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Quantum cosmology of classically constrained gravity

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Abstract

In [G. Gabadadze, Y. Shang, hep-th/0506040] we discussed a classically constrained model of gravity. This theory contains known solutions of General Relativity (GR), and admits solutions that are absent in GR. Here we study cosmological implications of some of these new solutions. We show that a spatially-flat de Sitter universe can be created from "nothing". This universe has boundaries, and its total energy equals to zero. Although the probability to create such a universe is exponentially suppressed, it favors initial conditions suitable for inflation. Then we discuss a finite-energy solution with a nonzero cosmological constant and zero space–time curvature. There is no tunneling suppression to fluctuate into this state. We show that for a positive cosmological constant this state is unstable—it can rapidly transition to a de Sitter universe providing a new unsuppressed channel for inflation. For a negative cosmological constant the space–time flat solutions is stable.

1. Introduction

In Ref. [1] a classically constrained General Relativity (CGR) was discussed. The gravitational part of the Lagrangian density consists of the conventional Einstein–Hilbert (EH) term amended by a term that enforces a constraint

$$\mathcal{L} = -\frac{\sqrt{-g}}{2}(R + 2\Lambda) - \sqrt{-g}\,g^{\nu\mu}\partial_\nu\lambda_\mu + \cdots. \tag{1.1}$$

Here λ_μ is a nondynamical Lagrange multiplier field, and we introduced a cosmological constant Λ. In most of the applications discussed below, Λ can be replaced by a "slow roll" inflationary potential $V(\phi)$, as $\Lambda \to V(\phi)$ (we put $M_{\rm Pl} = 1$).

The Lagrangian (1.1) is a part of the action used in path-integral quantization of GR.[1] The Lagrange multiplier term usually enforces the gauge fixing condition. For consistent quantization of small fluctuations this Lagrangian should be amended by appropriate boundary conditions for the fluctuations, and by the Faddeev–Popov (FP) ghosts. The main point of the approach of Ref. [1], which we follow here, was to allow for the boundary conditions on which the determinant of the FP operator has a zero-mode. This would make the path integral ill-defined, unless the zero-mode is treated separately from the fluctuations. The zero-mode is regarded as a classical background solution, and the small fluctuations are then quantized about that background. In Ref. [1] we considered only the background solutions on which the FP ghosts vanish, although they are present as quantum fluctuations.

The above approach, when it comes to classical solutions, reduces to the following simple algorithm. Considering (1.1) as a classically constrained theory. In this theory, Einstein's equations are modified due to the λ_μ field. The modified equations could allow for new solutions [1] that are absent in GR. To discuss those solutions we consider spaces with boundaries where the Gibbons–Hawking term [3] is implied and the following boundary conditions are imposed: $\delta g_{\mu\nu}|_{\rm boundary} = \delta\lambda_\mu|_{\rm boundary} = 0$. Then, the equations of motion take the form:

$$G_{\mu\nu} + (\partial_\mu\lambda_\nu + \partial_\nu\lambda_\mu) - g^{\sigma\tau}\partial_\sigma\lambda_\tau g_{\mu\nu} = \Lambda g_{\mu\nu}, \tag{1.2}$$

$$\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\right) = 0. \tag{1.3}$$

The above equations can admit solutions that are not present in GR. For instance, a theory with $\Lambda \neq 0$ has a solution with zero space–time curvature [1]: $g_{\mu\nu} = \eta_{\mu\nu}$, $\partial_\mu\lambda_\nu + \partial_\nu\lambda_\mu = -\Lambda\eta_{\mu\nu}$.

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[1] GR is not a renormalizable theory, however, it can be regarded as a low-energy effective quantum field theory with a cutoff. For an exposition of this point of view, see Ref. [2].

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The solution ends on a fixed boundary where the value of λ_μ, which is defined up to a constant, is adjusted to be zero so that the space is geodesically complete. We will call this a new flat solution below. On the other hand, putting $\lambda_\mu = 0$, the theory yields a conventional (anti)de Sitter solution written in a gauge (1.3) [1]. There are also other solutions, one of them being a zero-energy spatially-flat de Sitter (dS) space with a boundary, that we will discuss below in some detail.

The goal of the present work is to study these solutions and their relevance to cosmology. As a first example, we will look at a new possibility to create a universe into a state described by the *spatially-flat* dS solution. That quantum creation of a spatially-flat universe is possible if it has nontrivial topology, was first found by Zel'dovich and Starobinsky [4]. In our case, the spatially-flat universe that is being created has trivial topology, but comes with a fixed boundary on which the boundary conditions preserving completeness of the space are imposed. We calculate the probability of creation of such a universe out of "nothing", i.e., out of an initial state with no classical space–time. Linde's [5] and Vilenkin's (first reference in [6]) approaches give the same results in this case. We will find that the probabilistic arguments favor initial conditions needed for inflation, as opposed to the conditions that would favor universe sitting at the bottom of the potential. However, the probability itself is still exponentially small. This is somewhat similar to the emergence of a *closed* dS universe in a conventional approach [5,6].

Then we turn to a new flat solution described above. We study a process of producing a small region of primordial universe in a state of a nonzero energy described by the new flat solution. As we will see, in the minisuperspace approximation, there is no potential barrier to be penetrated in order to fluctuate into this state. Interestingly enough, if Λ is positive, this state is unstable—it can either collapse or with an almost equal probability, can rapidly transition into a spatially-flat dS universe with $H^2 = \Lambda/3$. The latter can be used to describe the required inflationary epoch. The above sequence of events, represents a new channel for obtaining an inflationary region in a primordial universe. The probability of these events to take place is not suppressed by the exponential factors. In that regard, the effect is similar to the one emphasized by Linde [7], in the context of the solution of [4].[2]

On the other hand, if $\Lambda < 0$, then the new flat solution is stable. Can this be used at late times for the adjustment of the cosmological constant? One could be contemplating a scenario in which a small region in a primordial universe first fluctuates into a state described by the new flat solution with a positive potential (positive Λ), then undergoes inflation as described above, and after that the potential drops to a negative value $\Lambda < 0$. One could use the new flat solution with $\Lambda < 0$ to obtain an (almost) flat universe today via this sequence. We will briefly comment on what it takes to have such a scenario.

Before we turn to quantum cosmology of CGR, we would like to make a few comment concerning the consistency of the theory (1.1) itself (this was discussed in detail in [1], here we just briefly summarize some main results):

- The Lagrangian (1.1) is not reparametrization invariant—the new term completely restricts the symmetry. Nevertheless, the equivalence principle is preserved. The gauge condition (1.3) allows *local*, point-dependent gauge transformations, that can be used to eliminate a nontrivial metric and connection in an infinitesimal neighborhood of any space–time point.
- The linearized theory has two propagating physical polarizations of a graviton. No negative-norm states or tachyons appear in the quadratic action.
- Bianchi identities enforce an additional condition on the Lagrange multiplier: $g^{\mu\nu}\partial_\mu\partial_\nu\lambda_\alpha = 0$. The latter has to be respected by all solutions of the theory.
- Conventional solutions of GR (the Schwarzschild solution, etc.) are also solutions of CGR. This is because the above solutions can be transformed to a gauge where (1.3) is fulfilled, and putting $\lambda_\mu = 0$, Eq. (1.2) is also satisfied.
- The structure of the Lagrangian (1.1) is not ruined by quantum loop corrections since for small fluctuations on a given background it can be completed to a BRST invariant form introducing the Faddeev–Popov ghost. The latter do not affect the classical solutions that we discuss.

2. Minisuperspace for constrained gravity

Computations in quantum cosmology are primarily performed in a minisuperspace approximation (for a review see, e.g., [8]). In this section we develop a minisuperspace approach to CGR. The metric for a spatially-flat universe in this approach takes the form:

$$ds^2 = N^2(t)\,dt^2 - a^2(t)\delta_{ij}\,dx^i\,dx^j. \tag{2.1}$$

Here both N and a are functions of t only, and $i, j = 1, 2, 3$. One difference from the conventional approach is that we will be working with general N not necessarily equal to the unity. We will show that N is determined by a because of the constraint.

The corresponding Lagrangian density (1.1) takes the form:

$$\mathcal{L} = -a^3 N\left[6\frac{1}{N^2}\left(\frac{\dot a}{a}\right)^2 + 2\Lambda\right] - 2\left(\frac{a^3}{N}\right)\dot\lambda_0 + 2aN\nabla\cdot\vec\lambda, \tag{2.2}$$

where $\vec\lambda = (\lambda_1, \lambda_2, \lambda_3)$, and $\nabla\cdot\vec\lambda \equiv \delta^{ij}\partial_i\lambda_j$. As a part of the rules of the minisuperspace reduction we require that λ_i's are time independent functions of spatial coordinates x_i only. This rule is justified by the complete Hamiltonian description of the theory (1.1) which is given in Appendix A. One can check that solutions to such a theory only exist when λ_0 is a function of the time coordinate t alone, and $\nabla\cdot\vec\lambda$ is a space–time constant.

It is straightforward to find the Hamiltonian density. For the canonical momentum conjugate to a, we obtain $\pi_a = -12a\dot a/N$. Moreover,

$$\pi_{\lambda_0} = -\frac{2a^3}{N}, \tag{2.3}$$

[2] We thank A. Vilenkin for bringing these references to our attention.

$$\pi_N = 0. \tag{2.4}$$

The above relations represent two primary constraints of the Hamiltonian formalism. Note that we are not introducing a conjugate momentum for $\vec{\lambda}$, since it is assumed to be time independent, and, therefore nondynamical. A more rigorous treatment is given in Appendix A. The total Hamiltonian density takes the form:

$$\mathcal{H}_{\text{total}} = -\frac{N}{24a}\pi_a^2 + 2a^3 N\left(\Lambda - \frac{\nabla\cdot\vec{\lambda}}{a^2}\right) + \alpha\left(\pi_{\lambda_0} + \frac{2a^3}{N}\right) + \beta\pi_N, \tag{2.5}$$

where α and β are Lagrange multipliers enforcing the primary constraints. Due to the Hamiltonian equations of motion $\alpha = \dot{\lambda}_0$ and $\beta = \dot{N}$. Requiring that the time variation of the two primary constraints vanishes, we obtain the equations of motion for the inexpressible velocities $\dot{N}$ and $\dot{\lambda}_0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3}{N}\right) = 0, \tag{2.6}$$

$$\dot{\lambda}_0 = \frac{N^2}{2a^3}\left[-\frac{\pi_a^2}{24a} + 2a^3\left(\Lambda - \frac{\nabla\cdot\vec{\lambda}}{a^2}\right)\right]. \tag{2.7}$$

As it could be checked directly, no further constraints emerge. On the surface of the existing constraints we can simplify the Hamiltonian density

$$\mathcal{H} = -\frac{N}{24a}\pi_a^2 + 2a^3 N\left(\Lambda - \frac{\nabla\cdot\vec{\lambda}}{a^2}\right). \tag{2.8}$$

Let us discuss classical solutions of such a theory first. From (2.7) we find that $a^3/N = b$, where b is an arbitrary constant. As we discussed already $\nabla\cdot\vec{\lambda} \equiv 3k$ is also a constant. From Eq. (2.7) we find

$$\dot{\lambda}_0 = \frac{N}{2a^3}\mathcal{H}. \tag{2.9}$$

Since both a^3/N and $\mathcal{H}$ itself commute with $\mathcal{H}$, so does $\dot{\lambda}_0$. Hence, for $b \neq 0$ we get that $\dot{\lambda}_0 = \mathcal{E}/2b$, which is a constant if $\mathcal{E}$ is an eigenvalue (energy density) of $\mathcal{H}$.

For further convenience we introduce the "conformal time"

$$\eta = \int^t N(t')\,\mathrm{d}t'. \tag{2.10}$$

Then, the equations of motion can be expressed in the following familiar form:

$$\left(\frac{a'}{a}\right)^2 + \frac{k}{a^2} = \frac{\Lambda}{3} - \frac{b\mathcal{E}}{6a^6}, \tag{2.11}$$

$$\frac{a''}{a} = \frac{\Lambda}{3} + \frac{b\mathcal{E}}{3a^6}, \tag{2.12}$$

where $' \equiv \mathrm{d}/\mathrm{d}\eta = \mathrm{d}/N\,\mathrm{d}t$. Interestingly, in these equations the quantity $\nabla\cdot\vec{\lambda} \equiv 3k$ plays the role similar to a three-dimensional spatial curvature of GR. Additional terms on the r.h.s. are also due to the λ_μ field. These terms act as a fluid with the equation of state $\rho = p = -b\mathcal{E}/2a^6$. Unlike other dynamical fields, there are no fluctuations of λ_μ.

We will consider the following three solutions of the equations of motion:

(1) $\mathcal{E} = 0$, $k \neq 0$, one finds a spatially flat inflating solution, where the scale factor a, as a function of conformal time η, is identical to that of a closed dS universe;
(2) $-b\mathcal{E} = 2k = \Lambda$ and $a = 1$, one finds a flat Minkowski space–time in spite of the fact that $\Lambda \neq 0$. This is the new flat solution described in the previous section. We consider two physically different cases: $\Lambda > 0$ and $\Lambda < 0$;
(3) $\mathcal{E} = k = 0$, gives a conventional, spatially-flat inflating de Sitter space–time.

Below we will study physical consequences of these solutions.[3]

3. Wave-function and creation probability

We now turn to the quantum mechanics of the Hamiltonian density given by (2.8). To do so we promote all the fields in (2.8) to operators with the prescription $\pi_a = -i\delta/\delta a$ and $\pi_{\lambda_0} = -i\delta/\delta\lambda_0$. The Lagrangian is an integral of the density $\mathcal{L}$ over the entire space on each time slice. To make the integral converge, we will be discussing a three-dimensionally flat space with a finite-size spatial boundary. Then, the integral

$$v = \int \mathrm{d}^3x, \tag{3.1}$$

is finite, and v denotes a spatial "comoving volume" on each time slice and is a fixed number. The physical 3-volume is $v_p(t) = \int\sqrt{\gamma}\,\mathrm{d}^3x = a^3(t)v$. So far we have ignored the factor vM_{Pl}^2 in the action. Restoring this factor, the Hamiltonian (2.8) reads

$$H = -\frac{N}{24vM_{\text{Pl}}^2 a}\pi_a^2 + 2vM_{\text{Pl}}^2 a^3 N\left(\Lambda - \frac{\nabla\cdot\vec{\lambda}}{a^2}\right). \tag{3.2}$$

Let us now suppose that $|\psi\rangle$ is an eigenstate of H with an energy eigenvalue E. As we discussed in the previous section, both $\pi_{\lambda_0} \sim a^3/N$ and $\nabla\cdot\vec{\lambda}$ commute with H. Therefore, one can always choose $|\psi\rangle$ to be an eigenstate of the above two operators with eigenvalues b and $3k$, respectively. On such a state, one can replace the operator N by a^3/b, and $\nabla\cdot\vec{\lambda}$ by $3k$. Therefore, most generically, we are looking for states

$$|\psi\rangle = \int \mathrm{d}a'\,\psi(a')|a'\rangle_a \otimes \left|a'^3/b\right\rangle_N \otimes |3k\rangle_{\nabla\cdot\vec{\lambda}}, \tag{3.3}$$

where $|a'\rangle_a$ represents the eigenstate of the operator a with eigenvalue a', and, etc. The "wave-function" ψ is determined by the Wheeler–De Witt equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}a^2} + 2A\left(3ka^2 - \Lambda a^4\right) + \frac{AbE}{a^2}\right]\psi(a) = 0, \tag{3.4}$$

[3] Note that a negative sign of the product $b\mathcal{E}$ corresponds to a positive energy density of the fluid. In general, on certain solutions $b\mathcal{E}$ could also take a positive sign producing a negative energy density fluid. However, this should not be a concern since the λ_μ field, that give rise to this fluid, is not dynamical and does not fluctuate.

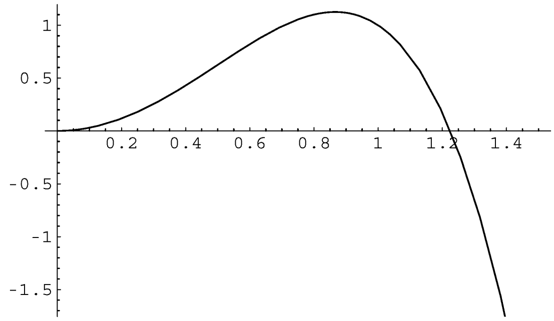

Fig. 1. The potential $U(a)$.

where $A \equiv 24M_{\rm Pl}^4 v^2$, and we have ignored the operator ordering ambiguity. The solution of this equation is equivalent to the wave-function of a particle with zero energy moving in a one-dimensional potential.

Let us look now at a probability of creation of the universe "from nothing", i.e., from a state with no classical space–time [6]. The solution describing this state should have zero energy $E = 0$. In this case the minisuperspace potential $U(a)$ is shown in Fig. 1. It has a classically forbidden region $0 \leqslant a \leqslant \sqrt{3/\Lambda}$, and a de Sitter region $a \geqslant \sqrt{3/\Lambda}$.

To calculate the probability of tunneling of the system from $a = 0$ to $a = \sqrt{3/\Lambda}$ we follow Vilenkin's tunneling wave-function approach (first reference in [6] and [8]). Linde's approach [5], although conceptually different, gives the same answer in this case. Taking the trace of equation (1.2) we easily find the action on the tunneling solutions

$$\mathcal{L} = -\frac{M_{\rm Pl}^2}{2}\sqrt{-g}\big(R + 2\Lambda + 2g^{\mu\nu}\partial_\mu\lambda_\nu\big) = 2M_{\rm Pl}^2\Lambda\sqrt{g}, \qquad (3.5)$$

and, introducing the euclidean time τ, we find

$$S_E = -2vM_{\rm Pl}^2\Lambda \int_0^{\sqrt{3/\Lambda}} \big|N(\tau)a^3(\tau)\big|\,\mathrm{d}\tau. \qquad (3.6)$$

Although this looks similar to the result in conventional quantum cosmology for the action of a closed dS universe, there is an essential difference. The comoving volume v is not fixed by the value of the cosmological constant Λ. As a result, if we are to maximize the probability of tunneling by creating a smallest size universe, then v is only to be bounded by the Planck scale. However, there is the following consideration to be taken into account. The fate of the universe after creation will depend on the boundary conditions chosen. For the universe created out of "nothing" we assume simple ones that the boundary surface has no tension, and that there is no exterior space–time.[4] Moreover, we adjust the value of the λ_j field on the boundary so that the space is complete (this is possible because the λ_μ field enters only linearly trough its first derivative in the Lagrangian (1.1)). Such a universe, to continue its inflationary expansion, should have a size bigger or equal to the scale of its dS horizon $\sqrt{3/\Lambda}$, otherwise it would collapse [10–13]. This puts a lower bound on the size of an acceptable initial universe $v_p \gtrsim 1/\Lambda^{3/2}$. For $\Lambda \ll M_{\rm Pl}^2$ we get that $|S_E| \gg 1$ and the quasi-classical arguments are well applicable. For a given value of Λ, the tunneling probability can be calculated using a conserved "Klein–Gordon" current $j_a = i(\psi^+\partial_a\psi - \psi\partial_a\psi^+)/2$ [6], and takes the form $P_T \propto \exp(-2|S_E|)$. The latter will be maximized by a smallest acceptable value of $v_p \sim 1/\Lambda^{3/2}$. This gives a results similar to the probability of creation of a closed dS universe in the tunneling approach $P_T \sim \exp(-3\pi^2 M_{\rm Pl}^2/\Lambda)$ [5,6]. However, as was emphasized above, in the present context the created universe has zero spatial curvature, while in the conventional approach only a closed dS universe can be materialized "from nothing".

The subsequent evolution of the created universe is clear. It will inflate and redshift away the contribution of the λ_j field that played the role of the spatial-curvature during the creation.

Note that if we instead followed a naive Euclidean continuation of the partition function, we would have obtained for the probability $\ln P \propto M_{\rm Pl}^2\Lambda v_p$. In such a case, largest values of Λ and v_p would have been preferred. This would favor a creation of an inflationary universe of a huge size. The above prescription is similar to the Hartle–Hawking (HH) approach [9] because of the euclidean continuation (it also somewhat differs from the HH no-boundary proposal since our solution has a boundary). However, it is not clear whether the obtained result has an interpretation of a probability of creation of a universe form "nothing".

4. Inflation through flat space

As it was discussed in Ref. [1], by choosing $\partial_\mu\lambda_\nu + \partial_\nu\lambda_\mu = -\Lambda g_{\mu\nu}$ we find a Minkowski solution even though Λ is nonzero. We will examine the properties of this solution more closely in the present section. Some results of the present section are similar to those of [7] obtained for topologically nontrivial compact universes [4].

In the minisuperspace approach the above solution is described by $k = \Lambda/2$, $bE = -\Lambda$, $a = 1$. Hence, the total energy of the solution is non-zero. Let us look at the mini-superspace potential U with the above values of k and E:

$$U(a) = A\Lambda\left(3a^2 - 2a^4 - \frac{1}{a^2}\right). \qquad (4.1)$$

Here, as before, $A = 24M_{\rm Pl}^4 v^2$, and we put back $M_{\rm Pl} = 1$ in this section. This potential is illustrated in Fig. 2. The new Minkowski solution is described by the point $a = 1$. Such a universe cannot be created out of "nothing" since it has a nonzero energy. However, during some stage of the primordial evolution, for instance at the Planck scale, a part of space can fluctuate into this state with an unsuppressed probability.

What is the cosmological evolution of such a state? Fluctuations in the system will destabilize this state. It will either roll down toward $a = 0$ corresponding to a contracting universe, or, with an almost equal probability, will roll toward $a \to \infty$ corresponding to an inflating de Sitter space. It is easy to estimate the time scale of this instability. Given any perturbation around $a = 1$, the time scale it takes for a to change significantly is determined by $(\sqrt{|U''(1)|/12A})^{-1} = \sqrt{2/\Lambda}$. There-

[4] This space can be "glued" to its own copy past the boundary.

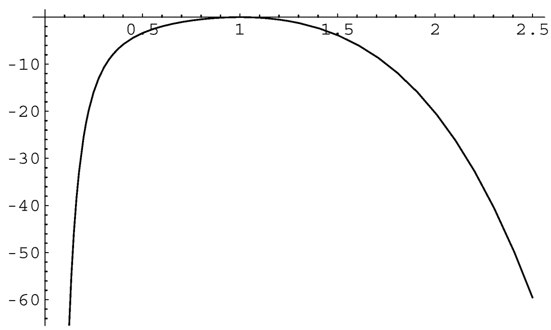

Fig. 2. Potential $U(a)$ with an unstable space–time flat solution.

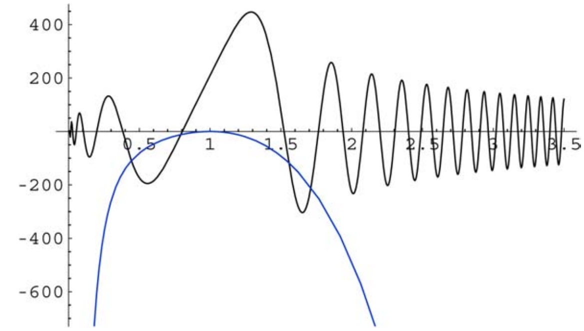

Fig. 3. Solution with $A\Lambda = 24$ and the cos-like initial condition.

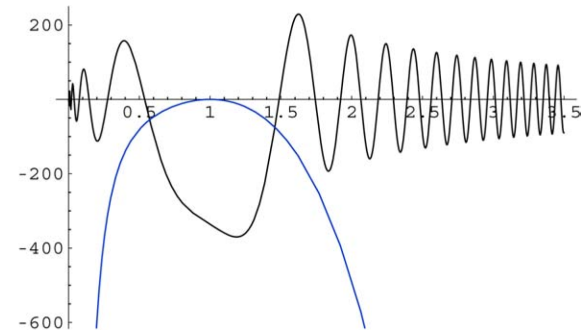

Fig. 4. Solution with $A\Lambda = 24$ and the sin-like initial condition.

fore, the process of obtaining dS universe through the above flat solution, provides a new channel for the inflationary phase.

Let us now discuss quantum mechanics of this model in more detail. The Wheeler–De Witt equation reads

$$\left[\frac{d^2}{da^2} + A\Lambda\left(2a^4 - 3a^2 + \frac{1}{a^2}\right)\right]\psi = 0. \tag{4.2}$$

When $a \to +\infty$, the term $2A\Lambda a^4$ dominates over the other terms in the potential. Therefore, the solution always asymptotes to a de Sitter universe and can be approximated by

$$\psi \sim C_1\sqrt{a}J_{+1/6}\left(\frac{\sqrt{2A\Lambda}}{3}a^3\right) + C_2\sqrt{a}J_{-1/6}\left(\frac{\sqrt{2A\Lambda}}{3}a^3\right), \tag{4.3}$$

where $J_{\pm 1/6}$ are Bessel functions of the first kind. The two linearly independent solutions are both oscillating and decaying.

The value of $A\Lambda$, however, can change the asymptotic behavior of ψ in the region of small a. There are two possibilities.

(1) If we assume that $\Lambda \sim O(1)$ and the volume of the universe at $a = 1$ is $\sim O(1)$, we find $A\Lambda \sim 24$ (this is not a realistic case since the absence of observed gravitational waves suggests that $H \sim \sqrt{\Lambda}$ has to be about 5 orders of magnitude below the Planck scale, nevertheless, we consider this as a theoretical example). Near this region it's typical that $1 - 4A\Lambda < 0$. In such a case the asymptotic behavior of ψ near $a = 0^+$ is given by

$$\psi \sim C_1\sqrt{a}\cos\left(\frac{\sqrt{4A\Lambda - 1}}{2}\ln a\right) + C_2\sqrt{a}\sin\left(\frac{\sqrt{4A\Lambda - 1}}{2}\ln a\right). \tag{4.4}$$

These are oscillating solutions. The amplitude scales as $\sqrt{a}$, and their frequencies increase to infinity toward the origin at $a = 0$. Close to the origin the wave-function has an infinitely many zeros. Since the amplitude of ψ vanishes at the origin, this behavior should not be a concern.

The above two solutions differ only by a pure phase, and, therefore, there is no physical reasons to favor one over the other. With both solutions allowed, one can smoothly interpolate the wave-function and its first derivative from $a \sim 0^+$ to the region $a \sim +\infty$. Typical solutions for ψ in this case are shown in Figs. 3 and 4, with the cos- and sin-like initial conditions near $a = 0$ respectively. The blue lines denote the potential $U(a)$.

(2) If Λ is much smaller, for example it is protected by supersymmetry at a scale much lower than the Planck scale, then the product $A\Lambda$ can be a very small number. In this case it is typical that $1 - 4A\Lambda > 0$ and the asymptotic behavior of ψ at $a = 0^+$ changes to

$$\psi \sim C_1 a^{\frac{1-\sqrt{1-4A\Lambda}}{2}} + C_2 a^{\frac{1+\sqrt{1-4A\Lambda}}{2}}. \tag{4.5}$$

The wave-function vanishes at $a = 0$ since both exponentials are positive and perfectly regular. Again, solutions that covers the entire region must exist since both solutions above are physically allowed.

In the limit where the quantity $A\Lambda$ is tiny one can ignore the existence of the potential $U(a)$ for a fairly long time, until a grows and $A\Lambda a^4$ becomes comparable with 1. When this happens, a is already so large that all the terms in the potential, besides $2A\Lambda a^4$, can be neglected. The wave function ψ should quickly turn into the Bessel functions described above. Before that happens, the Schrödinger equation takes a simple form $\psi(a)'' = 0$. As a result

$$\psi \sim C_1 + C_2 a. \tag{4.6}$$

In the present case, the contribution of the $1/a^2$ term in the potential is mostly ignorable except for the region very close to the origin. It becomes important there only to fix the initial value of ψ. Due to this term $\psi(0)$ can only be zero.

Typical properties of ψ are show in Figs. 5 and 6. One can see that the potential is extremely flat until $a \gg 1$, after which it quickly takes a form of $-2A\Lambda a^4$.

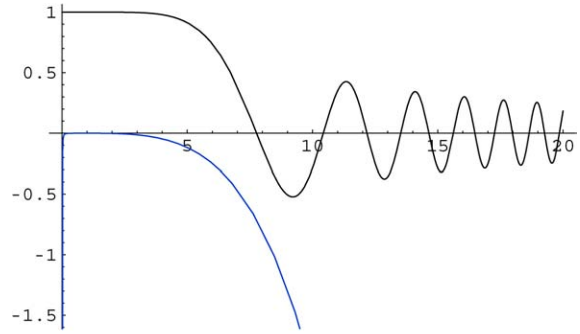

Fig. 5. Solution with $A = 10^{-4}$; ψ tends to 0 near $a = 0$ too fast to be shown in this figure.

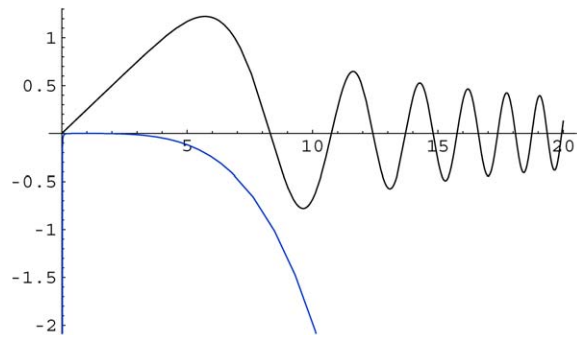

Fig. 6. Nearly linear solution with $A = 10^{-4}$.

5. Conclusions

Inflation provides a rather effective solution to the problems of the hot big-bang cosmology and successfully accounts for the observations (for a review and references, see [14]). Under reasonable physical conditions, inflationary universe is not *past-eternal* [15], and one would like to specify the past boundary of an inflating region of space–time. Quantum cosmology is one framework in which this issue can be addressed (for a review, see [8]). In this approach a *closed* dS universe can be materialized from "nothing", providing the initial conditions for inflation. The creation probability for such a state is exponentially suppressed, nevertheless, it favors inflationary initial conditions over the conditions for a universe sitting at the bottom of the potential [5,6]. On the other hand, if a *compact* spatially-flat dS universe of nontrivial topology is created [4, 7], the exponential suppression can go away.

In the present work we discussed classically constrained gravity [1]. This theory arises upon path-integral quantization of gravity as a low-energy field theory with certain boundary conditions (see discussions in Section 1 and in Ref. [1]). This approach gives rise to new solutions of equations of motion, some cosmological implications of which we studied in the present work. We showed that a *spatially-flat* dS universe with a boundary can be created form "nothing". With simple boundary conditions that we choose, the probability for creation of such a universe is exponentially suppressed, nevertheless, it favors inflationary initial conditions. This is similar to the result for a *closed* dS universe in the conventional approach [5,6].

Furthermore, we found a new interesting channel in which the probability for the inflationary initial conditions is not exponentially suppressed. The universe can fluctuate into a state with zero space–time curvature and then rapidly transitions to the inflating spatially-flat dS state. The fact that the probability is not exponentially suppressed in this case is similar to the finding of Ref. [7], however, the context in which our results are obtained, and the details of the dynamics are different.

There are a few questions that we left out for future detailed studies. It would be interesting to consider similar solutions in the presence of other dynamical fields, scalars, fermions, etc. For instance, if the cosmological constant in the Lagrangian (1.1) is negative, then the new flat solution is stable. One could imagine a scenario, in which the original inflationary universe eventually ends up in a state with a negative value of the potential. In that case, the Lagrange multiplier field can neutralize the negative potential energy and gives rise to a stable flat space–time. In general, however, it is hard to maintain this state intact during the course of cosmological evolution since any cosmological expansion of the universe redshifts the λ terms rather quickly. Under these circumstances fine-tuning might be needed to obtain a present-day universe in a state of the space–time flat solution with $\Lambda < 0$. The question of how severe this fine-tuning should be, and related issues will be discussed elsewhere.

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Appendix A. Hamiltonian formalism for CGR

To find the Hamiltonian we express the metrics in the ADM formalism:

$$g_{\mu\nu} = \begin{pmatrix} N^2 - h_{ij}N^iN^j & -h_{ij}N^j \\ -h_{ji}N^i & -h_{ij} \end{pmatrix},$$
$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{N^2} & -\frac{N^i}{N^2} \\ -\frac{N^j}{N^2} & -h^{ij} + \frac{N^iN^j}{N^2} \end{pmatrix}. \tag{A.1}$$

The Lagrangian density becomes (an overall factor of 2 is ignored below)

$$\mathcal{L} = \sqrt{\gamma}N\left(R^{(3)} + K_{ij}K^{ij} - K^2 - 2\Lambda\right) - 2\left[\left(\frac{\sqrt{\gamma}}{N}\right)\dot{\lambda}_0 - \left(\frac{\sqrt{\gamma}N^i}{N}\right)\partial_i\lambda_0\right] + 2\left[\left(\frac{\sqrt{\gamma}N^i}{N}\right)\dot{\lambda}_i + \left(N\sqrt{\gamma}h^{ij} - \frac{\sqrt{\gamma}N^iN^j}{N}\right)\partial_j\lambda_i\right], \tag{A.2}$$

where "$\cdot$" $\equiv \partial_0$. Here we have defined $\gamma = \det h_{ij}$ and the extrinsic curvature

$$K_{ij} = \frac{1}{2N}(\dot{h}_{ij} - \mathrm{D}_i N_j - \mathrm{D}_j N_i). \tag{A.3}$$

D denotes the spatial covariant derivative defined w.r.t. h_{ij}.

In order to simplify the formalism we introduce the following new variables

$$\tilde{N} = \frac{\sqrt{\gamma}}{N}, \qquad \tilde{N}^i = \frac{\sqrt{\gamma}N^i}{N}. \tag{A.4}$$

In terms of these new fields the Lagrangian density reads

$$\mathcal{L} = \frac{\gamma}{\tilde{N}}\big(R^{(3)} + K_{ij}K^{ij} - K^2 - 2\Lambda\big) - 2\big(\tilde{N}\dot{\lambda}_0 - \tilde{N}^i\partial_i\lambda_0\big) + 2\left[\tilde{N}^i\dot{\lambda}_i + \left(\frac{\gamma h^{ij}}{\tilde{N}} - \frac{\tilde{N}^i\tilde{N}^j}{\tilde{N}}\right)\partial_j\lambda_i\right]. \tag{A.5}$$

The conjugate momenta are:

$$\pi_{ij} = \sqrt{\gamma}(K_{ij} - Kh_{ij}), \tag{A.6}$$

$$\pi_{\lambda_0} = -2\tilde{N}, \tag{A.7}$$

$$\pi_{\lambda_i} = 2\tilde{N}^i, \tag{A.8}$$

$$\pi_{\tilde{N}} = \pi_{\tilde{N}^i} = 0. \tag{A.9}$$

Eqs. (A.7) through (A.9) are to be understood as eight primary constraints to be imposed on physical states. The total Hamiltonian density, including all the inexpressible velocities, is given by

$$\begin{aligned}\mathcal{H}_{\text{total}} = {} & -\frac{\gamma}{\tilde{N}}R^{(3)} + \frac{1}{\tilde{N}}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) + 2\pi^{ij}\mathrm{D}_i\left(\frac{\tilde{N}_j}{\tilde{N}}\right) \\ & + \frac{2\gamma}{\tilde{N}}\Lambda - 2\tilde{N}^i\partial_i\lambda_0 - 2\left(\frac{\gamma h^{ij}}{\tilde{N}} - \frac{\tilde{N}^i\tilde{N}^j}{\tilde{N}}\right)\partial_j\lambda_i \\ & + \pi_{\tilde{N}}\beta + \pi_{\tilde{N}^i}\gamma^i + (\pi_{\lambda_0} + 2\tilde{N})\alpha + \big(\pi_{\lambda_i} - 2\tilde{N}^i\big)\delta_i \\ \equiv {} & \mathcal{H}_0 + \pi_{\tilde{N}}\beta + \pi_{\tilde{N}^i}\gamma^i + (\pi_{\lambda_0} + 2\tilde{N})\alpha + \big(\pi_{\lambda_i} - 2\tilde{N}^i\big)\delta_i.\end{aligned} \tag{A.10}$$

The definition of $\mathcal{H}_0$ can easily be read off the expression above, and the Lagrange multipliers $\beta, \gamma_j, \alpha, \delta_i$, are determined by the Hamilton equations in terms of the velocities $\beta = \dot{\tilde{N}}$, $\gamma_j = \dot{\tilde{N}}_j$, $\alpha = \dot{\lambda}_0$, $\delta_i = \dot{\lambda}_i$. All the eight inexpressible velocities are resolvable:

$$\dot{\tilde{N}} = \partial_i\tilde{N}^i, \tag{A.11}$$

$$\dot{\tilde{N}}^i = -\partial_j\left(\frac{\gamma h^{ij}}{\tilde{N}} - \frac{\tilde{N}^i\tilde{N}^j}{\tilde{N}}\right), \tag{A.12}$$

$$\dot{\lambda}_0 = -\frac{1}{2}\frac{\delta\mathcal{H}_0}{\delta\tilde{N}}, \tag{A.13}$$

$$\dot{\lambda}_i = \frac{1}{2}\frac{\delta\mathcal{H}_0}{\delta\tilde{N}^i}. \tag{A.14}$$

These results are very different from what one finds in GR. After eliminating the inexpressible velocities, the Hamiltonian density reads

$$\mathcal{H} = -\frac{\gamma}{\tilde{N}}R^{(3)} + \frac{1}{\tilde{N}}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) + 2\pi^{ij}\mathrm{D}_i\left(\frac{\tilde{N}_j}{\tilde{N}}\right) + \frac{2\gamma}{\tilde{N}}\Lambda - 2\tilde{N}^i\partial_i\lambda_0 - 2\left(\frac{\gamma h^{ij}}{\tilde{N}} - \frac{\tilde{N}^i\tilde{N}^j}{\tilde{N}}\right)\partial_j\lambda_i, \tag{A.15}$$

where $\tilde{N}$ and $\tilde{N}^i$ *must be identified with* $-\pi_{\lambda_0}/2$ *and* $\pi_{\lambda_i}/2$ *respectively while taking Poisson brackets*. We have thrown away terms that are proportional to $\pi_{\tilde{N}}$ and $\pi_{\tilde{N}^i}$, which necessarily vanish in any case.

Upon quantization one should impose the constraints $\pi_{\tilde{N}}|\psi\rangle = \pi_{\tilde{N}^i}|\psi\rangle = 0$ on physical states and proceed with the usual canonical procedure using the Hamiltonian density given above. Notice that since $-2\tilde{N} \equiv \pi_{\lambda_0}$ and $2\tilde{N}^i \equiv \pi_{\lambda_i}$, and they both appear in $\mathcal{H}$, λ_0 and λ_i do not commute with the Hamiltonian and therefore are not conserved in general. However, Eq. (A.14) can be expressed as

$$\dot{\lambda}_i = -\frac{1}{\tilde{N}}\mathrm{D}_j\pi_i^j + \frac{\tilde{N}^j}{\tilde{N}}(\partial_j\lambda_i + \partial_i\lambda_j), \tag{A.16}$$

therefore $\dot{\lambda}_i = 0$ as long as $N^i = 0$, and π_j^i depends on time t only. In such a case, λ_i do commute with the Hamiltonian density. It is because of this reason that we have imposed this condition in the minisuperspace formalism.

Mathematically such a constraint can be enforced more rigorously by introducing a term $A_i\dot{\lambda}_i$ in the Lagrangian density with Lagrange multipliers A_i. Using this Lagrangian density one can work out $\mathcal{H}_{\text{total}}$ in a similar fashion as we did above. After all the constraints are taken into account consistently, one finds that the Hamiltonian density is exactly the same as (2.8) with extra constraints $\pi_{A_i} = 0$. In the quantum mechanics of such a theory one must then impose the constraints $\pi_{A_i}|\psi\rangle = 0$ on physical states $|\psi\rangle$. This says that any physical state must be independent to the Lagrange multipliers A_i, which is what one should have expected.

Appendix B. Equations for λ_μ from Hamiltonian

In the Hamiltonian formalism for CGR, time derivatives of some of the primary constraints give rise to Eqs. (A.11) and (A.12). We will be using these conditions in the following derivations without mentioning them explicitly. Time derivatives of the rest of the primary constraints generate the equations of motion for the Lagrange multipliers λ_μ (up to a surface term) as we will illustrate in this appendix.

From (A.13) and (A.14) we have

$$\dot{\lambda}_0 = -\frac{1}{2}\frac{\partial\mathcal{H}_0}{\partial\tilde{N}} = \frac{\mathcal{H}_0 + 2\tilde{N}^i\partial_i\lambda_0}{2\tilde{N}}, \tag{B.1}$$

$$\dot{\lambda}_i = \frac{\partial\mathcal{H}_0}{2\partial\tilde{N}^i} = -\frac{\sqrt{\gamma}}{\tilde{N}}\mathrm{D}_j\big[(\sqrt{\gamma})^{-1}\pi_i^j\big] - \partial_i\lambda_0 + \frac{\tilde{N}^j}{\tilde{N}}(\partial_j\lambda_i + \partial_i\lambda_j). \tag{B.2}$$

From the first equation above one immediately finds that up to a surface term

$$\mathcal{H}_0 = 2(\tilde{N}\dot{\lambda}_0 + \lambda_0\dot{\tilde{N}}) = 2\partial_0(\tilde{N}\lambda_0). \tag{B.3}$$

Notice that $\partial_0\mathcal{H}_0 = 0$, and further time derivative of this equation gives

$$0 = \int d^3x \left[\tilde{N}\ddot{\lambda}_0 + 2\dot{\tilde{N}}\dot{\lambda}_0 + \lambda_0 \partial_i \dot{\tilde{N}}^i\right]$$
$$= \int d^3x \left[\tilde{N}\ddot{\lambda}_0 - 2\tilde{N}^i \partial_i \dot{\lambda}_0 - \left(\frac{\gamma h^{ij}}{\tilde{N}} - \frac{\tilde{N}^i \tilde{N}^j}{\tilde{N}}\right)\partial_i \partial_j \lambda_0\right], \tag{B.4}$$

which, up to a surface term, reproduces $g^{\mu\nu}\partial_\mu\partial_\nu\lambda_0 = 0$.

To make further use of Eq. (B.2) we first notice that up to a surface term

$$2\int d^3x \sqrt{\gamma}\, D_j\left[(\sqrt{\gamma})^{-1}\pi_i^j\right] = \int d^3x\, h_{jk}\partial_i \pi^{jk}. \tag{B.5}$$

Therefore, using the identities $\pi_{\lambda_0} = -2\tilde{N}$ and $\pi_{\lambda_i} = 2\tilde{N}^i$, the spatial integral of (B.2) can be simplified as

$$-2\int d^3x \left[\tilde{N}\dot{\lambda}_i - \tilde{N}^j \partial_j \lambda_i\right]$$
$$= \int d^3x \left[h_{jk}\partial_i \pi^{jk} + \lambda_0 \partial_i \pi_{\lambda_0} + \lambda_j \partial_i \pi_{\lambda_j}\right]. \tag{B.6}$$

Likewise, we find that the time derivative of the l.h.s. of this equation gives

$$-2\int d^3x \left[\tilde{N}\ddot{\lambda}_i - 2\tilde{N}^j \partial_j \dot{\lambda}_i - \left(\frac{\gamma h^{jk}}{\tilde{N}} - \frac{\tilde{N}^j \tilde{N}^k}{\tilde{N}}\right)\partial_j \partial_k \lambda_i\right]. \tag{B.7}$$

To compute the time derivative of the r.h.s. of Eq. (B.6) one only needs to notice that (up to a surface term)

$$\int d^3x \left[\dot{h}_{jk}\partial_i \pi^{jk} - \partial_i h_{jk}\dot{\pi}^{jk}\right]$$
$$= \int d^3x \left[\frac{\partial \mathcal{H}}{\partial \pi^{jk}}\partial_i \pi^{jk} + \frac{\partial \mathcal{H}}{\partial h_{jk}}\partial_i h_{jk}\right]. \tag{B.8}$$

If we apply this same trick to the last two terms on the r.h.s. of Eq. (B.6) we find that its time derivative is simply

$$\int d^3x \left[\frac{\partial \mathcal{H}}{\partial \pi^{jk}}\partial_i \pi^{jk} + \frac{\partial \mathcal{H}}{\partial h_{jk}}\partial_i h_{jk} + \frac{\partial \mathcal{H}}{\partial \pi_{\lambda_0}}\partial_i \pi_{\lambda_0}\right.$$
$$\left. + \frac{\partial \mathcal{H}}{\partial \lambda_0}\partial_i \lambda_0 + \frac{\partial \mathcal{H}}{\partial \pi_{\lambda_j}}\partial_i \pi_{\lambda_j} + \frac{\partial \mathcal{H}}{\partial \lambda_j}\partial_i \lambda_j\right] = \int d^3x\, \partial_i \mathcal{H}. \tag{B.9}$$

Here $\mathcal{H}$ is the Hamiltonian density given by (A.15) in which $\tilde{N}$ and $\tilde{N}^i$ are understood as conjugate momenta to λ_0 and λ_i, respectively. Therefore, we find

$$\tilde{N}\ddot{\lambda}_i - 2\tilde{N}^j \partial_j \dot{\lambda}_i - \left(\frac{\gamma h^{jk}}{\tilde{N}} - \frac{\tilde{N}^j \tilde{N}^k}{\tilde{N}}\right)\partial_j \partial_k \lambda_i = 0, \tag{B.10}$$

which is indeed equivalent to $g^{\mu\nu}\partial_\mu\partial_\nu\lambda_i = 0$.

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