Extended M-Matrices and Subtangentiality

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ABSTRACT

The concept of a singular M-matrix A with respect to a proper cone $\mathscr{K}$ is extended, by replacing the usual regularity condition $A = \alpha I - B$ for a $\mathscr{K}$-nonnegative matrix B with the weaker condition, exponential nonnegativity of $-A$. As in earlier work which dealt with the nonsingular case, in the present characterizations the lack of regularity is overcome by employing subtangentiality.

1. INTRODUCTION

Let $\mathscr{K} \subset R^n$ be a proper cone, and let A be a real $n \times n$ matrix which is $\mathscr{K}$-*regular*; that is, $A = \alpha I - B$ for some $\alpha \in R$ and some matrix B which is $\mathscr{K}$-nonnegative. Then A is called a *K-general M-matrix* provided that the eigenvalues of A all have nonnegative real parts. Our general reference on both singular and nonsingular $\mathscr{K}$-general M-matrices is Berman and Plemmons [2], which also contains further bibliographic information.

The purpose of the present work is to generalize certain results on $\mathscr{K}$-general M-matrices given in [2], when the condition of $\mathscr{K}$-regularity is replaced by the *weaker* condition $\mathscr{K}$-*exponential nonnegativity* of $-A$; that is, $e^{-tA}\mathscr{K} \subset \mathscr{K}$ $\forall t \geqslant 0$. Unlike several well-known results on M-matrices, in the present work conditions on "extended" M-matrices involving spectral radii are not relevant, as regularity may not hold. Also, since we will work with general proper cones, it is not surprising that we will consider only

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"operator theoretic" properties (spectral conditions, types of monotonicity and semipositivity, etc.) as opposed to properties involving "internal structure" such as are known in particular for $\mathscr{K} = R^n_+$ (e.g. conditions involving diagonal dominance, principal minors, etc.).

The next section contains definitions and preliminary results. Then in Section 3, the concept of a $\mathscr{K}$-extended M-matrix is introduced. In the main results of that section, we obtain characterizations of $\mathscr{K}$-extended M-matrices which generalize results in Stern [8], which dealt with the nonsingular case. In the present work, as in [8], the lack of regularity is overcome by making use of the concept of subtangentiality, which is a geometric condition imposed by exponential nonnegativity. Some further results are given in Section 4.

2. DEFINITIONS AND PRELIMINARY RESULTS

A nonempty set $\mathscr{K} \subset R^n$ is said to be a *cone* if $\alpha\mathscr{K} \subset \mathscr{K}$ $\forall\alpha \geqslant 0$. The cone $\mathscr{K}$ is *polyhedral* if it is the intersection of a finite number of closed half spaces (or equivalently if it is generated by a finite set of vectors). A cone $\mathscr{K}$ is *proper* if it is closed, convex, pointed (i.e. $\mathscr{K} \cap \{-K\} = \{0\}$), and solid (i.e. has a nonempty interior, denoted by int $\mathscr{K}$).

Now we introduce some required terminology.

DEFINITION 2.1. Let $\mathscr{K} \subset R^n$ be a proper cone. Then for a real $n \times n$ matrix A we denote

$$\mathscr{S}_A = \bigcap_{m=0}^{\infty} \mathscr{R}(A^m),$$

where $\mathscr{R}(\cdot)$ denotes range. We say that A is:

(2.1.1) *$\mathscr{K}$-nonnegative* if $A\mathscr{K} \subset \mathscr{K}$.

(2.1.2) *$\mathscr{K}$-regular* if there exist $\alpha \in R$ and a $\mathscr{K}$-nonnegative matrix B such that $A = \alpha I - B$.

(2.1.3) *$\mathscr{K}$-exponentially nonnegative* if $e^{tA}\mathscr{K} \subset \mathscr{K}$ $\forall t \geqslant 0$.

(2.1.4) *$\mathscr{K}$-monotone on $\mathscr{S}_A$* if $Ax \in \mathscr{K}$, $x \in \mathscr{S}_A \Rightarrow x \in \mathscr{K}$.

(2.1.5) *Weakly stable* if Re[Spectrum(A)] $\leqslant 0$.

(2.1.6) *$\mathscr{K}$-semipositive on $\mathscr{S}_A$* if there exists $x \in \mathscr{K} \cap \mathscr{S}_A$ such that $Ax \in \{\text{int}\,\mathscr{K}\} \cap \mathscr{S}_A$.

(2.1.7) *$(\mathscr{K} \cap \mathscr{S}_A)$-zeroed* if $\{x \in \mathscr{K} \cap \mathscr{S}_A : Ax \in \mathscr{K}\} = \{0\}$.

REMARK 2.2.

(2.2.1) If $\mathscr{K} = R^n_+$, the nonnegative orthant, then A is $\mathscr{K}$-regular if and only if $a_{ij} \leqslant 0$ for $i \neq j$.

(2.2.2) It was proven in [7] that $\mathscr{K}$-regularity of A implies $\mathscr{K}$-exponential nonnegativity of $-A$, with equivalence holding in case $\mathscr{K}$ is polyhedral.

Next we review some required basic definitions and known results on generalized inverses. (Our reference in this regard is Ben-Israel and Grenville [1].) The *index* of a square matrix A is the smallest nonnegative integer k such that $\operatorname{rank}(A^{k+1}) = \operatorname{rank}(A^k)$. Then $\mathscr{S}_A = \bigcap_{m=0}^{k} \mathscr{R}(A^m)$. A real $n \times n$ matrix X which satisfies $XAX = X$, $AX = XA$, $A^{p+1}X = A^pX\ \forall p \geqslant \operatorname{index}(A)$ exists uniquely and is called the *Drazin inverse* of A, denoted by A^D. The Drazin inverse of A is a *generalized left inverse of* A; that is, $A^DAx = x$ for all $x \in \mathscr{S}_A$. We also note that

$$\mathscr{S}_A \cap \mathscr{N}(A) = \{0\}. \tag{2.3}$$

DEFINITION 2.4. Let A be an $n \times n$ matrix, and let $\mathscr{K} \subset R^n$ be a proper cone. Then a generalized left inverse of A, say Y, is said to be *$\mathscr{K}$-nonnegative on $\mathscr{S}_A$* if $Y(\mathscr{K} \cap \mathscr{S}_A) \subset \mathscr{K}$.

The following result was proven in Neumann and Plemmons [6] for $\mathscr{K} = R^n_+$. (See also Theorem 5.4.24 in [2].) Since the extension to general proper cones is straightforward, we omit the proof.

THEOREM 2.5. *For a real $n \times n$ matrix A and proper cone $\mathscr{K} \subset R^n$, the following statements are equivalent:*

(i) *A has a generalized left inverse which is $\mathscr{K}$-nonnegative on $\mathscr{S}_A$.*

(ii) *Every generalized left inverse of A is $\mathscr{K}$-nonnegative on $\mathscr{S}_A$. In particular $A^D(\mathscr{K} \cap \mathscr{S}_A) \subset \mathscr{K}$.*

(iii) *A is $\mathscr{K}$-monotone on $\mathscr{S}_A$.*

DEFINITION 2.6. For an $n \times n$ real matrix A, consider the linear autonomous differential equation

$$\dot{x}(t) = Ax(t). \tag{2.7}$$

A set $\Gamma \subset R^n$ is said to be *positively invariant with respect to* A if $x(0) \in \Gamma$ implies that $x(t) = e^{tA}x(0) \in \Gamma\ \forall t \geqslant 0$. (If Γ is a proper cone, the property is the same as $\mathscr{K}$-exponential nonnegativity of A.)

If $\Gamma \subset R^n$ is closed and convex, we define the set of *nonzero outward unit normal vectors to* Γ at a point $x \in \partial\Gamma$ (the boundary) as

$$N_\Gamma(x) = \{\nu \in R^n : \langle \nu, y - x\rangle \leqslant 0 \ \forall y \in \Gamma, \ \|\nu\| = 1\},$$

where $\|\cdot\|$ denotes the euclidean norm.

DEFINITION 2.8. For a closed convex set $\Gamma \subset R^n$, a vector $z \in R^n$ is *subtangential* to Γ at $x \in \partial\Gamma$ if $\langle z, \nu\rangle \leqslant 0 \ \forall \nu \in N_\Gamma(x)$.

The following theorem characterizes positive invariance of a closed convex set as equivalent to the velocity vector Ax being "tangent to or pointing into the set" for each point x on the boundary of the set.

THEOREM 2.9. *A closed convex set $\Gamma \subset R^n$ is positively invariant with respect to A if and only if Ax is subtangential to x for every $x \in \partial\Gamma$.*

We shall require the following lemma.

LEMMA 2.10. *Let $\mathscr{K} \subset R^n$ be a proper cone. Then:*

(2.10.1) $\langle \nu, x\rangle = 0 \ \forall x \in \partial\mathscr{K}, \ \forall \nu \in N_{\mathscr{K}}(x)$.

(2.10.2) *If A is $\mathscr{K}$-exponentially nonnegative and $x \in R^n$ is such that $Ax \in \mathscr{K}$, then the shifted cone $\{x + \mathscr{K}\}$ is positively invariant.*

Theorem 2.9 is proven in Stern [9], and is based on a result of Nagumo [5]. The proof of the Lemma 2.10 can be found in [8]. We also shall make use of the following.

THEOREM 2.11 (Elsner [3], Schneider and Vidyasagar [7]). *Let $\mathscr{K} \subset R^n$ be a proper cone. If $e^{tA}\mathscr{K} \subset \mathscr{K} \ \forall t \geqslant 0$ then*

$$\lambda_A = \max\{\operatorname{Re}\lambda : \lambda \in \operatorname{Spectrum}(A)\}$$

is an eigenvalue of A and has an associated eigenvector in $\mathscr{K}$.

The *dual cone* of a set $\Gamma \subset R^n$ is denoted

$$\Gamma^* = \{y \in R^n : \langle y, x\rangle \geqslant 0 \ \forall x \in \Gamma\}.$$

The proof of the next lemma (which can be found in [8]) follows readily from the fact that

$$(\mathscr{K}^*)^* = \mathscr{K} \qquad \text{for any proper cone } \mathscr{K}.$$

LEMMA 2.12. *Let $\mathscr{K} \subset R^n$ be a proper cone. Then for any $t \in R$*

$$e^{tA}\mathscr{K} \subset \mathscr{K} \quad \Leftrightarrow \quad e^{tA^T}\mathscr{K}^* \subset \mathscr{K}^*.$$

Finally, we have the following lemma, which is a straightforward consequence of (2.10.1).

LEMMA 2.13. *Let $\mathscr{K} \subseteq R^n$ be a proper cone. Then*

$$N_{\mathscr{K}}(x) \subset -(\mathscr{K}^*) \qquad \forall x \in \partial\mathscr{K}.$$

3. EXTENDED M-MATRICES

In the following theorem we shall use the fact that if Re[Spectrum(A)] < 0, then the origin is a *stable equilibrium* of the differential equation (2.7.1); that is, $e^{tA}x \to 0$ as $t \to \infty$ for every $x \in R^n$.

THEOREM 3.1. *Let $\mathscr{K} \subset R^n$ be a proper cone, and let A be a real $n \times n$ matrix such that $-A$ is $\mathscr{K}$-exponentially nonnegative. Then the following are equivalent:*

(i) *$-A$ is $(\mathscr{K} \cap S_A)$-zeroed.*
(ii) *A is $\mathscr{K}$-monotone on S_A.*
(iii) *$-A$ is weakly stable.*

Proof. (i) ⇒ (iii): Suppose that (i) holds but that (iii) did not hold. Then there exists $\lambda \in \text{Spectrum}(-A)$ such that $\text{Re}\,\lambda > 0$, whence $\lambda_{-A} > 0$. Furthermore, in view of Theorem 2.11, the $\mathscr{K}$-exponential nonnegativity of $-A$ implies that λ_{-A} is an eigenvalue of $-A$ with an associated eigenvector $x \in \mathscr{K}$. Since $(-A)^m x = (\lambda_{-A})^m x$ for all $m = 0, 1, 2, \ldots$, we have $0 \neq x \in \mathscr{K} \cap \mathscr{S}_A$ and $-Ax \in \mathscr{K}$, which violates (i).

(iii) ⇒ (ii): If $\mathscr{K} \cap \mathscr{S}_A = \{0\}$ then A is trivially $\mathscr{K}$-monotone on $\mathscr{S}_A$. Hence we shall assume that $\mathscr{K} \cap \mathscr{S}_A \neq \{0\}$ and that (iii) holds. Suppose that A were not $\mathscr{K}$-monotone on $\mathscr{S}_A$. Then there exists $x \in R^n$ such that $Ax \in \mathscr{K}$, $x \in \mathscr{S}_A$, and $x \notin \mathscr{K}$. According to Lemma 2.10, the shifted cone $\{-x + \mathscr{K}\}$ is positively invariant with respect to $-A$. Since $0 \notin \{-x + \mathscr{K}\}$, the closedness of $\{-x + \mathscr{K}\}$ then implies that it is impossible for $e^{-tA}x \to 0$ as $t \to \infty$. Now viewing $\mathscr{K} \cap \mathscr{S}_A$ as a proper cone in the A-invariant subspace $\mathscr{S}_A$, and upon letting $\bar{A}$ denote the restriction of A to $\mathscr{S}_A$, Theorem 2.11 implies that there exists $0 \neq \hat{x} \in \mathscr{K} \cap \mathscr{S}_A$ such that $-A\hat{x} = \lambda_{-\bar{A}}\hat{x}$. Since $x \in \mathscr{S}_A$ and $e^{-tA}x \nrightarrow 0$ as $t \to \infty$, it follows that $-\bar{A}$ is not a stability matrix, and since $\mathscr{S}_A \cap \mathscr{N}(A) = \{0\}$, we have $\lambda_{-\bar{A}} \neq 0$. Hence $0 < \lambda_{-\bar{A}} \leqslant \lambda_{-A}$, which implies that some eigenvalue of A has negative real part, thus violating (iii).

(ii) $\Rightarrow$ (i): If (i) does not hold, then there exists $0 \neq x \in \mathscr{K} \cap \mathscr{S}_A$ such that $-Ax \in \mathscr{K}$. Then (ii) implies $-x \in \mathscr{K}$, which violates the pointedness of $\mathscr{K}$. ■

DEFINITION 3.2. Let $\mathscr{K} \subset R^n$ be a proper cone. If A is $\mathscr{K}$-exponentially nonnegative and A satisfies any of the equivalent conditions in Theorem 3.1, then A is called *$\mathscr{K}$-extended M-matrix.*

REMARK 3.3. In view of Remark 2.2.2, the concepts of $\mathscr{K}$-general and $\mathscr{K}$-extended M-matrices are identical in case $\mathscr{K}$ is polyhedral. A example of a singular $\mathscr{K}$-extended M-matrix which is *not* a $\mathscr{K}$-general M-matrix is provided by the ice-cream cone $\mathscr{K} = \{x \in R^3 : x_1^2 + x_2^2 \leqslant x_3^2, x_3 \geqslant 0\}$ and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

since as is readily checked, $-A$ is $\mathscr{K}$-exponentially nonnegative and weakly stable, while A is not $\mathscr{K}$-regular.

THEOREM 3.4. *Let $\mathscr{K} \subset R^n$ be a proper cone, and let $-A$ be $\mathscr{K}$-exponentially nonnegative. Assume further that*

$$\{\operatorname{int} \mathscr{K}\} \cap \mathscr{S}_A \neq \varnothing. \tag{3.5}$$

Then A is a $\mathscr{K}$-extended M-matrix if and only if A is $\mathscr{K}$-semipositive on $\mathscr{S}_A$.

Proof. First assume that A is a $\mathscr{K}$-extended M-matrix. Then A is $\mathscr{K}$-monotone on $\mathscr{S}_A$. In view of Theorem 2.5, the Drazin inverse A^D is then $\mathscr{K}$-nonnegative on $\mathscr{S}_A$. Let $0 \neq d \in \{\operatorname{int} \mathscr{K}\} \cap \mathscr{S}_A$. Then

$$x = A^D d \in \mathscr{K} \cap \mathscr{S}_A \quad \Rightarrow \quad Ax = AA^D d = d.$$

Since $x \in \mathscr{K} \cap \mathscr{S}_A$ and $Ax \in \{\operatorname{int} \mathscr{K}\} \cap \mathscr{S}_A$, we conclude that A is $\mathscr{K}$-semipositive on $\mathscr{S}_A$.

Now assume that A is $\mathscr{K}$-semipositive on $\mathscr{S}_A$. Let $x \in \mathscr{K} \cap \mathscr{S}_A$ be such that $Ax \in \{\operatorname{int} \mathscr{K}\} \cap \mathscr{S}_A$. We will show that A^T is a $\mathscr{K}^*$-extended M-matrix. (In view of Lemma 2.12 and Theorem 3.1, this will suffice.) Suppose by way of contradiction that A^T were not a $\mathscr{K}^*$-extended M-matrix. Then because of Theorem 3.1 [and condition (i) in particular] there would necessarily exist a vector $0 \neq u \in \mathscr{K}^* \cap \mathscr{S}_{A^T}$ such that $-A^T u \in \mathscr{K}^*$. Since $x \in \mathscr{K}$, it follows that $\langle x, A^T u \rangle \leqslant 0$. But

$$0 \neq u \in \mathscr{K}^*, \quad Ax \in \operatorname{int} \mathscr{K} \quad \Rightarrow \quad \langle Ax, u \rangle = \langle x, A^T u \rangle > 0,$$

which provides the required contradiction. ■

4. FURTHER RESULTS

We begin with a generalization of Lemma 6.4.1 of [2].

LEMMA 4.1. *Let $\mathscr{K} \subset R^n$ be a proper cone. Then A is a $\mathscr{K}$-extended M-matrix if and only if $A + \varepsilon I$ is a nonsingular $\mathscr{K}$-extended M-matrix $\forall \varepsilon > 0$.*

Proof. Let A be a $\mathscr{K}$-extended M-matrix, and let $\varepsilon > 0$. Then the eigenvalues of $A + \varepsilon I$ all have positive real parts. Furthermore, since $e^{-t(A+\varepsilon I)}\mathscr{K} = e^{-t\varepsilon} e^{-tA}\mathscr{K} \subset \mathscr{K}$, it follows that $A + \varepsilon I$ is a nonsingular $\mathscr{K}$-extended M-matrix. This proves the "only if" part of the lemma. In order to prove the "if," assume that $A + \varepsilon I$ is a $\mathscr{K}$-extended M-matrix for every $\varepsilon > 0$, and let $\varepsilon \to 0$. Then clearly the eigenvalues of A all have nonnegative real parts. Furthermore, since

$$e^{-tA}\mathscr{K} = e^{-tA - t\varepsilon + t\varepsilon}\mathscr{K} = e^{t\varepsilon} e^{-t(A+\varepsilon I)}\mathscr{K} \subset \mathscr{K},$$

it follows that A is a $\mathscr{K}$-extended M-matrix. ■

REMARK 4.2. Since the set of $\mathscr{K}$-exponentially nonnegative matrices is the closure of the set of $\mathscr{K}$-regular matrices (see [7]), the above lemma readily implies that every $\mathscr{K}$-extended M-matrix is the limit of $\mathscr{K}$-general M-matrices.

The "if" part of the next result follows from *Satz* 1 in Elsner [4]. It generalizes part of Theorem 6.4.7 in [2], where it was assumed that $\mathscr{K} = R^n_+$ and regularity was explicitly used. We present a proof which makes use of subtangentiality.

THEOREM 4.3. *Let $\mathscr{K} \subset R^n$ be a proper cone. Then A is an extended M-matrix if and only if $A + \varepsilon I$ is $\mathscr{K}$-monotone on R^n $\forall \varepsilon > 0$.*

Proof. "Only if": Let A be a $\mathcal{K}$-extended M-matrix. Then according to Lemma 4.1, $A + \varepsilon I$ is a nonsingular $\mathcal{K}$-extended M-matrix $\forall \varepsilon > 0$, and consequently $A + \varepsilon I$ is $\mathcal{K}$-monotone on $\mathcal{S}_A = R^n$.

"If": Assume that $A + \varepsilon I$ is $\mathcal{K}$-monotone on R^n for every $\varepsilon > 0$. First we will prove that $-A$ is $\mathcal{K}$-exponentially nonnegative. Suppose not. Then by Theorem 2.9 there exists $g \in \partial\mathcal{K}$ such that $\langle \nu, -Ag \rangle > 0$ for a vector $\nu \in N_{\mathcal{K}}(g)$. For sufficiently small $\varepsilon > 0$, $\varepsilon A + I$ is nonsingular and the second term dominates the series expansion.

$$(\varepsilon A + I)^{-1} = L - \varepsilon A + (\varepsilon A)^2 - (\varepsilon A)^3 + \cdots .$$

Then

$$\nu^T\left(A + \frac{1}{\varepsilon} I\right)^{-1} g = \varepsilon \nu^T(\varepsilon A + I)^{-1} g = \varepsilon\left[\nu^T g - \varepsilon \nu^T A g + \varepsilon^2 \nu^T A^2 g - \cdots\right] > 0,$$

since $\nu^T g = 0$ (by Lemma 2.10). In view of Lemma 2.13, this implies that $[A + (1/\varepsilon)I]^{-1} g \notin \mathcal{K}$ for small $\varepsilon > 0$, contradicting the fact that $g \in \mathcal{K}$ and the $\mathcal{K}$-monotonicity of $A + (1/\varepsilon)I$. Hence $-A$ is $\mathcal{K}$-exponentially nonnegative, and it readily follows that $-(A + \varepsilon I)$ is $\mathcal{K}$-exponentially nonnegative as well, $\forall \varepsilon > 0$. Hence $(A + \varepsilon I)$ is a $\mathcal{K}$-extended M-matrix $\forall \varepsilon > 0$. Since $\mathcal{K}$-monotonicity on R^n implies nonsingularity, an application of Lemma 4.1 completes the proof. ∎

A matrix $A \in R^{n \times n}$ was called *almost monotone* in [2] if $Ax \geq 0 \Rightarrow Ax = 0$. The analagous property for general proper cones is given next.

DEFINITION 4.4. Let $\mathcal{K} \subset R^n$ be a proper cone. Then a real $n \times n$ matrix is said to be *almost $\mathcal{K}$-monotone* provided that $Ax \in \mathcal{K} \Rightarrow Ax = 0$.

THEOREM 4.5. *Let $\mathcal{K} \in R^n$ be a proper cone, and let A be a singular $\mathcal{K}$-extended M-matrix such that A has no left eigenvector in $\partial(\mathcal{K}^*)$. Then A is almost $\mathcal{K}$-monotone.*

Proof. Since A^T is a singular $\mathcal{K}^*$-extended M-matrix, we have $\lambda_{A^T} = 0$. Our hypotheses imply that there exists $y \in \operatorname{int}(\mathcal{K}^*)$ such that $A^T y = 0$. Now let $x \in R^n$ be such that $Ax \in \mathcal{K}$. If $Ax \neq 0$ then $\langle y, Ax \rangle = \langle A^T y, x \rangle > 0$, a contradiction. ∎

Next we generalize a result of Varga [10] (see also [2, Section 6.3]), which was obtained for $\mathscr{K} = R^n_+$ by using analytic function methods.

LEMMA 4.6. *Let A be a nonsingular real $n \times n$ matrix, and let $\mathscr{K} \subset R^n$ be a proper cone. Then:*

(4.6.1) $A^{-1}[e^{tA} - I]\mathscr{K} \subset \mathscr{K}$ $\forall t \geqslant 0$ $\Leftrightarrow$ *$\mathscr{K}$ is positively invariant with respect to A.*

(4.6.2) $A^{-1}[e^{tA} - I](\mathscr{K}/\{0\}) \subset \operatorname{int} \mathscr{K}$ $\forall t > 0 \Rightarrow \mathscr{K}$ *is positively invariant with respect to A, and A contains no eigenvector in $\partial\mathscr{K}$.*

Proof. "$\Leftarrow$" in (4.6.1): For $g \in R^n$ and $t \geqslant 0$ let us define the R^n-valued function

$$x(t,g) = A^{-1}[e^{tA} - I]g.$$

Note that $\dot{x}(t,g) = e^{tA}g$ and that $x(0,g) = 0$. Therefore

$$x(t,g) = \int_0^t e^{sA}g\,ds.$$

Now let $g \in \mathscr{K}$. Since $e^{tA}g \in \mathscr{K}$ for all $t \geqslant 0$, we have $x(t,g) = \int_0^t e^{sA}g\,ds \in \mathscr{K}$ for all $t \geqslant 0$, by Riemann sum approximation and conicity.

"$\Rightarrow$" in (4.6.1): Suppose that $\mathscr{K}$ were not positively invariant with respect to A. Then there exists $g \in \partial\mathscr{K}$ such that Ag is not subtangential to $\mathscr{K}$ at g; that is, there exists $\nu \in \mathscr{N}_{\mathscr{K}}(g)$ such that $\langle \nu, Ag \rangle > 0$, and therefore $\langle \nu, \ddot{x}(0,g)\rangle = \langle \nu, Ag \rangle > 0$. In view of Lemma 2.10.1 we have $\langle \nu, g \rangle = \langle \dot{x}(0,g)\rangle = 0$. Hence there exists $T > 0$ such that

$$\langle \nu, \dot{x}(t,g)\rangle = \left\langle \nu, \int_0^t \ddot{x}(s,g)\,ds \right\rangle > 0 \qquad \text{for all} \quad t \in (0,T].$$

Since $x(0,g) = 0$, it follows that $\langle \nu, x(t,g)\rangle > 0$ for all $t \in (0,T]$, and by Lemma 2.13, $x(t,g) \notin \mathscr{K}$ for all $t \in (0,T]$, which yields the required contradiction.

Proof of (4.6.2): From (4.6.1) we already know that $\mathscr{K}$ is positively invariant with respect to A. Suppose by way of contradiction that the (nonsingular) matrix A had an eigenvector in the boundary of $\mathscr{K}$. That is, suppose that there exists $0 \neq \lambda \in R$ and $0 \neq g \in \partial \mathscr{K}$ such that $Ag = \lambda g$. Then

$$x(t, g) = \frac{1}{\lambda}[e^{t\lambda} - 1]g \in \partial \mathscr{K} \qquad \text{for all} \quad t \geqslant 0,$$

which violates the left hand side of (4.6.2). ∎

The following two results are immediate consequences of Lemma 4.6.

THEOREM 4.7. *Let $\mathscr{K} \subset R^n$ be a proper cone, and suppose that A is a nonsingular $\mathscr{K}$-extended M-matrix. Then*

$$(4.8) \qquad A^{-1}[I - e^{-tA}](\mathscr{K} \backslash \{0\}) \subset \mathscr{K} \backslash \{0\} \qquad \textit{for all} \quad t > 0.$$

Furthermore, we have

$$(4.9) \qquad A^{-1}[I - e^{-tA}](\mathscr{K} \backslash \{0\}) \subset \operatorname{int} \mathscr{K} \qquad \textit{for all} \quad t > 0$$

only if A has no eigenvector in $\partial \mathscr{K}$.

THEOREM 4.10. *Let $\mathscr{K} \subset R$ be a proper cone, and let A be a real $n \times n$ nonsingular matrix. Assume that all the eigenvalues of A have nonnegative real parts. Then* (4.8) *holds if and only if A is a nonsingular $\mathscr{K}$-extended M-matrix. Furthermore,* (4.9) *holds only if A is a nonsingular $\mathscr{K}$-extended M-matrix which has no eigenvector in $\partial \mathscr{K}$.*

As a concluding comment we have the following.

REMARK 4.11. It is possible to prove singular versions of Theorems 4.7 and 4.10 in which A^D replaces A^{-1}. In the expressions (4.8) and (4.9) one replaces $\mathscr{K}$ with $\mathscr{K} \cap \mathscr{S}_A$, which is then treated as a proper cone in $\mathscr{S}_A$. (It is then required to work with boundary and interior relative to $\mathscr{S}_A$).

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