\]

or
\[
t_1 \cdot C_1^{(1)} = t_1 \cdot C_4^{(2)} = t_1 \cdot C_2^{(3)} = t_1 \cdot C_3^{(4)} = 1 \quad \text{and} \quad t_1 \cdot C_i^{(j)} = 0 \quad \text{otherwise}.
\]

Observe that these two cases describe the same situation if we change the “orientation” on the last two fibers, so we assume to be in the first case. Let \( \sigma_5 \) be the automorphism of order five which
leaves each fiber invariant and is the translation by \( t_1 \), so \( \sigma_3^s(s) = t_1, \sigma_3^s(t_1) = t_2, \sigma_3^s(t_2) = t_3, \sigma_3^s(t_3) = t_4, \sigma_3^s(t_4) = s \).

**Proposition 4.3.** A \( \mathbb{Z} \)-basis for the lattice \( NS(X) \) is given by

\[
s, t_1, t_2, t_3, t_4, F, C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)}, C_1^{(2)}, C_2^{(2)}, C_3^{(2)}, C_4^{(2)}, C_1^{(3)}, C_2^{(3)}, C_3^{(3)}, C_4^{(3)}.
\]

Let \( U \oplus A_4^4 \) be the lattice generated by the section, the fiber and the irreducible components of the four fibers \( I_5 \) which do not intersect the zero section \( s \). It has index five in the Néron–Severi group of \( X, NS(X) \).

The lattice \( NS(X) \) has discriminant \(-5^2\) and its discriminant form is

\[
\mathbb{Z}_5 \left( \frac{2}{5} \right) \oplus \mathbb{Z}_5 \left( -\frac{2}{5} \right).
\]

The transcendental lattice is

\[
T_X = U \oplus U(5)
\]

and has a unique primitive embedding in the lattice \( \Lambda_K^3 \).

**Proof.** The proof is similar to the proof of Proposition 4.1. So we sketch it briefly. The classes \( s, F, C_i^{(j)}, i = 1, \ldots, 4, j = 1, \ldots, 4 \), generate \( U \oplus A_4^4 \). By using the intersection form, or by the result of [Mi2, p. 299], we find

\[
t_1 = s + 2F - \frac{1}{5} \left[ \sum_{i=1}^{2} (4C_1^{(i)} + 3C_2^{(i)} + 2C_3^{(i)} + C_4^{(i)}) + \sum_{j=3}^{4} (3C_1^{(j)} + 6C_2^{(j)} + 4C_3^{(j)} + 2C_4^{(j)}) \right]. \tag{12}
\]

A \( \mathbb{Z} \)-basis is \( s, t_1, t_2, t_3, t_4, F, C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)}, C_1^{(2)}, C_2^{(2)}, C_3^{(2)}, C_4^{(2)}, C_1^{(3)}, C_2^{(3)}, C_3^{(3)}, C_4^{(3)} \).

Since \( d(NS(X)) = -5^2 \) and \( d(U \oplus A_4^4) = -5^4 \), the index of \( U \oplus A_4^4 \) in \( NS(X) \) is five. Let \( w_1 \) and \( w_2 \) be

\[
w_1 = \frac{1}{5} (2C_1^{(1)} + 4C_2^{(1)} + C_3^{(1)} + 3C_4^{(1)} + 4C_1^{(3)} + 3C_2^{(3)} + 2C_3^{(3)} + C_4^{(3)});
\]

\[
w_2 = \frac{1}{5} (3C_1^{(2)} + C_2^{(2)} + 4C_3^{(2)} + 2C_4^{(2)} + C_1^{(3)} + 2C_2^{(3)} + 3C_3^{(3)} + 4C_4^{(3)}).
\]

The classes \( v_1 = w_1 - w_2, v_2 = w_1 + w_2 \) are orthogonal classes and generate the discriminant group of \( NS(X) \), the discriminant form is

\[
\mathbb{Z}_5 \left( \frac{2}{5} \right) \oplus \mathbb{Z}_5 \left( -\frac{2}{5} \right).
\]
The transcendental lattice $T_X$ has rank four, signature $(2, 2)$ and discriminant form $\mathbb{Z}_5(-\frac{2}{5}) \oplus \mathbb{Z}_5(\frac{1}{5})$. Since in this case $T_X$ is uniquely determined by signature and discriminant form (cf. [Ni2, Corollary 1.13.3]) this is the lattice

$$T_X = U \oplus U(5).$$

The transcendental lattice has a unique embedding in the lattice $\Lambda_{K3}$ by [Ni2, Theorem 1.14.4] or [Mo, Corollary 2.10].

### 4.4.1. The invariant lattice and its orthogonal complement

**Proposition 4.4.** The invariant sublattice of the Néron–Severi lattice is isometric to the lattice $U(5)$ and it is generated by the classes $F$ and $s + t_1 + t_2 + t_3 + t_4$.

The invariant lattice $H^2(X, \mathbb{Z})^s$ is isometric to $U \oplus U(5) \oplus U(5)$ and its orthogonal complement $\Omega_5 = (H^2(X, \mathbb{Z})^s)^\perp$ is the negative definite sixteen dimensional lattice $\{\mathbb{Z}^{16}, M\}$ where $M$ is the bilinear form

$$
\begin{pmatrix}
-4 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 1 & -1 & 0 \\
2 & -4 & 2 & 0 & 5 & 2 & -1 & 0 & 0 & 2 & -1 & 0 & 1 & 1 \\
0 & 2 & -4 & 2 & -5 & -1 & 2 & -1 & 0 & -1 & 2 & -1 & 1 & -1 \0 & 0 & 2 & -4 & 0 & 0 & -1 & 2 & 0 & 0 & -1 & 2 & -1 & 1 & -1 \\
0 & 5 & -5 & 0 & -50 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -15 \\
-1 & 2 & -1 & 0 & 0 & -6 & 4 & -1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 4 & -6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & -1 & 4 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 1 & 0 & -4 & 3 & -1 & 0 & 2 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 3 & -6 & 4 & -1 & -3 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 2 & -3 & 1 & 0 & -4 & 3 & -1 & 0 \\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -6 & 4 & -1 & 0 \\
-1 & 1 & 0 & 1 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 & 0 \\
0 & 1 & -1 & -1 & -15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & -6 & 0 & 0
\end{pmatrix}
$$

and it is equal to the lattice $(\text{NS}(X)^s)^\perp$.

The lattice $\Omega_5$ admits a unique primitive embedding in the lattice $\Lambda_{K3}$.

The discriminant of $\Omega_5$ is $\mathbb{Z}^4$ and its discriminant form is $(\mathbb{Z}_5(\frac{2}{5}))^{\oplus 4}$.

The isometry $\sigma_5^*$ acts on the discriminant group $\Omega_5^\vee / \Omega_5$ as the identity.

**Proof.** As in the case of an elliptic fibration with a section of order three, it is clear that $\sigma_5^*$ fixes the classes $F$ and $s + t_1 + t_2 + t_3 + t_4$. These classes generate the lattice $U(5)$, and so $H^2(X, \mathbb{Z})^s \supseteq U(5) \oplus T_X = U(5) \oplus U(5) \oplus U$. Using Nikulin’s result in [Ni1, p. 133] we find that the lattice $H^2(X, \mathbb{Z})^s$ has determinant $-5^4$, which is exactly the determinant of $U(5) \oplus U(5) \oplus U$. Since these have the same rank, we conclude that $H^2(X, \mathbb{Z})^s = (U(5) \oplus U(5) \oplus U$.

The orthogonal complement $(H^2(X, \mathbb{Z})^s)^\perp$ is equal to $(\text{NS}(X)^s)^\perp$ as in Proposition 4.2. It has signature $(0, 16)$ and by [Ni1, p. 133] the discriminant group is $(\mathbb{Z}/5\mathbb{Z})^{\oplus 4}$. Hence by [Ni2, Theorem 1.14.4] there is a unique primitive embedding of $(H^2(X, \mathbb{Z})^s)^\perp$ in the $K3$-lattice. By using the orthogonality conditions one finds the following basis of $\Omega_5 = (\text{NS}(X)^s)^\perp$:
The lattice theorem

4.4. The lattice $\Omega_5$ are orthogonal to each other and have self-intersection 2.

4.5. The lattice $\Omega_5$ but the situation is more complicated because the section $t_1$ does not meet all the fibers $I_5$ in the same component. For this reason the hermitian form of the lattice $\mathcal{L}_5$ is not the standard hermitian form on all the components. It is possible to repeat the construction used in the case of order three, but with the hermitian form (13).

\[ h_{\mathcal{L}_5}(\alpha, \beta) = \sum_{i=1}^{2} \alpha_i \bar{\beta}_i + \sum_{j=3}^{4} f \alpha_j \bar{f} \beta_j = \sum_{i=1}^{2} \alpha_i \bar{\beta}_i + \tau \sum_{j=3}^{4} \alpha_j \bar{\beta}_j, \quad (13) \]

where $\alpha, \beta \in \mathcal{L}_5 \subset \mathbb{Z}[\omega_5]\oplus^4$, $f = 1 - (\omega_5^2 + \omega_5^3)$ and $\tau = f \bar{f} = 2 - 3(\omega_5^2 + \omega_5^3)$.

**Proof.** The strategy of the proof is the same as in the case with an automorphism of order three, but the situation is more complicated because the section $t_1$ does not meet all the fibers $I_5$ in the same component. For this reason the hermitian form of the $\mathbb{Z}[\omega_5]$-lattice $L_5$ is not the standard hermitian form on all the components. It is possible to repeat the construction used in the case of order three, but with the hermitian form (13). We explain now how we find this hermitian form.

Let $F := F_5^3$ be the lattice generated by the elements $C_i^{(j)} - C_{i+1}^{(j)}$, $i = 0, \ldots, 4$, $j = 1, \ldots, 4$. This is a sublattice of $(\text{NS}(X)\oplus^5)^\perp$. A basis is
\[d_{1+i} = (\sigma_5^*)^i (C_1^{(1)} - C_2^{(1)}) , \quad d_{5+i} = (\sigma_5^*)^i (C_1^{(2)} - C_2^{(2)}) , \quad d_{9+i} = (\sigma_5^*)^i (C_1^{(3)} - C_2^{(3)}) , \quad d_{13+i} = (\sigma_5^*)^i (C_1^{(4)} - C_2^{(4)}) , \quad i = 0 , \ldots , 3 ,
\]

and the bilinear form is thus the diagonal block matrix \(Q = \text{diag}(A, A, B, B)\)

\[
A = \begin{pmatrix}
-6 & 4 & -1 & -1 \\
4 & -6 & 4 & -1 \\
-1 & 4 & -6 & 4 \\
-1 & -1 & 4 & -6 \\
\end{pmatrix}, \quad B = \begin{pmatrix}
-6 & -1 & 4 & 4 \\
-1 & -6 & -1 & 4 \\
4 & -1 & -6 & -1 \\
4 & 4 & -1 & -6 \\
\end{pmatrix}.
\]

We want to identify the multiplication by \(\omega_5\) in the lattice \(G\) with the action of the isometry \(\sigma_5^*\) on the lattice \(F\). We consider the \(\mathbb{Z}[\omega_5]\)-module \(G = (1 - \omega)^2 \mathbb{Z}[\omega]^{\oplus 4}\). Now we consider the \(\mathbb{Z}\)-module \(G_\mathbb{Z}\). The map

\[
\phi : (\sigma_5^*)^i (C_1^{(1)} - C_2^{(1)}) \mapsto (1 - \omega_5)^2 \omega_i (1, 0, 0, 0),
\]

\[
(\sigma_5^*)^i (C_1^{(2)} - C_2^{(2)}) \mapsto (1 - \omega_5)^2 \omega_i (0, 1, 0, 0),
\]

\[
(\sigma_5^*)^i (C_1^{(3)} - C_2^{(3)}) \mapsto (1 - \omega_5)^2 \omega_i (0, 0, 1, 0),
\]

\[
(\sigma_5^*)^i (C_1^{(4)} - C_2^{(4)}) \mapsto (1 - \omega_5)^2 \omega_i (0, 0, 0, 1)
\]

is an isomorphism between the \(\mathbb{Z}\)-modules \(G_\mathbb{Z}\) and \(F\).

Now we have to find a bilinear form \(b_G\) on \(G\) such that \(\{G_\mathbb{Z}, b_G\}\) is isometric to \(\{F, Q\}\). On the first and second fiber the action of \(\sigma_5^*\) is \(\sigma_5^*(C_i^{(j)}) = C_{i+1}^{(j)}\), \(j = 1, 2\), \(i = 0, \ldots , 4\), so \((\sigma_5^*)^i (C_1^{(j)} - C_2^{(j)}) = C_{i+1}^{(j)} - C_{i+2}^{(j)}\). Hence the map \(\phi\) operates on the first two fibers in the following way:

\[
\phi : C_{i+1}^{(1)} - C_{i+2}^{(1)} \mapsto (1 - \omega_5)^2 \omega_i (1, 0, 0, 0),
\]

\[
C_{i+1}^{(2)} - C_{i+2}^{(2)} \mapsto (1 - \omega_5)^2 \omega_i (0, 1, 0, 0).
\]

This identification is exactly the identification described in Lemma 2.1, so on these generators of the lattices \(F\) and \(G\) we can choose exactly the form described in the lemma.

On the third and fourth fiber the action of \(\sigma_5^*\) is different (because \(\sigma_5^*\) is the translation by \(t_1\) and it meets the first and second fiber in the component \(C_1\) and the third and fourth fiber in the component \(C_2\)). In fact \((\sigma_5^*)^i (C_1^{(j)} - C_2^{(j)}) = C_{2i+1}^{(j)} - C_{2i+2}^{(j)}\), \(j = 3, 4\), \(i = 0, \ldots , 4\), and so

\[
\phi : C_{2i+1}^{(3)} - C_{2i+2}^{(3)} \mapsto (1 - \omega_5)^2 \omega_i (0, 0, 1, 0),
\]

\[
C_{2i+1}^{(4)} - C_{2i+2}^{(4)} \mapsto (1 - \omega_5)^2 \omega_i (0, 0, 0, 1) \quad (14)
\]

A direct verification shows that the map \(\phi\) defines an isometry between the module generated by \((\sigma_5^*)^i (C_1^{(j)} - C_2^{(j)})\), \(i = 0, \ldots , 4\), and \((1 - \omega_5)^2 \mathbb{Z}[\omega_5]\) \((j = 3, 4)\), if one considers on \((1 - \omega_5)^2 \mathbb{Z}[\omega_5]\) the bilinear form associated to the hermitian form \(h(\alpha, \beta) = \tau \alpha \bar{\beta}\) where \(\tau = (2 - 3(\omega_5^2 + \omega_5^3))\). The real number \(\tau\) is the square of \(f = 1 - (\omega_5^3 + \omega_5^3)\), so the hermitian form above is also \(h(\alpha, \beta) = \tau \alpha \bar{\beta} = f \alpha \bar{f} \beta\). So now we consider the \(\mathbb{Z}[\omega_5]\)-lattice \(\mathbb{Z}[\omega_5]^{\oplus 4}\) with
the hermitian form $h$ given in (13) and $G$ as a sublattice of $\{\mathbb{Z}[\omega_5]^{\oplus 4}, h\}$. We show that $L_5 = \Omega_5$.

We have to add to the lattice $F$ some classes to obtain the lattice $\Omega_5$, and so we have to add some vectors to the lattice $G$ to obtain the lattice $L_5$. It is sufficient to add to $F$ the classes $s - t_1$, $C_1^{(1)} - C_1^{(2)}$, $C_1^{(2)} - C_1^{(3)}$, $C_1^{(3)} - C_1^{(4)}$ and their images under $\sigma_5^*$. These classes correspond to the following vectors in $\mathbb{Z}[\omega_5]^{\oplus 4}$:

$$s - t_1 = (1, 1, c, c),$$

$$C_1^{(1)} - C_1^{(2)} = (1 - \omega_5)(1, -1, 0, 0),$$

$$C_1^{(2)} - C_1^{(3)} = (1 - \omega_5)(0, 1, -(1 + \omega_3^2), 0),$$

$$C_1^{(3)} - C_1^{(4)} = (1 - \omega_5)(0, 0, (1 + \omega_3^3), -(1 + \omega_3^3))$$

where $c = \omega_5(2\omega_5^2 - \omega_5 + 2)$. A basis for the lattice $L_5$ is then

$$l_1 = (1, 1, c, c),$$

$$l_3 = \omega_5^2 l_1, \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ l_2 = \omega_5 l_1,$$

$$l_5 = (1 - \omega_5)^2(0, 0, 2 + 4\omega_5 + \omega_5^2 + 3\omega_5^3, 0), \ \ \ \ l_6 = (1 - \omega_5)^2(1, 0, 0, 0),$$

$$l_7 = \omega_5 l_6, \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ l_8 = \omega_5^2 l_6,$$

$$l_9 = (1 - \omega_5)(1, -1, 0, 0), \ \ \ \ l_9 = (1 - \omega_5)^2(0, 1, 0, 0),$$

$$l_{11} = \omega_5 l_{10}, \ \ \ \ l_{10} = (1 - \omega_5)^2(0, 1, 0, 0),$$

$$l_{13} = (1 - \omega_5)(0, 1, -(1 + \omega_3^2), 0), \ \ \ \ l_{13} = (1 - \omega_5)^2(0, 0, 1, 0),$$

$$l_{15} = \omega_5 l_{14}, \ \ \ \ l_{14} = (1 - \omega_5)^2(0, 0, 1, 0),$$

The identification between $\Omega_5$ and $L_5$ is given by the map $b_i \mapsto l_i$. After this identification the intersection form on $\Omega_5$ is exactly the form $b_{|L_5}$ on $L_5$. □

**Remark.** (1) We recall that the density of a lattice $L$ of rank $n$ is $\Delta = V_n/\sqrt{\det L}$ where $V_n$ is the volume of the $n$ dimensional sphere of radius $r$ (called packing radius of the lattice), $V_n = r^n\pi^{n/2}/(n/2)!$, $r = \sqrt{\mu}/2$ and $\mu$ is the minimal norm of a vector of the lattice.

The density of $\Omega_5$ is $\Delta = \frac{\pi^8}{815}\approx 0.0094$.

(2) The lattice $\Omega_5$ does not admit vectors of norm $-2$ and can be generated by vectors of norm $-4$, and a basis is $b_1, b_2, b_3, b_4, b_5 - b_13 - 2b_{14} - 3b_{15} - 4b_{16}, b_6, b_7, b_8, b_9, b_{10} + b_{11}, b_{11} + b_{12}, b_{10} + b_{11} + b_{12}, b_{13}, b_{14} + b_{15}, b_{15} + b_{16}, b_{14} + b_{15} + b_{16}$.

4.6. *Section of order seven*

Let $X$ be a K3 surface with an elliptic fibration which admits a section of order seven as described in Section 3. We recall that $X$ has three reducible fibers of type $I_7$ and three singular irreducible fibers of type $I_1$. We have seen that the rank of the Néron–Severi group is 20. We determine now $NS(X)$ and $T_X$.

We label the fibers and their components as described in Section 3. Let $t_1$ denote the section of order seven which meets the first fiber in $C_1^{(1)}$. Again by the formula (4) of Section 3 we have

$$t_1 \cdot C_1^{(1)} = 1, \ \ \ t_1 \cdot C_2^{(1)} = 1, \ \ \ t_1 \cdot C_3^{(1)} = 1, \ \ \ \text{and} \ \ t_1 \cdot C_i^{(j)} = 0 \ \ \text{otherwise.}$$
Let $\sigma_7$ denote the automorphism of order seven which leaves each fiber invariant and is the translation by $t_1$, so $\sigma_7^2(s) = t_1$, $\sigma_7^3(t_1) = t_2$, $\sigma_7^4(t_2) = t_3$, $\sigma_7^5(t_3) = t_4$, $\sigma_7^6(t_4) = t_5$, $\sigma_7^7(t_5) = t_6$, $\sigma_7^8(t_6) = s$. The proofs of the next two propositions are very similar to those of the similar propositions in the case of the automorphisms of order three and five, so we omit them.

**Proposition 4.5.** A $\mathbb{Z}$-basis for the lattice $\text{NS}(X)$ is given by

$$s, t_1, t_2, t_3, t_4, t_5, t_6, F, C_1, C_2, C_3, C_4, C_5, C_6,$$

Let $U \oplus A_3^2$ be the lattice generated by the section, the fiber and the irreducible components of the three fibers $I_7$ which do not intersect the zero section $s$. It has index seven in the Néron–Severi group of $X$, $\text{NS}(X)$.

The lattice $\text{NS}(X)$ has discriminant $-7$ and its discriminant form is $\mathbb{Z}_7(-\frac{4}{7})$.

The transcendental lattice $T_X$ is the lattice $\{\mathbb{Z}^2, \mathfrak{r}\}$ where

$$\mathfrak{r} := \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

and it has a unique primitive embedding in the lattice $\Lambda_{K_3}$.

4.6.1. The invariant lattice and its orthogonal complement

**Proposition 4.6.** The invariant sublattice of the Néron–Severi lattice is isometric to the lattice $U(7)$ and it is generated by the classes $F$ and $s + t_1 + t_2 + t_3 + t_4 + t_5 + t_6$.

The invariant lattice $H^2(X, \mathbb{Z})_{\sigma_7}$ is isometric to $U(7) \oplus T_X$. Its orthogonal complement $\Omega_7 := (H^2(X, \mathbb{Z})_{\sigma_7})^\perp$ is the negative definite eighteen dimensional lattice $\{\mathbb{Z}^{18}, M\}$ where $M$ is the bilinear form

$$\begin{pmatrix}
-4 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
2 & -4 & 2 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \\
0 & 2 & -4 & 2 & 0 & -7 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\
0 & 0 & 2 & -4 & 2 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 1 & 1 & -1 & 0 \\
0 & 0 & 0 & 2 & -4 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & -98 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & -21 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & -1 & 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & -1 & 3 & -4 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 1 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 4 & 0 \\
0 & 0 & 1 & -1 & 0 & -1 & -21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -6 & 6
\end{pmatrix}$$

and it is equal to the lattice $(\text{NS}(X)_{\sigma_7})^\perp$. 

Theorem 4.5. The lattice $Ω_7$ admits a unique primitive embedding in the lattice $Λ_{K3}$.

The discriminant of $Ω_7$ is $7^3$ and its discriminant form is $(\mathbb{Z}/7\mathbb{Z})^{\oplus 3}$.

The isometry $σ_7^*$ acts on the discriminant group $Ω_7^*/Ω_7$ as the identity.

The basis of $(\text{NS}(X)^{\sigma_7^*})^\perp$ associated to the matrix $M$ is \( b_1 = s - t_1, b_2 = t_1 - t_2, b_3 = t_2 - t_3, b_4 = t_3 - t_4, b_5 = t_4 - t_5, b_6 = t_5 - t_6, b_7 = F - 7C_6^{(2)}, \) \( b_i = C_i^{(1)} - C_i^{(1)}(1 + \omega_7), \) \( b_i = C_i^{(2)} - C_i^{(2)}, \) \( i = 8, \ldots, 12, b_{13} = C_1^{(1)} - C_1^{(2)}, \) \( b_i = C_i^{(2)} - C_i^{(2)}, i = 14, \ldots, 18. \)

4.7. The lattice $Ω_7$

Let $ω_7$ be a primitive seventh root of the unity. In this section we prove the following result.

**Theorem 4.5.** The lattice $Ω_7$ is isometric to the $\mathbb{Z}$-lattice associated to the $\mathbb{Z}[ω_7]$-lattice $\{L_7, h_{L_7}\}$ where

\[
L_7 = \left\{ (x_1, x_2, x_3) \in (\mathbb{Z}[ω_7])^{\oplus 3} : x_1 + x_2 + 6x_3 \equiv 0 \mod (1 - ω_7), (1 + 5ω_7)x_1 + 3x_2 + 2x_3 \equiv 0 \mod (1 - ω_7)^2 \right\}
\]

with the hermitian form

\[
h_{L_7}(α, β) = α_1β_1 + f_1α_2β_2 + f_2α_3β_3,
\]

where \( f_1 = 3 + 2(ω_7 + ω_7^5) + (ω_7^2 + ω_7^5), f_2 = 2 + (ω_7 + ω_7^5). \)

**Proof.** As in the previous cases we define the lattice $F := F_7$. We consider the hermitian form

\[
h(α, β) = α_1β_1 + f_1α_2β_2 + f_2α_3β_3
\]

on the lattice $\mathbb{Z}[ω_7]^{\oplus 3}$, and define $G$ to be the sublattice $G = (1 - ω_7)^2\mathbb{Z}[ω_7]^{\oplus 3}$ of $\{\mathbb{Z}[ω_7]^{\oplus 3}, h\}$.

The map $φ : F \to G$

\[
φ : (σ_7^*)^i(C_1^{(1)} - C_1^{(2)}), (1 - ω_7)^2ω_7^i(1, 0, 0),
\]

\[
(σ_7^*)^i(C_1^{(2)} - C_1^{(2)}), (1 - ω_7)^2ω_7^i(0, 1, 0),
\]

\[
(σ_7^*)^i(C_1^{(3)} - C_1^{(2)}), (1 - ω_7)^2ω_7^i(0, 0, 1)
\]

is an isomorphism between the $\mathbb{Z}$-lattice $G_Z$, with the bilinear form induced by the hermitian form, and $F$ with the intersection form. We have to add to $G$ some vectors to find a lattice $L_7$ isomorphic to $Ω_7$. These vectors are

\[
s - t_1 = (1, c, k),
\]

\[
C_1^{(1)} - C_1^{(2)} = (1 - ω_7)(1, -(1 + ω_7^4), 0),
\]

\[
C_1^{(2)} - C_1^{(3)} = (1 - ω_7)(0, (1 + ω_7^4), -(1 + ω_7^3 + ω_7^5)).
\]
where \( c = 1 + 3\omega_7 + 3\omega_7^4 + \omega_7^5 \) and \( k = -5 + 5\omega_7^2 - 3\omega_7^4 - 3\omega_7^5 \). A basis for the lattice \( L_7 \) is

\[
\begin{align*}
l_1 &= (1, c, k), \\
l_3 &= \omega_7^2 l_1, \\
l_5 &= \omega_7^4 l_1, \\
l_7 &= (1 - \omega_7)^2(0, 2 + 4\omega_7 + 6\omega_7^2 + \omega_7^3 + 3\omega_7^4 + 5\omega_7^5, 0), \\
l_9 &= \omega_7 l_8, \\
l_{11} &= \omega_7^3 l_8, \\
l_{13} &= (1 - \omega_7)(1, -(1 + \omega_7^2), 0), \\
l_{15} &= \omega_7^7 l_{14}, \\
l_{17} &= \omega_7^3 l_{14}, \\
l_2 &= \omega_7 l_1, \\
l_4 &= \omega_7^3 l_1, \\
l_6 &= \omega_7^5 l_1, \\
l_8 &= (1 - \omega_7)^2(1, 0, 0), \\
l_{10} &= \omega_7^2 l_8, \\
l_{12} &= \omega_7^4 l_8, \\
l_{14} &= (1 - \omega_7)^2(0, 1, 0), \\
l_{16} &= \omega_7^2 l_{14}, \\
l_{18} &= \omega_7^4 l_{14}. 
\end{align*}
\]

The identification between \( \Omega_7 \) and \( L_7 \) is given by the map \( b_i \mapsto l_i \). After this identification the intersection form on \( \Omega_7 \) is exactly the form \( b_i l_j \) on \( L_7 \) induced by the hermitian form (15).

\[ \square \]

**Remark.** (1) The density of \( \Omega_7 \) is \( \Delta = \frac{\pi^9}{9!} \frac{1}{\sqrt{3}} \approx 0.0044 \).

(2) As in the previous cases the lattice \( \Omega_7 \) does not admit vectors of norm \(-2\) and can be generated by vectors of norm \(-4\), and a basis is \( b_1, b_2, b_3, b_4, b_5, b_6, b_7 - b_{13} - 2b_{14} - 3b_{15} - 4b_{16} - 5b_{17} - 6b_{18}, b_8 + b_9, b_9 + b_{10}, b_{10} + b_{11}, b_{11} + b_{12}, b_{10} + b_{11} + b_{12}, b_{13}, b_{14} + b_{15}, b_{15} + b_{16}, b_{16} + b_{17}, b_{17} + b_{18}, b_{16} + b_{17} + b_{18} \).

5. Families of K3 surfaces with a symplectic automorphism of order \( p \)

In the previous sections we used elliptic K3 surfaces to describe some properties of the automorphism \( \sigma_p \). All these K3 surfaces have Picard number \( \rho_p + 1 \), where \( \rho_p \) is the minimal Picard number found in Proposition 1.1. In this section we want to describe algebraic K3 surfaces with symplectic automorphism of order \( p \) and with the minimal possible Picard number. Recall that the values of \( \rho_p \) are

\[
p: \quad 3 \quad 5 \quad 7,
\]

\[
\rho_p: \quad 13 \quad 17 \quad 19,
\]

and \( \Omega_p \) denote the lattices described in Sections 4.1, 4.4, 4.6.

**Proposition 5.1.** Let \( X \) be a K3 surface with symplectic automorphism of order \( p = 3, 5, 7 \) and Picard number \( \rho_p \) as above. Let \( L \) be a generator of \( \Omega_p^+ \subset NS(X) \), with \( L^2 = 2d > 0 \) and let

\[
\mathcal{L}_{2d}^p := \mathbb{Z}L \oplus \Omega_p.
\]

Then we may assume that \( L \) is ample and

(1) if \( L^2 \equiv 2, 4, \ldots, 2(p - 1) \mod 2p \), then \( \mathcal{L}_{2d}^p = NS(X) \),

(2) if \( L^2 \equiv 0 \mod 2p \), then either \( \mathcal{L}_{2d}^p = NS(X) \) or \( NS(X) = \widetilde{\mathcal{L}}_{2d}^p \) with \( \widetilde{\mathcal{L}}_{2d}^p \mathcal{L}_{2d}^p \simeq \mathbb{Z}/p\mathbb{Z} \) and in particular \( \widetilde{\mathcal{L}}_{2d}^p \) is generated by an element \( (L/p, v/p) \) with \( v^2 \equiv 0 \mod 2p \) and \( L^2 + v^2 \equiv 0 \mod 2p^2 \).
Proof. Since $L^2 > 0$ by Riemann-Roch theorem we can assume $L$ or $-L$ effective. Hence we assume $L$ effective. Let $N$ be an effective $(−2)$ curve then $N = αL + v'$, with $v' ∈ Ω_p$ and $α > 0$ since $Ω_p$ do not contains $(−2)$-curves. We have $L · N = αL^2 > 0$, and so $L$ is ample. Moreover recall that $L$ and $Ω_p$ are primitive sublattices of $NS(X)$. Since the discriminant group of $L^p_2 := ZL ⊕ Ω_p$ is $(Z/2dZ) ⊕ (Z/pZ)_{2n}$, with $n_3 = 6, n_5 = 4, n_7 = 3$ an element in $NS(X)$ not in $L^p_2$ is of the form $(αL/2d, v/p)$, $v ∈ Ω_p$ and satisfy the following conditions:

(a) $p · (αL/2d, v/p) ∈ NS(X)$,
(b) $(αL/2d, v/p) · L ∈ Z$,
(c) $(αL/2d, v/p)^2 ∈ Z$.

By using the condition (a) we obtain $p · (αL/2d, v/p) − (0, v) ∈ NS(X)$ and so

$$pαL/2d ∈ NS(X).$$

Hence by the primitivity of $L$ in $NS(X)$ follows that $d ≡ 0 \mod p, d = pd', d' ∈ Z_{>0}$ and so

$$αL/2d' ∈ NS(X)$$

which gives $α = 2d'$ and the class (if there is) is $(L/p, v/p)$. Now condition (b) gives

$$(L/p, v/p) · L = L^2/p ∈ Z$$

and so $L^2 = 2p · r, r ∈ Z_{>0}$, since the lattice is even. And so if $NS(X) = L^p_2$, then $L^2 ≡ 0 \mod 2p$. Finally condition (c) gives

$$(L/p, v/p)^2 = L^2/v^2 + v^2/p^2$$

and so since a square is even $L^2 + v^2 ≡ 0 \mod 2p^2$. □

In Sections 4.1, 4.4, 4.6 we defined a symplectic automorphism $σ_p, p = 3, 5, 7$, of order $p$ on some special K3 surfaces and we found the lattices $Ω_p = (Λ_p^*)^⊥$. Now we consider more in general an isometry on $Λ_{K3}$ defined as $σ_p^*$ (we call it again $σ_p^*$). In the next theorem we prove that if $X$ is a K3 surface such that $NS(X) = L^p_2$ or $NS(X) = L^p_2$, then this isometry is induced by a symplectic automorphism of the surface $X$.

Proposition 5.2. Let $L_p = L^p_2$ or $L_p = L^p_2$ if $p = 3, 5$ and let $L_7 = L^7_2$. Then there exists a K3 surface $X$ with symplectic automorphism $σ_p$ of order $p$ such that $NS(X) = L_p$ ($p = 3, 5, 7$) and $(H^2(X, Z))^{σ_p^*} = Ω_p$.

Moreover there are no K3 surfaces with Néron-Severi group isometric to $L^7_{14d}$.

Proof. Let $σ^*_p, p = 3, 5, 7$, be an isometry as in Sections 4.1, 4.4, 4.6. We make the proof in several steps.
Step 1: there exists a marked K3 surface $X$ such that $NS(X)$ is isometric to $\mathcal{L}_p$, and there are no K3 surfaces with Néron–Severi group isometric to $\mathcal{L}_p^7$. By [Ni2, Corollary 1.12.3] the lattices $\mathcal{L}_2^3$, $\mathcal{L}_2^5$, $\mathcal{L}_2^7$ have a primitive embedding in the K3 lattice. The lattice $T_5 = U(5) \oplus U(5) \oplus (-2d)$ has a unique primitive embedding in $\Lambda_{K3}$, by [Ni2, Theorem 1.14.4]. Its signature is $(2,3)$ and its discriminant form is the opposite of the discriminant form of $\mathcal{L}_2^5$. Since, by [Ni2, Corollary 1.13.3], $\mathcal{L}_2^5$ is uniquely determined by its signature and discriminant form, it is the orthogonal complement of $T_5$ in $\Lambda_{K3}$ and then clearly $\mathcal{L}_2^5$ admits a primitive embedding in $\Lambda_{K3}$.

The lattice $\mathcal{L}_2^7$ is a primitive sublattice of the Néron–Severi group of the K3 surface described in Section 4.6, so it is a primitive sublattice of $\Lambda_{K3}$ (the same argument could be applied to the lattices $\mathcal{L}_2^p$, $p = 3, 5$ too). Let now $\omega \in \mathcal{L}_p^C \subseteq \Lambda_{K3} \subseteq C$, with $\omega \omega = 0$, $\omega \omega > 0$. We choose $\omega$ generic with these properties. By the surjectivity of the period map of K3 surfaces, $\omega$ is the period of a K3 surface $X$ with $NS(X) = \omega \cdot \Lambda_{K3} = \mathcal{L}_p$.

The rank of the lattice $\mathcal{L}_p^7$ is 19 and its discriminant group has four generators. If $\mathcal{L}_p^7$ was the Néron–Severi group of a K3 surface, the transcendental lattice of this surface should be a rank three lattice with a discriminant group generated by four elements. This is clearly impossible.

Step 2: the isometry $\sigma_p^*$ fixes the sublattice $\mathcal{L}_p$. Since $\sigma_p^*(\Omega_p) = \Omega_p$ and $\sigma_p^*(L) = L$ (because $L \in \Omega_p^\perp$ which is the invariant sublattice of $\Lambda_{K3}$), if $\mathcal{L}_p = \mathcal{L}_p^2 = ZL \oplus \Omega_p$ it is clear that $\sigma_p^*(\mathcal{L}_p) = \mathcal{L}_p$. Now we consider the case $\mathcal{L}_p = \mathcal{L}_p^2$. The isometry $\sigma_p^*$ acts trivially on $\Omega_p^\perp / \Omega_p$ (cf. Propositions 4.2, 4.4, 4.6) and on $(ZL)^\perp / ZL$. Let $\frac{1}{p}(L, v') \in \mathcal{L}_p$, with $v' \in \Omega_p$. This is also an element in $(\Omega_p \oplus ZL)^\perp / (\Omega_p \oplus ZL)$. So we have $\sigma_p^*(\frac{1}{p}(L, v')) \equiv \frac{1}{p}(L, v') \mod (\Omega_p \oplus ZL)$, which means

$$\sigma_p^*(\frac{1}{p}(L, v')) = \frac{1}{p}(L, v') + (\beta L, v''), \quad \beta \in \mathbb{Z}, \ v'' \in \Omega_p.$$ 

Hence we have $\sigma_p^*(\mathcal{L}_p) = \mathcal{L}_p$.

Step 3: The isometry $\sigma_p^*$ is induced by an automorphism of the surface $X$. The isometry $\sigma_p^*$ fixes the sublattice $\mathcal{L}_p^\perp$ of $\Lambda_{K3}$, so it is an Hodge isometry. By the Torelli theorem an effective Hodge isometry of the lattice $\Lambda_{K3}$ is induced by an automorphism of the K3 surface (cf. [BPV, Theorem 11.1]). To apply this theorem we have to prove that $\sigma_p^*$ is an effective isometry. An effective isometry on a surface $X$ is an isometry which preserves the set of effective divisors. By [BPV, Corollary 3.11] $\sigma_p^*$ preserves the set of the effective divisors if and only if it preserves the ample cone. So if $\sigma_p^*$ preserves the ample cone it is induced by an automorphism of the surface. This automorphism is symplectic by construction (it is the identity on the transcendental lattice $T_X \subset \Omega_p^\perp$), and so if $\sigma_p^*$ preserves the ample cone, the theorem is proven.

Step 4: the isometry $\sigma_p^*$ preserves the ample cone $\mathcal{A}_X$. Let $\mathcal{C}_X^\perp$ be one of the two connected components of the set $\{x \in H^{1,1}(X, \mathbb{R}) \mid (x, x) > 0\}$. The ample cone of a K3 surface $X$ can be described as the set $\mathcal{A}_X = \{x \in \mathcal{C}_X^\perp \mid (x, d) > 0 \text{ for each } d \text{ such that } (d, d) = -2, \text{ } d \text{ effective}\}$. First we prove that $\sigma_p^*$ fixes the set of the effective $(-2)$-curves. Since there are no $(-2)$-curves in $\Omega_p$, if $N \in \mathcal{L}_p$ has $N^2 = -2$ then $N = \frac{1}{p}(aL, v')$, $v' \in \Omega_p$, for an integer $a \neq 0$. Since $\frac{1}{p}aL^2 = L \cdot N > 0$, because $L$ and $N$ are effective divisor, we obtain $a > 0$. The curve $N' = \sigma_p^*(N)$ is a $(-2)$-curve because $\sigma_p^*$ is an isometry, hence $N'$ or $-N'$ is effective. Since $\frac{1}{p}aL^2 = L \cdot N > 0$ and so $-N'$ is not effective. Using the fact that $\sigma_p^*$ has finite order it is clear that $\sigma_p^*$ fixes the set of the effective $(-2)$-curves.
Now let $x \in \mathcal{A}_Y$ then $\sigma^*_p(x) \in \mathcal{A}_X$, in fact $(\sigma^*_p(x), \sigma^*_p(x)) = (x, x) > 0$ and for each effective $(−2)$-curve $d$ there exists an effective $(−2)$-curve $d'$ with $d = \sigma^*_p(d')$, so we have $(\sigma^*_p(x), d) = (\sigma^*_p(x), \sigma^*_p(d')) = (x, d') > 0$. Hence $\sigma^*_p$ preserves $\mathcal{A}_X$ as claimed. □

**Corollary 5.1.** The coarse moduli space of $\mathcal{L}_p$-polarized K3 surfaces (cf. [Do] for the definition) $p = 3, 5, 7$ has dimension seven, three, respectively one and is a quotient of

$$\mathcal{D}_{\mathcal{L}_p} = \{\omega \in \mathbb{P}(\mathcal{L}_p^\perp \otimes \mathbb{Z} \mathbb{C}) : \omega^2 = 0, \omega\bar{\omega} > 0\}$$

by an arithmetic group $O(\mathcal{L}_p)$.

**Remark.** In particular the moduli space of K3 surfaces admitting a symplectic automorphism of order $p = 3, 5, 7$ has dimension respectively seven, three and one.

6. Final remarks

1. In Proposition 5.2 it would be interesting to prove the unicity of the lattices $\mathcal{L}_{2d}$, this requires some careful analysis of the automorphism group of the lattices $\Omega_{2d}$, $p = 3, 5$.

2. It is not difficult to give examples of K3 surfaces (not elliptic) in some projective space with a symplectic automorphism of order three or five. Consider for example the surfaces of $\mathbb{P}^3$:

$$(S1): \quad q_4(x_0, x_1) + q_2(x_0, x_1)x_2x_3 + l_1(x_0, x_1)x_3^3 + l'_1(x_0, x_1)x_3^2 + ax_2^2x_3^2 = 0,$$

$$(S2): \quad a_0x_0^2x_1^2 + a_2x_0^2x_2^2 + a_0x_0x_1x_2x_3 + a_0x_0^2x_2 + a_1x_1^2x_3 + a_1x_1x_2^2 + a_0x_0x_3x_3^3 = 0,$$

where $q_i$ is homogeneous of degree $i$, $l_1, l'_1$ are linear forms, and $a_{ij} \in \mathbb{C}$. The surfaces $S_1$ respectively $S_2$ admit symplectic automorphisms of order three respectively of order five induced by the automorphisms of $\mathbb{P}^3$ given by $\sigma_3 : (x_0 : x_1 : x_2 : x_3) \to (x_0 : x_1 : \omega_3x_2 : \omega_2^2x_3)$ and $\sigma_5 : (x_0 : x_1 : x_2 : x_3) \to (\omega_5x_0 : \omega_5^4x_1 : \omega_5^2x_2 : \omega_5^3x_3)$. The automorphisms of $\mathbb{P}^3$ commuting with $\sigma_3$ respectively $\sigma_5$ form a space of dimension six, respectively four, since the equations depend on 13, respectively seven parameters the dimension of the moduli space is seven, respectively three as expected (this is the minimal possible dimension). In a similar way one can construct many more examples. In the case of order seven automorphisms it is more difficult to give such examples.

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References


