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# Filtrations and completions of certain positive level modules of affine algebras

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**Abstract**

We define a filtration indexed by the integers on the tensor product of a simple highest weight module and a loop module for a quantum affine algebra. We prove that such a filtration is either trivial or strictly decreasing and give sufficient conditions for this to happen. In the first case we prove that the tensor product is simple and in the second case we prove that the intersection of all the modules in the filtration is zero, thus allowing us to define the completed tensor product. In certain special cases, we identify the subsequent quotients of filtration.


*Keywords:* Quantum affine algebra; Integrable module

## 0. Introduction

This paper was motivated by an effort to understand the representation theoretic meaning of the results of [14,15,21] on realizations of (pseudo-)crystal bases of certain quantum loop modules in the framework of Littelmann's path model. These papers showed in particular, that one could write the tensor product of a crystal basis of a highest weight integrable module with a (pseudo-)crystal basis of such a quantum loop module as a union of highest weight crystals. The obvious and natural interpretation

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would be that the decomposition of the crystals gave rise to a direct sum decomposition of the tensor product of the corresponding modules for the quantum affine algebra. It is however, not very difficult to see that such a tensor product never contains a copy of a highest weight module. In addition, the corresponding classical situation which was studied in [7] and more recently in [1,18] did not exclude the possibility that such tensor products might in fact be irreducible.

In this paper, we are able to show that the tensor product of an integrable highest weight representation with the quantum loop module associated to the natural representation admits a filtration such that the successive quotients are highest weight integrable modules with multiplicity and highest weight given by the path model.

We now describe the main results of the paper. In Section 2 we recall some well-known properties of highest weight modules and modules of level zero. We also establish several new results on the structure of an irreducible finite-dimensional module $V$, and in particular introduce a function $n : V \to \mathbf{N}$ which plays an important role in Section 7. In Section 3 we establish (Theorem 1) the quantum analogue of one of the main results of [7] (Theorem 4.2). Namely, we prove that the tensor product of a simple highest weight module with a finite-dimensional module is simple. In this situation, we work over the smaller version of the quantum affine algebra which does not contain an analogue of the Euler operator. The result is the same as the classical one proved in [7] but, the absence (in general) of the evaluation map and the non-cocommutativity of the comultiplication in the quantum case makes it harder to establish.

In the rest of the paper we study the more complicated and interesting situation of the tensor product of a highest weight module $V(\Lambda)$ with a quantum loop module $L(V)$. We begin by introducing (Section 4) a filtration $\mathscr{V}_n \supseteq \mathscr{V}_{n+1}$, $n \in \mathbf{Z}$ on $V(\Lambda) \otimes L(V)$. We prove that this filtration is either strictly decreasing, i.e. $\mathscr{V}_n \supsetneq \mathscr{V}_{n+1}$ for all $n \in \mathbf{Z}$, and $\bigcap_{n\in\mathbf{Z}} \mathscr{V}_n = 0$, or trivial, i.e. $\mathscr{V}_n = \mathscr{V}_m$ for all $n, m \in \mathbf{Z}$. Furthermore, for all $n \in \mathbf{Z}$ the quotients $\mathscr{V}_n/\mathscr{V}_{n+1}$ are modules in the category $\mathscr{O}$ for $\widehat{\mathbf{U}}_q$. In the case when $V(\Lambda)$ is the Verma module, the filtration $\mathscr{V}_n$ is always strictly decreasing. If $V(\Lambda)$ is irreducible, then $V(\Lambda) \otimes L(V)$ is irreducible if and only if the filtration is trivial.

In the next two sections we study the filtration $\mathscr{X}_n$, $n \in \mathbf{Z}$, of $X(\Lambda) \otimes L(V)$ where $X(\Lambda)$ is the irreducible integrable module with highest weight $\Lambda$. We give sufficient conditions for the filtration to be trivial or strictly decreasing. In the latter case, the quotients $\mathscr{X}_n/\mathscr{X}_{n+1}$ are integrable modules in the category $\mathscr{O}$ and hence isomorphic to finite direct sums of irreducible highest weight integrable modules $X(\mu)$.

A particularly interesting family of quantum loop modules are the loop spaces of the so-called Kirillov–Reshetikhin representations (cf. for example [4]). These modules are indexed by multiples $n\varpi_k$ of fundamental weights of $\mathfrak{g}$. A consequence of Theorem 2 is that the tensor product of a highest weight module of level one with the affinization of the Kirillov–Reshetikhin module $n\varpi_k$ is irreducible if $n > 1$. If $n = 1$, then Theorem 3 shows that the filtration is strictly decreasing.

In the last section, we let $L(V)$ be the loop module associated to the natural representation of the quantum affine algebra of classical type and study the filtration on $X(\Lambda) \otimes L(V)$. This case is not covered by either of the sufficient conditions given in the previous sections. We are still able to show that $\mathcal{X}_n \supsetneq \mathcal{X}_{n+1}$ if $\dim X(\Lambda) > 1$. We also identify the highest weight and multiplicities of the irreducible modules in $\mathcal{X}_n/\mathcal{X}_{n+1}$.

It follows from our results that one can complete the modules $M(\Lambda) \otimes L(V)$ (and $X(\Lambda) \otimes L(V)$ if $\mathcal{X}_n \supsetneq \mathcal{X}_{n+1}$) with respect to the topology induced by the filtration. Further, $M(\Lambda) \otimes L(V)$ (resp. $X(\Lambda) \otimes L(V)$) embeds canonically into $M(\Lambda)\widehat{\otimes}L(V)$ (resp. $X(\Lambda)\widehat{\otimes}L(V)$). This should be compared with the results of [12,13]. The referee has pointed out to us that there are several natural questions arising from the results of this paper, for instance the results of Section 7 together with Theorem 3.1.1 of [12] indicate that a similar result on the decomposition of $\mathcal{X}_n/\mathcal{X}_{n+1}$ should be true in a more general situation.

## 1. Preliminaries

Throughout this paper $\mathbf{N}$ (respectively, $\mathbf{N}^+$) denotes the set of non-negative (respectively, positive) integers.

**1.1.** Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra of rank $\ell$ with a Cartan subalgebra $\mathfrak{h}$. Set $I = \{1, 2, \ldots, \ell\}$ and let $A = (d_i a_{ij})_{i,j \in I}$, where the $d_i$ are positive co-prime integers, be the $\ell \times \ell$ symmetrized Cartan matrix of $\mathfrak{g}$. Let $\{\alpha_i : i \in I\} \subset \mathfrak{h}^*$ (respectively $\{\varpi_i : i \in I\} \subset \mathfrak{h}^*$) be the set of simple roots (respectively, of fundamental weights) of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\theta$ be the highest root of $\mathfrak{g}$. As usual, $Q$ (respectively, $P$) denotes the root (respectively, weight) lattice of $\mathfrak{g}$. Let $P^+ = \sum_{i \in I} \mathbf{N}\varpi_i$ be the set of dominant weights and set $Q^+ = \sum_{i \in I} \mathbf{N}\alpha_i$. Given $\gamma = \sum_{i \in I} k_i \alpha_i \in Q^+$, set $\operatorname{ht}\gamma = \sum_{i \in I} k_i$. Let $W$ be the Weyl group of $\mathfrak{g}$ and let $s_\alpha \in W$ denote the reflection with respect to the root $\alpha$. It is well-known that $\mathfrak{h}^*$ admits a non-degenerate symmetric $W$-invariant bilinear form which will be denoted by $(\cdot \,|\, \cdot)$. We assume that $(\alpha_i \,|\, \alpha_i) = d_i a_{ij}$ for all $i, j \in I$. Given a root $\beta$ of $\mathfrak{g}$, denote by $\beta^\vee \in \mathfrak{h}$ the corresponding co-root.

**1.2.** Let

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d$$

be the untwisted extended affine algebra associated with $\mathfrak{g}$ and let $\widehat{A} = (d_i a_{ij})_{i,j \in \widehat{I}}$, where $\widehat{I} = I \cup \{0\}$ be the extended symmetrized Cartan matrix. Set $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbf{C}c \oplus \mathbf{C}d$.

From now on we identify $\mathfrak{h}^*$ with the subspace of $\widehat{\mathfrak{h}}^*$ consisting of elements which are zero on $c$ and $d$. Define $\delta \in \widehat{\mathfrak{h}}^*$ by

$$\delta(\mathfrak{h} \oplus \mathbf{C}c) = 0, \quad \delta(d) = 1.$$

Set $\alpha_0 = \delta - \theta$. Then $\{\alpha_i : i \in \widehat{I}\}$ is a set of simple roots for $\widehat{\mathfrak{g}}$ with respect to $\widehat{\mathfrak{h}}$, $\alpha_0^\vee = c - \theta^\vee$ and $\delta$ generates its imaginary roots.

Let $\widehat{W}$ be the Weyl group of $\widehat{\mathfrak{g}}$. The bilinear form on $\mathfrak{h}^*$ extends to a $\widehat{W}$-invariant bilinear form on $\widehat{\mathfrak{h}}^*$ which we continue to denote by $(\cdot \,|\, \cdot)$. One has $(\delta \,|\, \alpha_i) = 0$ and $(\alpha_i \,|\, \alpha_j) = d_i a_{ij}$, for all $i, j \in \widehat{I}$. Define a set of fundamental weights $\{\omega_i : i \in$

$\widehat{I}\} \subset \widehat{\mathfrak{h}}^*$ of $\widehat{\mathfrak{g}}$ by the conditions $(\omega_i|\alpha_j) = d_i\delta_{i,j}$ and $\omega_i(d) = 0$ for all $i, j \in \widehat{I}$. Let $\widehat{P} = \sum_{i\in\widehat{I}} \mathbf{Z}\omega_i \oplus \mathbf{Z}\delta$ (respectively, $\widehat{P}^+ = \sum_{i\in\widehat{I}} \mathbf{N}\omega_i \oplus \mathbf{Z}\delta$) be the corresponding set of integral (respectively, dominant) weights. We have $\varpi_i = \omega_i - a_i^\vee\omega_0$ where $a_i^\vee$ is the coefficient of $\alpha_i^\vee$ in $\theta^\vee$. Identify $P$ with the free abelian subgroup of $\widehat{P}$ generated by the $\varpi_i$, $i \in I$. Denote by $\widehat{Q}$ the root lattice of $\widehat{\mathfrak{g}}$ and set $\widehat{Q}^+ = \sum_{i\in\widehat{I}} \mathbf{N}\alpha_i$. Given $\widehat{\gamma} = \sum_{i\in\widehat{I}} k_i\alpha_i \in \widehat{Q}^+$, set $\operatorname{ht}\widehat{\gamma} = \sum_{i\in\widehat{I}} k_i$. Given $\lambda, \mu \in \widehat{P}^+$ (respectively, $\lambda, \mu \in P^+$) we say that $\lambda \leqslant \mu$ if $\mu - \lambda \in \widehat{Q}^+$ (respectively, $\mu - \lambda \in Q^+$). For all $\lambda \in \widehat{P}$ set $\lambda_i = \lambda(\alpha_i^\vee)$, $i \in \widehat{I}$.

**1.3.** Let $q$ be an indeterminate and let $\mathbf{C}(q)$ be the field of rational functions in $q$ with complex coefficients. For $r, m \in \mathbf{N}$, $m \geqslant r$, define

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]_q! = [m]_q[m-1]_q \ldots [2]_q[1]_q, \quad \begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q!}{[r]_q![m-r]_q!}.$$

For $i \in \widehat{I}$, set $q_i = q^{d_i}$ and $[m]_i = [m]_{q^i}$.

The quantum affine algebra $\widehat{\mathbf{U}}_q(\mathfrak{g})$ (cf. [2,3,11,16]) associated to $\mathfrak{g}$, which will be further denoted as $\widehat{\mathbf{U}}_q$, is an associative algebra over $\mathbf{C}(q)$ with generators $x_{i,r}^{\pm}$, $h_{i,k}$, $K_i^{\pm 1}$, $C^{\pm 1/2}$, $D^{\pm 1}$, where $i \in I$, $k, r \in \mathbf{Z}$, $k \neq 0$, and the following defining relations:

$$C^{\pm 1/2} \text{ are central},$$
$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad C^{1/2}C^{-1/2} = C^{-1/2}C^{1/2} = 1, \qquad DD^{-1} = D^{-1}D = 1,$$
$$K_iK_j = K_jK_i, \quad DK_i = K_iD,$$
$$K_ih_{j,r} = h_{j,r}K_i, \quad Dh_{j,r}D^{-1} = q^r h_{j,r},$$
$$K_ix_{j,r}^{\pm}K_i^{-1} = q_i^{\pm a_{ij}}x_{j,r}^{\pm}, \qquad Dx_{j,r}^{\pm}D^{-1} = q^r x_{j,r}^{\pm},$$
$$[h_{i,r}, h_{j,s}] = \delta_{r,-s}\,\frac{1}{r}[ra_{ij}]_i\,\frac{C^r - C^{-r}}{q_j - q_j^{-1}},$$
$$[h_{i,r}, x_{j,s}^{\pm}] = \pm\frac{1}{r}[ra_{ij}]_i\,C^{\mp|r|/2}x_{j,r+s}^{\pm},$$
$$x_{i,r+1}^{\pm}x_{j,s}^{\pm} - q_i^{\pm a_{ij}}x_{j,s}^{\pm}x_{i,r+1}^{\pm} = q_i^{\pm a_{ij}}x_{i,r}^{\pm}x_{j,s+1}^{\pm} - x_{j,s+1}^{\pm}x_{i,r}^{\pm},$$
$$[x_{i,r}^{+}, x_{j,s}^{-}] = \delta_{i,j}\,\frac{C^{(r-s)/2}\,\psi_{i,r+s}^{+} - C^{-(r-s)/2}\,\psi_{i,r+s}^{-}}{q_i - q_i^{-1}},$$
$$\sum_{\pi\in\Sigma_m}\sum_{k=0}^{m}(-1)^k\begin{bmatrix} m \\ k \end{bmatrix}_i x_{i,r_{\pi(1)}}^{\pm}\cdots x_{i,r_{\pi(k)}}^{\pm}x_{j,s}^{\pm}x_{i,r_{\pi(k+1)}}^{\pm}\cdots x_{i,r_{\pi(m)}}^{\pm} = 0, \quad \text{if } i \neq j$$

for all sequences of integers $r_1, \ldots, r_m$, where $m = 1 - a_{ij}$, $\Sigma_m$ is the symmetric group on $m$ letters, and the $\psi_{i,r}^{\pm}$ are determined by equating powers of $u$ in the formal power series

$$\sum_{r=0}^{\infty}\psi_{i,\pm r}^{\pm}u^{\pm r} = K_i^{\pm 1}\exp\left(\pm(q_i - q_i^{-1})\sum_{s=1}^{\infty}h_{i,\pm s}u^{\pm s}\right).$$

The subalgebra of $\widehat{\mathbf{U}}_q$ generated by the elements $x_{i,0}^{\pm 1}$, $K_i^{\pm 1}$, $i \in I$ is isomorphic to the quantized enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$.

Let $\widehat{\mathbf{U}}_q(\gg)$ (respectively $\widehat{\mathbf{U}}_q(\ll)$) be the subalgebra of $\widehat{\mathbf{U}}_q$ generated by the $x_{i,s}^+$ (respectively, by the $x_{i,s}^-$) for all $i \in I$, $s \in \mathbf{Z}$. Given $r \in \mathbf{Z}$, let $\widehat{\mathbf{U}}_q^r(\gg)$ (respectively $\widehat{\mathbf{U}}_q^r(\ll)$) be the subalgebra of $\widehat{\mathbf{U}}_q(\gg)$ (respectively, of $\widehat{\mathbf{U}}_q(\ll)$) generated by the $x_{i,s}^+$ (respectively, by the $x_{i,s}^-$) for all $i \in I$ and for all $s \geqslant r$. Furthermore, let $\widehat{\mathbf{U}}_q(0)$ (respectively, $\widehat{\mathbf{U}}_q^r(0)$) be the subalgebra of $\widehat{\mathbf{U}}_q$ generated by the $h_{i,s}$, for all $i \in I$ and for all $s \in \mathbf{Z}$ (respectively, for all $s \geqslant r$), $s \neq 0$. Finally, let $\widehat{\mathbf{U}}_q^\circ$ be the subalgebra generated by the $K_i^{\pm 1}$, $i \in I$, $D^{\pm 1}$ and $C^{\pm 1/2}$.

**1.4.** Define a $\mathbf{Z}$-grading on $\widehat{\mathbf{U}}_q$ by setting $\deg x_{i,r}^{\pm} = r$, $\deg h_{i,k} = k$ for all $i \in I$ and for all $r \in \mathbf{Z}$, $k \in \mathbf{Z} \setminus \{0\}$ and $\deg K_i = \deg D = \deg C^{\pm 1/2} = 0$ for all $i \in I$. Equivalently, we say that $x \in \widehat{\mathbf{U}}_q$ is homogeneous of degree $k = \deg x$ if $DxD^{-1} = q^k x$. Given $z \in \mathbf{C}(q)^\times$, let $\phi_z$ be the automorphism of $\widehat{\mathbf{U}}_q$ defined by extending $\phi_z(x) = z^{\deg x} x$ for $x \in \widehat{\mathbf{U}}_q$ homogeneous.

On the other hand, the algebra $\widehat{\mathbf{U}}_q$ is graded by the root lattice $\widehat{Q}$, the elements $x_{i,r}^{\pm}$, $i \in I$, $r \in \mathbf{Z}$ being of weight $r\delta \pm \alpha_i$, the $h_{i,k}$, $i \in I$, $k \in \mathbf{Z} \setminus \{0\}$ being of weight $k\delta$ and the other generators being of weight zero. Given $\nu \in \widehat{Q}$, we denote the corresponding weight subspace of $\widehat{\mathbf{U}}_q$ by $(\widehat{\mathbf{U}}_q)_\nu$. Observe also that if $x \in (\widehat{\mathbf{U}}_q)_{r\delta+\gamma}$, $\gamma \in Q$, $r \in \mathbf{Z}$ then $\deg x = r$.

**1.5.** We will also need another presentation of $\widehat{\mathbf{U}}_q$. Namely, after [2,16], the algebra $\widehat{\mathbf{U}}_q$ is isomorphic to an associative $\mathbf{C}(q)$-algebra generated by $E_i, F_i, K_i^{\pm 1} : i \in \widehat{I}$, $D^{\pm 1}$ and central elements $C^{\pm 1/2}$ satisfying the following relations:

$$C = K_0 \prod_{i \in I} K_i^{a_i}, \text{ where } \theta = \sum_{i \in I} a_i \alpha_i,\, a_i \in \mathbb{N}^+,$$

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \qquad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j,$$

$$D E_j D^{-1} = q^{\delta_{j0}} E_j, \qquad D F_j D^{-1} = q^{-\delta_{j0}} F_j,$$

$$[E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_i (E_i)^r E_j (E_i)^{1-a_{ij}-r} = 0 \qquad \text{if } i \neq j,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_i (F_i)^r F_j (F_i)^{1-a_{ij}-r} = 0 \qquad \text{if } i \neq j.$$

The element $E_i$ (respectively, $F_i$), $i \in I$ corresponds to $x_{i,0}^+$ (respectively $x_{i,0}^-$). In particular, the elements $E_i, F_i, K_i^{\pm 1} : i \in I$ generate a subalgebra of $\widehat{\mathbf{U}}_q$ isomorphic to $U_q(\mathfrak{g})$.

Let $\widehat{\mathbf{U}}_q^+$ (respectively, $\widehat{\mathbf{U}}_q^-$) be the $\mathbf{C}(q)$-subalgebra of $\widehat{\mathbf{U}}_q$ generated by the $E_i$ (respectively, by the $F_i$), $i \in \widehat{I}$. Let $\widehat{\mathbf{U}}'_q$ be the subalgebra of $\widehat{\mathbf{U}}_q$ generated by the $E_i, F_i, K_i^{\pm 1}$, $i \in \widehat{I}$ and by $C^{\pm 1/2}$.

We will need the following result which was established in [3].

**Proposition.** *We have* $\widehat{\mathbf{U}}_q^+ \subset \widehat{\mathbf{U}}_q^0(\ll)\widehat{\mathbf{U}}_q^0(0)\widehat{\mathbf{U}}_q^0(\gg)$ *and* $\widehat{\mathbf{U}}_q^r(\ll), \widehat{\mathbf{U}}_q^s(\gg) \subset \widehat{\mathbf{U}}_q^+$ *for all* $r \in \mathbf{N}^+$, $s \in \mathbf{N}$.

**1.6.** It is well-known that $\widehat{\mathbf{U}}_q$ is a Hopf algebra over $\mathbf{C}(q)$ with the co-multiplication being given in terms of generators $E_i, F_i, K_i^{\pm 1} : i \in \widehat{I}$ by the following formulae:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \qquad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

the $K_i^{\pm 1}$, $D^{\pm 1}$, $C^{\pm 1/2}$ being group-like. Notice that $\widehat{\mathbf{U}}'_q$ is a Hopf subalgebra of $\widehat{\mathbf{U}}_q$. Let $\widehat{\mathscr{U}}_q^+$ (respectively, $\widehat{\mathscr{U}}_q^-$) be the subalgebra of $\widehat{\mathbf{U}}_q$ and $\widehat{\mathbf{U}}'_q$ generated by the $E_i$ (respectively, by the $F_i$) and by the $K_i^{\pm 1}$, $i \in \widehat{I}$. Obviously, the $\widehat{\mathscr{U}}_q^\pm$ are Hopf algebras and $\widehat{\mathbf{U}}_q^\pm \subset \widehat{\mathscr{U}}_q^\pm$.

Although explicit formulae for the co-multiplication on generators $x_{i,r}^\pm$, $h_{i,r}$ are not known, we have the following partial results [10] which are enough for this paper.

**Lemma.** *For* $i \in I$, $r \in \mathbf{N}$, $s \in \mathbf{N}^+$, *we have*

$$\Delta(h_{i,s}) = h_{i,s} \otimes 1 + 1 \otimes h_{i,s} + \textit{ terms in } \widehat{\mathbf{U}}_q^\circ((\widehat{\mathbf{U}}_q^+)_+ \otimes (\widehat{\mathbf{U}}_q^+)_+), \tag{1.1}$$

$$\Delta(x_{i,r}^+) = x_{i,r}^+ \otimes 1 + K_i \otimes x_{i,r}^+ + \textit{ terms in } \widehat{\mathbf{U}}_q^\circ((\widehat{\mathbf{U}}_q^+)_+ \otimes (\widehat{\mathbf{U}}_q^0(\gg))_+), \tag{1.2}$$

$$\Delta(x_{i,s}^-) = x_{i,s}^- \otimes K_i + 1 \otimes x_{i,s}^- + \textit{ terms in } \widehat{\mathbf{U}}_q^\circ((\widehat{\mathbf{U}}_q^+)_+ \otimes (\widehat{\mathbf{U}}_q^1(\ll))_+), \tag{1.3}$$

*where* $(\widehat{\mathbf{U}}_q^+)_+$ *denotes the augmentation ideal of* $\widehat{\mathbf{U}}_q^+$.

For $i \in I$, set

$$P_i^\pm(u) = \exp\left(-\sum_{k=1}^\infty \frac{q^{\pm k} h_{i,\pm k}}{[k]_i} u^k\right).$$

Let $P_{i,\pm r}$ be the coefficient of $u^r$ in $P_i^\pm(u)$. It is easy to see that the elements $h_{i,r}$ belong to the subalgebra of $\widehat{\mathbf{U}}_q$ generated by the elements $P_{i,r}$, $i \in I$, $r \in \mathbf{Z}$. Further, one can deduce from Lemma 1.6 as in [3] that, for all $s \in \mathbf{N}$,

$$\Delta(P_{i,s}) = \sum_{r=0}^{s} P_{i,s-r} \otimes P_{i,r} + \text{ terms in } \widehat{\mathbf{U}}_q^\circ((\widehat{\mathbf{U}}_q^+)_+) \otimes (\widehat{\mathbf{U}}_q^+)_+ \tag{1.4}$$

## 2. The modules $M(\Lambda)$, $X(\Lambda)$, $V(\pi)$ and $L(V(\pi))$

In this section we recall the definition and some properties of several families of integrable modules for $\widehat{\mathbf{U}}_q$ and $\widehat{\mathbf{U}}'_q$. For modules of level zero we also establish some results which we need in later sections.

**2.1.** A $\widehat{\mathbf{U}}_q$-module $M$ is said to be of type 1 if $M = \bigoplus_{\mu \in \widehat{P}} M_\mu$, where

$$M_\mu = \{m \in M \,:\, K_i m = q_i^{\mu(\alpha_i^\vee)} m,\ \forall\ i \in \widehat{I},\ Dm = q^{\mu(d)} m\}.$$

Type 1-modules for $\widehat{\mathbf{U}}'_q$ are defined in the obvious way. If $m \in M_\mu \setminus \{0\}$, we say that $m$ is of weight $\mu$ and write $\mathrm{wt}\, m = \mu$. Set $\Omega(M) = \{\nu \in \widehat{P} : M_\nu \neq 0\}$.

A $\widehat{\mathbf{U}}_q$- or a $\widehat{\mathbf{U}}'_q$-module $M$ of type 1 is said to be integrable if the elements $E_i$, $F_i$, $i \in \widehat{I}$ act locally nilpotently on $M$. Evidently, a $\widehat{\mathbf{U}}_q$-module $M$ can be viewed as a $\widehat{\mathbf{U}}'_q$-module $M'$ and $M'_\nu = \bigoplus_{r \in \mathbf{Z}} M_{\nu + r\delta}$.

**2.2.** Let $\mathcal{O}$ be the category of $\widehat{\mathbf{U}}_q$-modules satisfying the following properties. A $\widehat{\mathbf{U}}_q$-module $M$ is an object in $\mathcal{O}$ if and only if

(i) $M$ is a module of type 1 and $\dim M_\mu < \infty$ for all $\mu \in \widehat{P}$.
(ii) The set $\Omega(M)$ is contained in the set $\bigcup_{k=1}^r \{\lambda_k - \widehat{\gamma} : \widehat{\gamma} \in \widehat{Q}^+\}$ for some $r \in \mathbf{N}^+$ and for some $\lambda_k \in \widehat{P}$.

Given $\Lambda \in \widehat{P}$, let $M(\Lambda)$ denote the Verma module of highest weight $\Lambda$. It is generated as a $\widehat{\mathbf{U}}_q$-module by an element $m_\Lambda$ of weight $\Lambda$ with defining relation

$$(\widehat{\mathbf{U}}_q^+)_+ m_\Lambda = 0.$$

It is well-known that $M(\Lambda)$ has a unique simple quotient which we denote by $X(\Lambda)$. Let $v_\Lambda$ be the canonical image of $m_\Lambda$ in $X(\Lambda)$.

The next result is well-known and follows immediately from [17,19].

**Proposition.**

(i) *For all $\Lambda \in \widehat{P}$, $M(\Lambda) \in \mathcal{O}$ and is a free $\widehat{\mathbf{U}}_q^-$-module. In particular* $\mathrm{Ann}_{\widehat{\mathbf{U}}_q^-} m_\Lambda = 0$ *and $\Omega(M(\Lambda)) \subset \Lambda - \widehat{Q}^+$.*
(ii) *For all $\Lambda \in \widehat{P}^+$, $X(\Lambda)$ is an integrable $\widehat{\mathbf{U}}_q$-module in the category $\mathcal{O}$ and is generated as a $\widehat{\mathbf{U}}_q$-module by the element $v_\Lambda$. Moreover,* $\mathrm{Ann}_{\widehat{\mathbf{U}}_q^-} v_\Lambda = \sum_{i \in \widehat{I}} \widehat{\mathbf{U}}_q^- F_i^{\Lambda(\alpha_i^\vee)+1}$.
(iii) *Let $M \in \mathcal{O}$ be integrable. Then $M$ is isomorphic to a finite direct sum of modules of the form $X(\Lambda)$, $\Lambda \in \widehat{P}^+$. In particular, if $M \in \mathcal{O}$ is simple and integrable, then it is isomorphic to $X(\Lambda)$ for some $\Lambda \in \widehat{P}^+$.*

Regarded as modules for $\widehat{\mathbf{U}}'_q$, the $X(\Lambda)$ remain simple, although they no longer have finite-dimensional weight spaces. Indeed, by [17] if $\lambda \in \Omega(X(\Lambda))$ then

$\lambda - n\delta \in \Omega(X(\Lambda))$ for all $n \in \mathbf{N}$ and we can write

$$X(\Lambda) = \bigoplus_{\gamma \in Q^+} \bigoplus_{n \in \mathbf{N}} X(\Lambda)_{\Lambda - \gamma - n\delta}. \tag{2.1}$$

Obviously, for $\gamma \in Q^+$ fixed, $\bigoplus_{n \in \mathbf{N}} X(\Lambda)_{\Lambda-\gamma-n\delta}$ is a weight space of $X(\Lambda)$ viewed as a $\widehat{\mathbf{U}}'_q$-module. Observe also that

$$M(\Lambda) \cong M(\Lambda + r\delta), \qquad X(\Lambda) \cong X(\Lambda + r\delta)$$

as $\widehat{\mathbf{U}}'_q$-modules for all $r \in \mathbf{Z}$.

**2.3.** The next important family of modules we consider is that of the irreducible finite-dimensional representations $V(\boldsymbol{\pi})$ of $\widehat{\mathbf{U}}'_q$. Let $\boldsymbol{\pi} = (\pi_i(u))_{i \in I}$ be an $\ell$-tuple of polynomials with coefficients in $\mathbf{C}(q)$ and with constant term 1. Set $\lambda_{\boldsymbol{\pi}} := \sum_{i \in I} (\deg \pi_i) \varpi_i \in P$. Let $W(\boldsymbol{\pi})$ be the $\widehat{\mathbf{U}}'_q$-module generated by an element $v_{\boldsymbol{\pi}}$ satisfying

$$x^+_{i,r} v_{\boldsymbol{\pi}} = 0, \quad (x^-_{i,r})^{\lambda_{\boldsymbol{\pi}}(\alpha_i^\vee)+1} v_{\boldsymbol{\pi}} = 0,$$
$$K_i v_{\boldsymbol{\pi}} = q_i^{\lambda_{\boldsymbol{\pi}}(\alpha_i^\vee)} v_{\boldsymbol{\pi}}, \quad C v_{\boldsymbol{\pi}} = v_{\boldsymbol{\pi}},$$
$$P_{i,\pm s} v_{\boldsymbol{\pi}} = \pi^{\pm}_{i,s} v_{\boldsymbol{\pi}}$$

for all $i \in I$, $r \in \mathbf{Z}$,$s \in \mathbf{N}$ where $\pi_i^{\pm}(u) = \sum_s \pi^{\pm}_{i,s} u^s$ and

$$\pi_i^+(u) = \pi_i(u), \qquad \pi_i^-(u) = u^{\deg \pi_i} \frac{\pi_i(u^{-1})}{\pi^+_{i,\deg \pi_i}}.$$

The following proposition was proved in [9].

**Proposition.**

(i) *The* $\widehat{\mathbf{U}}'_q$*-modules* $W(\boldsymbol{\pi})$ *are finite-dimensional.*
(ii) $W(\boldsymbol{\pi}) = \widehat{\mathbf{U}}_q(\ll) v_{\boldsymbol{\pi}}$. *In particular,* $\Omega(W(\boldsymbol{\pi})) \subset \lambda_{\boldsymbol{\pi}} - Q^+$.
(iii) $\dim W(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}} = \dim W(\boldsymbol{\pi})_{w_\circ \lambda_{\boldsymbol{\pi}}} = 1$, *where* $w_\circ$ *is the longest element of the Weyl group of* $\mathfrak{g}$. *Let* $v^*_{\boldsymbol{\pi}}$ *be a non-zero element in* $W(\boldsymbol{\pi})_{w_\circ \lambda_{\boldsymbol{\pi}}}$. *Then*

$$x^-_{i,r} v^*_{\boldsymbol{\pi}} = 0, \quad (x^+_{i,r})^{-(w_\circ \lambda_{\boldsymbol{\pi}})(\alpha_i^\vee)+1} v^*_{\boldsymbol{\pi}} = 0, \quad W(\boldsymbol{\pi}) = \widehat{\mathbf{U}}_q(\gg) v^*_{\boldsymbol{\pi}},$$

*and* $\Omega(W(\boldsymbol{\pi})) \subset w_\circ \lambda_{\boldsymbol{\pi}} + Q^+$.

(iv) *$W(\boldsymbol{\pi})$ has a unique simple quotient $V(\boldsymbol{\pi})$ and all simple finite dimensional $\widehat{\mathbf{U}}'_q$-modules are obtained that way.*
(v) *Denote the images of the elements $v_{\boldsymbol{\pi}}$, $v^*_{\boldsymbol{\pi}}$ in $V(\boldsymbol{\pi})$ by the same symbols. Then*

$$V(\boldsymbol{\pi}) = \widehat{\mathbf{U}}_q(\ll)v_{\boldsymbol{\pi}}, \quad x^+_{i,r}v_{\boldsymbol{\pi}} = 0, \quad (x^-_{i,r})^{\lambda_\pi(\alpha_i^\vee)+1}v_{\boldsymbol{\pi}} = 0,$$
$$\Omega(V(\boldsymbol{\pi})) \subset \lambda_{\boldsymbol{\pi}} - Q^+$$

*and analogous statements hold for $v^*_{\boldsymbol{\pi}}$.*

Given $z \in \mathbf{C}^\times$ and an $\ell$-tuple of polynomials $\boldsymbol{\pi} = (\pi_i(u))_{i\in I}$, one can introduce on $V(\boldsymbol{\pi})$ another $\widehat{\mathbf{U}}'_q$-module structure by twisting the action by automorphism $\phi_z$. Then $\phi_z^* V(\boldsymbol{\pi}) \cong V(\boldsymbol{\pi}_z)$ where $\boldsymbol{\pi}_z = (\pi_i(zu))_{i\in I}$.

**2.4.** We now establish some facts about $V(\boldsymbol{\pi})$ which will be needed later.

**Lemma.** *Let $\boldsymbol{\pi}$ be an $\ell$-tuple of polynomials with coefficients in $\mathbf{C}(q)$ and constant term 1. Let $k_i$ (respectively, $k^*_j$), $i, j \in I$, be the dimension of $V(\boldsymbol{\pi})_{\lambda_\pi - \alpha_i}$ (respectively, of $V(\boldsymbol{\pi})_{w_\circ\lambda_\pi + \alpha_j}$). Suppose that $k_i, k^*_j > 0$. Then*

(i) *$\{x^-_{i,s}v_{\boldsymbol{\pi}}, \dots, x^-_{i,s+k_i-1}v_{\boldsymbol{\pi}}\}$ is a basis of $V(\boldsymbol{\pi})_{\lambda_\pi-\alpha_i}$ for all $s \in \mathbf{Z}$.*
(ii) *$\{x^+_{j,s}v^*_{\boldsymbol{\pi}}, \dots, x^+_{j,s+k^*_j-1}v^*_{\boldsymbol{\pi}}\}$ is a basis of $V(\boldsymbol{\pi})_{w_\circ\lambda_\pi+\alpha_j}$ for all $s \in \mathbf{Z}$.*

**Proof.** We prove only (i), the proof of (ii) being similar. Since $V(\boldsymbol{\pi}) = \widehat{\mathbf{U}}_q(\ll)v_{\boldsymbol{\pi}}$, the elements $x^-_{i,k}v_{\boldsymbol{\pi}}$, $k \in \mathbf{Z}$ span $V(\boldsymbol{\pi})_{\lambda_\pi-\alpha_i}$. Next, observe that $x^-_{i,k}v_{\boldsymbol{\pi}} \neq 0$ for all $k \in \mathbf{Z}$. Indeed, if $x^-_{i,n}v_{\boldsymbol{\pi}} = 0$ for some $n \in \mathbf{Z}$ then, since $v_{\boldsymbol{\pi}}$ is an eigenvector for the $h_{i,s}$, $s \in \mathbf{Z}$ we get, using the defining relations of $\widehat{\mathbf{U}}_q$

$$0 = h_{i,s}x^-_{i,n}v_{\boldsymbol{\pi}} = -\frac{[2s]_i}{s}\, x^-_{i,n+s}v_{\boldsymbol{\pi}}.$$

It follows that $x^-_{i,k}v_{\boldsymbol{\pi}} = 0$ for all $k \in \mathbf{Z}$. Therefore, $V(\boldsymbol{\pi})_{\lambda_\pi-\alpha_i} = 0$, which is a contradiction.

It remains to prove that the set $\{x^-_{i,s}v_{\boldsymbol{\pi}}, \dots, x^-_{i,s+k_i-1}v_{\boldsymbol{\pi}}\}$ is linearly independent for all $s \in \mathbf{Z}$. If $k_i = 1$ then, since $x^-_{i,s}v_{\boldsymbol{\pi}} \neq 0$ for all $s \in \mathbf{Z}$, there is nothing to prove. Assume that $k_i > 1$ and that $\sum_{r=0}^{k_i-1} a_r x^-_{i,r+s}v_{\boldsymbol{\pi}} = 0$ for some $a_r \in \mathbf{C}(q)$, $r = 1, \dots, k$ and for some $s \in \mathbf{Z}$. Applying $h_{i,m}$ as above we conclude that

$$\sum_{r=0}^{k_i-1} a_r x^-_{i,r+s}v_{\boldsymbol{\pi}} = 0, \qquad \forall s \in \mathbf{Z}. \tag{2.2}$$

Let $r_1$ (respectively, $r_2$) be the minimal (respectively, the maximal) $r$, $0\leqslant r\leqslant k_i - 1$ such that $a_r \neq 0$. Then, using (2.2) with $s = 0$ and $s = 1$, we obtain

$$x^-_{i,r_1} v_{\boldsymbol{\pi}} = -a_{r_1}^{-1} \sum_{r=r_1+1}^{r_2} a_r x^-_{i,r} v_{\boldsymbol{\pi}}, \qquad x^-_{i,r_2+1} v_{\boldsymbol{\pi}} = -a_{r_2}^{-1} \sum_{r=r_1+1}^{r_2} a_{r-1} x^-_{i,r} v_{\boldsymbol{\pi}}.$$

Observe that both sums contain at least one non-zero term. Then it follows by induction on $r_1 - k$ (respectively, on $k - r_2$) from the above formulae and (2.2) that the $x^-_{i,k} v_{\boldsymbol{\pi}}$ lie in the linear span of vectors $x^-_{i,r_1+1} v_{\boldsymbol{\pi}}, \ldots, x^-_{i,r_2} v_{\boldsymbol{\pi}}$ for all $k < r_1$ (respectively, for all $k > r_2$). Therefore, $\dim V(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}-\alpha_i} < k_i$ which is a contradiction. □

**2.5.**

**Lemma.** *Define*

$$k(\boldsymbol{\pi}) = \min_{i\in I}\{\dim V(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}-\alpha_i} \;:\; V(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}-\alpha_i} \neq 0\}.$$

*Then*

$$k(\boldsymbol{\pi}) = \min_{i\in I}\{\dim V(\boldsymbol{\pi})_{w_\circ\lambda_{\boldsymbol{\pi}}+\alpha_i} \;:\; V(\boldsymbol{\pi})_{w_\circ\lambda_{\boldsymbol{\pi}}+\alpha_i} \neq 0\}.$$

**Proof.** Let $k^* = \min_{i\in I}\{\dim V(\boldsymbol{\pi})_{w_\circ\lambda_{\boldsymbol{\pi}}+\alpha_i} \;:\; V(\boldsymbol{\pi})_{w_\circ\lambda_{\boldsymbol{\pi}}+\alpha_i} \neq 0\}$. Choose $i \in I$ such that $k(\boldsymbol{\pi}) = \dim V(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}-\alpha_i}$ for some $i \in I$. Since $V(\boldsymbol{\pi})$ is an integrable $\widehat{\mathbf{U}}'_q$-module, its character is $W$-invariant (cf. say [20]). Since $w_\circ\alpha_i = -\alpha_j$ for some $j \in I$ we conclude that $\dim V(\boldsymbol{\pi})_{w_\circ\lambda_{\boldsymbol{\pi}}+\alpha_j} = k(\boldsymbol{\pi})$ and so $k^*\leqslant k(\boldsymbol{\pi})$. A similar argument shows that $k(\boldsymbol{\pi})\leqslant k^*$. □

**2.6.**

**Lemma.** *For any* $v \in V(\boldsymbol{\pi})$, $E_0^{\lambda_{\boldsymbol{\pi}}(\theta^\vee)+1} v = 0 = F_0^{\lambda_{\boldsymbol{\pi}}(\theta^\vee)+1} v$.

**Proof.** Since $V(\boldsymbol{\pi})$ is finite dimensional, it decomposes, uniquely, as a direct sum of simple finite dimensional highest weight modules $V(\lambda)$ over $U_q(\mathfrak{g})$ with $\lambda \in \lambda_{\boldsymbol{\pi}} - Q^+$. Therefore, in order to prove the assertion it is sufficient to show that, if $\lambda - \gamma$, $\gamma \in Q^+$ is a weight of $V(\lambda)$, then $\mu = \lambda - \gamma - (\lambda(\theta^\vee) + 1)\theta$ is not a weight of $V(\lambda)$. Indeed, otherwise, since the formal character of $V(\lambda)$ is $W$-invariant, $s_\theta\mu = \lambda - \gamma + (\gamma(\theta^\vee) + 1)\theta$ is also a weight of $V(\lambda)$. It follows that $\gamma' = \gamma - (\gamma(\theta^\vee) + 1)\theta \in Q^+$.

Let $J = \{i \in I \;:\; \alpha_i(\theta^\vee) > 0\}$ and observe that $J$ is not empty. Write $\gamma = \sum_{i\in I} n_i\alpha_i$. Suppose first that $\gamma(\theta^\vee) = 0$. Then $n_i = 0$ for all $i \in J$. Yet $\theta = \sum_{i\in I} a_i\alpha_i$ and $a_i > 0$ for all $i \in I$. It follows that the $\alpha_i$, $i \in J$ occur in $\gamma' = \gamma - \theta$ with strictly negative coefficients. Therefore, $\gamma' \notin Q^+$ which is a contradiction.

Finally, suppose that $\gamma(\theta^\vee) > 0$. Then there exists $i \in J$ such that $n_i \neq 0$. It follows that $\alpha_i$ occurs in $(\gamma(\theta^\vee)+1)\theta$ with the coefficient at least $a_i(n_i+1) > n_i$. Thus, $\alpha_i$ occurs in $\gamma'$ with a negative coefficient and so $\gamma' \notin Q^+$.

A similar argument shows that $F_0^{-w_\circ\lambda_\pi(\theta^\vee)+1} v = 0$ for all $v \in V(\boldsymbol{\pi})$. It remains to observe that $-w_\circ\lambda_{\boldsymbol{\pi}}(\theta^\vee) = \lambda_{\boldsymbol{\pi}}(\theta^\vee)$. □

**2.7.**

**Proposition.** *For all* $r \in \mathbf{N}$, $V(\boldsymbol{\pi}) = \widehat{\mathbf{U}}_q^r(\ll) v_{\boldsymbol{\pi}} = \widehat{\mathbf{U}}_q^r(\gg) v_{\boldsymbol{\pi}}^*$.

**Proof.** It is sufficient to prove the statement for $v_{\boldsymbol{\pi}}$, the proof of the other one being similar. Recall that all weights of $V(\boldsymbol{\pi})$ are of the form $\lambda_{\boldsymbol{\pi}} - \gamma$, $\gamma \in Q^+$. We prove by induction on $\operatorname{ht}\gamma$ that

$$V(\boldsymbol{\pi})_{\lambda_\pi - \gamma} \subset \widehat{\mathbf{U}}_q^r(\ll) v_{\boldsymbol{\pi}}, \qquad \forall\, r \in \mathbf{N}.$$

If $\operatorname{ht}\gamma = 0$ then there is nothing to prove. Assume that $\operatorname{ht}\gamma = 1$, that is $\gamma = \alpha_i$ for some $i \in I$. Then $k = \dim V(\boldsymbol{\pi})_{\lambda_\pi - \alpha_i} > 0$ and by Lemma 2.4, $V(\boldsymbol{\pi})_{\lambda_\pi - \alpha_i}$ is spanned by $x_{i,r}^- v_{\boldsymbol{\pi}}, \ldots, x_{i,r+k-1}^- v_{\boldsymbol{\pi}}$ for all $r \geqslant 0$. In particular,

$$V(\boldsymbol{\pi})_{\lambda_\pi - \alpha_i} \subset \widehat{\mathbf{U}}_q^r(\ll) v_{\boldsymbol{\pi}}$$

for all $r \geqslant 0$.

Suppose that $v \in V(\boldsymbol{\pi})_{\lambda_\pi - \gamma}$ with $\operatorname{ht}\gamma \geqslant 1$ and that $v \in \widehat{\mathbf{U}}_q^r(\ll) v_{\boldsymbol{\pi}}$ for all $r \in \mathbf{N}$. Fix some $r \in \mathbf{N}$. For the inductive step, it suffices to prove that $x_{i,k}^- v \in \widehat{\mathbf{U}}_q^r(\ll) v_{\boldsymbol{\pi}}$ for all $k \in \mathbf{Z}$ and for all $i \in I$. We may assume, without loss of generality, that $v = x_{j,n}^- w$ for some $w \in \widehat{\mathbf{U}}_q^{r+1}(\ll) v_{\boldsymbol{\pi}}$, $j \in I$ and $n > r$. It follows from the defining relations of the algebra $\widehat{\mathbf{U}}_q$ that

$$x_{i,k}^- x_{j,n}^- w = q_i^{a_{ij}} x_{j,n}^- x_{i,k}^- w + q_i^{a_{ij}} x_{i,k+1}^- x_{j,n-1}^- w - x_{j,n-1}^- x_{i,k+1}^- w.$$

If $k \geqslant r-1$ then all terms in the right-hand side lie in $\widehat{\mathbf{U}}_q^r(\ll) v_{\boldsymbol{\pi}}$ by the assumption on $w$ and by the induction hypothesis. Then it follows from the above formula by induction on $r-k$ that $x_{i,k}^- x_{j,n}^- w \in \widehat{\mathbf{U}}_q^r(\ll) v_{\boldsymbol{\pi}}$ for all $k < r$ and the proposition is proved. □

**Corollary.** *We have* $V(\boldsymbol{\pi}) = \widehat{\mathbf{U}}_q^+ v_{\boldsymbol{\pi}} = \widehat{\mathbf{U}}_q^+ v_{\boldsymbol{\pi}}^*$.

**Proof.** This follows immediately from the above and Proposition 1.5. □

**2.8.** As a consequence of Proposition 2.7, we can define a map $n : V(\boldsymbol{\pi}) \to \mathbf{N}$ in the following way. Given $v \in V(\boldsymbol{\pi})$, $v \neq 0$ let $n(v)$ be the minimal $r \in \mathbf{N}$ such that $v$ can be written as a linear combination of homogeneous elements of $\widehat{\mathbf{U}}'_q$ of degree $\leqslant r$ applied to $v_{\boldsymbol{\pi}}$. Such a number is well-defined since $V(\boldsymbol{\pi}) = \widehat{\mathbf{U}}^0_q(\ll)v_{\boldsymbol{\pi}}$ by Proposition 2.7. Set $n(0) = -\infty$ with the convention that $-\infty < n$ for all $n \in \mathbf{Z}$. Finally, set $n(\boldsymbol{\pi}) = \max\{n(v) : v \in V(\boldsymbol{\pi})\}$.

**Lemma.** *We have, for all* $v \in V(\boldsymbol{\pi})$,

$$\begin{aligned} &n(E_i v) \leqslant n(v), \quad i \in I\\ &n(E_0 v) \leqslant n(v) + 1,\\ &n(F_i v) \leqslant n(v) + \dim V(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}-\alpha_i}, \quad i \in I\\ &n(F_0 v) \leqslant n(v) - 1. \end{aligned}$$

**Proof.** The first two statements are obvious. For the next two, observe that since $V(\boldsymbol{\pi})$ is spanned by vectors of the form $Xv_{\boldsymbol{\pi}}$ where $X$ is a monomial in the $E_i$, $i \in \widehat{I}$, it suffices by the relations in $\widehat{\mathbf{U}}_q$ to prove the assertion for $v = v_{\boldsymbol{\pi}}$. If $i = 0$, then $F_0 v_{\boldsymbol{\pi}} = 0$ and there is nothing to prove. So assume that $i \neq 0$, and that $F_i v_{\boldsymbol{\pi}} \neq 0$ (if $F_i v_{\boldsymbol{\pi}} = 0$, there is again nothing to prove). Then, by Lemma 2.4, $F_i v_{\boldsymbol{\pi}}$ is contained in the linear span of the $x^-_{i,s} v_{\boldsymbol{\pi}}$, $s = 1, \ldots, \dim V(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}-\alpha_i}$. $\square$

**2.9.** Let $m = m(\boldsymbol{\pi}) \in \mathbf{N}^+$ be maximal such that $\boldsymbol{\pi} \in (\mathbf{C}(q)[u^m])^\ell$. Then $\boldsymbol{\pi}$ can be written uniquely as $\boldsymbol{\pi}^0 \boldsymbol{\pi}^0_\zeta \cdots \boldsymbol{\pi}^0_{\zeta^{m-1}}$ where $\boldsymbol{\pi}^0$ is an $\ell$-tuple of polynomials with constant term 1, $\zeta$ is an $m$th primitive root of unity and the product is taken component-wise. By [5], $V(\boldsymbol{\pi}) \cong V(\boldsymbol{\pi}^0) \otimes \cdots \otimes V(\boldsymbol{\pi}^0_{\zeta^{m-1}})$. It follows that $\dim V(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}-\alpha_i} = m \dim V(\boldsymbol{\pi}^0)_{\lambda_{\boldsymbol{\pi}^0}-\alpha_i}$ for all $i \in I$.

Denote by $\tau_{\boldsymbol{\pi}}$ the unique isomorphism of $\widehat{\mathbf{U}}'_q$-modules

$$V(\boldsymbol{\pi}^0) \otimes \cdots \otimes V(\boldsymbol{\pi}^0_{\zeta^{m-1}}) \longrightarrow V(\boldsymbol{\pi}^0_{\zeta^{m-1}}) \otimes V(\boldsymbol{\pi}^0) \otimes \cdots \otimes V(\boldsymbol{\pi}^0_{\zeta^{m-2}})$$

which sends $v_{\boldsymbol{\pi}} = v_{\boldsymbol{\pi}^0} \otimes \cdots v_{\boldsymbol{\pi}^0_{\zeta^{m-1}}}$ to the corresponding permuted tensor product of highest weight vectors. Set $\eta_{\boldsymbol{\pi}} = (\phi^*_\zeta)^{\otimes m} \circ \tau_{\boldsymbol{\pi}}$, where $\phi^*_\zeta$ is the pull-back by the automorphism $\phi_\zeta$ of $\widehat{\mathbf{U}}_q$. Then $\eta_{\boldsymbol{\pi}}(xv) = \zeta^{-\deg x} x \eta_{\boldsymbol{\pi}}(v)$ and $\eta_{\boldsymbol{\pi}}(v_{\boldsymbol{\pi}}) = v_{\boldsymbol{\pi}}$, whence $\eta^m_{\boldsymbol{\pi}} = \mathrm{id}$ and

$$V(\boldsymbol{\pi}) = \bigoplus_{k=0}^{m-1} V(\boldsymbol{\pi})^{(k)}, \qquad \text{where } V(\boldsymbol{\pi})^{(k)} = \{v \in V(\boldsymbol{\pi}) : \eta_{\boldsymbol{\pi}}(v) = \zeta^k v\}.$$

Notice also that, since $\deg K_i = 0$, $\eta_{\boldsymbol{\pi}}$ preserves weight spaces of $V(\boldsymbol{\pi})$.

**Lemma.** *Let* $v \in V(\boldsymbol{\pi})^{(k)}$ *and suppose that* $v = \sum_{s=1}^N X_s v_{\boldsymbol{\pi}}$ *with* $X_s \in \widehat{\mathbf{U}}'_q$ *homogeneous. Then* $X_s v_{\boldsymbol{\pi}} \neq 0$ *only if* $\deg X_s + k = 0 \,(\mathrm{mod}\, m)$. *In particular,* $n(v) + k = 0 \,(\mathrm{mod}\, m)$.

**Proof.** This is immediate since $V(\boldsymbol{\pi}) = \bigoplus_{r=0}^{m-1} V(\boldsymbol{\pi})^{(r)}$ and $X_s v_{\boldsymbol{\pi}} \in V(\boldsymbol{\pi})^{(l)}$ where $l = -\deg X_s \pmod m$. □

**2.10.** Let $L(V(\boldsymbol{\pi})) = V(\boldsymbol{\pi}) \otimes_{\mathbf{C}(q)} \mathbf{C}(q)[t, t^{-1}]$ be the loop space of $V(\boldsymbol{\pi})$. Define the $\widehat{\mathbf{U}}_q$-module structure on $L(V(\boldsymbol{\pi}))$ by

$$x(v \otimes t^n) = xv \otimes t^{n+\deg x}, \quad D(v \otimes t^n) = q^n v \otimes t^n, \quad C^{\pm 1/2}(v \otimes t^n) = v \otimes t^n$$

for all $x \in \widehat{\mathbf{U}}_q$ homogeneous, $v \in V(\boldsymbol{\pi})$ and $n \in \mathbf{Z}$. Henceforth we write $vt^n$ for the element $v \otimes t^n$, $v \in V(\boldsymbol{\pi})$, $n \in \mathbf{Z}$ of $L(V(\boldsymbol{\pi}))$.

Let $m = m(\boldsymbol{\pi})$. By [6], $L(V(\boldsymbol{\pi}))$ is a direct sum of simple submodules $L^r(V(\boldsymbol{\pi}))$, $r = 0, \ldots, m-1$ where $L^r(V(\boldsymbol{\pi})) = \widehat{\mathbf{U}}_q(v_{\boldsymbol{\pi}} t^r) = \widehat{\mathbf{U}}_q(v_{\boldsymbol{\pi}}^* t^r)$.

Define $\widehat{\eta}_{\boldsymbol{\pi}}(vt^r) = \zeta^r \eta_{\boldsymbol{\pi}}(v) t^r$. Then by [6], $\widehat{\eta}_{\boldsymbol{\pi}} \in \operatorname{End}_{\widehat{\mathbf{U}}_q} L(V(\boldsymbol{\pi}))$ and the simple submodule $L^s(V(\boldsymbol{\pi}))$ is just the eigenspace of $\widehat{\eta}_{\boldsymbol{\pi}}$ corresponding to the eigenvalue $\zeta^s$.

**Lemma.** *For all $s = 0, \ldots, m-1$, the $\mathbf{C}(q)$-subspace $\widehat{\mathbf{U}}_q^+(v_{\boldsymbol{\pi}} t^s)$ is spanned by elements of the form $vt^{s+n(v)+k}$, $v \in V(\boldsymbol{\pi})$, $k \in \mathbf{N}$.*

**Proof.** The statement follows immediately from Corollary 2.7 and from the definition of $n(v)$ in 2.8. □

## 3. Irreducibility of $X(\Lambda) \otimes V(\boldsymbol{\pi})$

In this section we prove the following:

**Theorem 1.** *Let $\Lambda \in \widehat{P}$ and let $V(\boldsymbol{\pi})$ be a finite dimensional simple $\widehat{\mathbf{U}}_q'$-module corresponding to an $\ell$-tuple $\boldsymbol{\pi}$ of polynomials in one variable with constant term 1. Then $X(\Lambda) \otimes V(\boldsymbol{\pi})$ is a simple $\widehat{\mathbf{U}}_q'$-module.*

This result is a quantum version of [7, Theorem 4.2].

**3.1.** By Proposition 1.5, $\widehat{\mathbf{U}}_q^r(\gg)$, $r \geqslant 0$ and $\widehat{\mathbf{U}}_q^r(\ll)$, $r > 0$ are contained in $\widehat{\mathbf{U}}_q^+$ which is in turn contained in the Hopf subalgebra $\widehat{\mathscr{U}}_q^+$ of $\widehat{\mathbf{U}}_q'$.

**Proposition.** *Let $\Lambda \in \widehat{P}$.*

(i) *As $\widehat{\mathbf{U}}_q'$-module, we have*

$$M(\Lambda) \otimes V(\boldsymbol{\pi}) = \widehat{\mathbf{U}}_q'(m_\Lambda \otimes v_{\boldsymbol{\pi}}) = \widehat{\mathbf{U}}_q'(m_\Lambda \otimes v_{\boldsymbol{\pi}}^*)$$

*and*

$$X(\Lambda) \otimes V(\boldsymbol{\pi}) = \widehat{\mathbf{U}}'_q(v_\Lambda \otimes v_{\boldsymbol{\pi}}) = \widehat{\mathbf{U}}'_q(v_\Lambda \otimes v^*_{\boldsymbol{\pi}}).$$

(ii) *As* $\widehat{\mathbf{U}}_q$*-modules, we have*

$$M(\Lambda) \otimes L^s(V(\boldsymbol{\pi})) = \sum_{n \in \mathbf{Z}} \widehat{\mathbf{U}}_q(m_\Lambda \otimes v_{\boldsymbol{\pi}} t^{mn+s}) = \sum_{n \in \mathbf{Z}} \widehat{\mathbf{U}}_q(m_\Lambda \otimes v^*_{\boldsymbol{\pi}} t^{mn+s})$$

*and*

$$X(\Lambda) \otimes L^s(V(\boldsymbol{\pi})) = \sum_{n \in \mathbf{Z}} \widehat{\mathbf{U}}_q(v_\Lambda \otimes v_{\boldsymbol{\pi}} t^{mn+s}) = \sum_{n \in \mathbf{Z}} \widehat{\mathbf{U}}_q(v_\Lambda \otimes v^*_{\boldsymbol{\pi}} t^{mn+s})$$

*for all* $s = 0, \ldots, m-1$.

**Proof.** The argument repeats that of the proof of [7, Lemma 2.1] and is included here for the reader's convenience.

Observe first that, since $\widehat{\mathscr{U}}_q^+$ is a Hopf subalgebra of $\widehat{\mathbf{U}}'_q$, we have by Lemma 2.7

$$\widehat{\mathscr{U}}_q^+(m_\Lambda \otimes v_{\boldsymbol{\pi}}) = m_\Lambda \otimes \widehat{\mathbf{U}}_q^+ v_{\boldsymbol{\pi}} = m_\Lambda \otimes V(\boldsymbol{\pi}).$$

We prove by induction on $\operatorname{ht}\widehat{\gamma}$ that

$$M(\Lambda)_{\Lambda - \widehat{\gamma}} \otimes V(\boldsymbol{\pi}) \subset \widehat{\mathbf{U}}'_q(m_\Lambda \otimes V(\boldsymbol{\pi})).$$

If $\operatorname{ht}\widehat{\gamma} = 0$ then there is nothing to prove. Suppose that $\operatorname{ht}\widehat{\gamma} = 1$ that is $\widehat{\gamma} = \alpha_i$ for some $i \in \widehat{I}$. Since $M(\Lambda)_{\Lambda-\alpha_i}$ is spanned by $F_i m_\Lambda$, we have

$$F_i(m_\Lambda \otimes V(\boldsymbol{\pi})) = F_i m_\Lambda \otimes V(\boldsymbol{\pi}) + m_\Lambda \otimes V(\boldsymbol{\pi}),$$

whence

$$F_i m_\Lambda \otimes V(\boldsymbol{\pi}) \subset \widehat{\mathbf{U}}'_q(m_\Lambda \otimes V(\boldsymbol{\pi})).$$

The inductive step is proved similarly.

Thus, $M(\Lambda) \otimes V(\boldsymbol{\pi}) \subset \widehat{\mathbf{U}}'_q(m_\Lambda \otimes v_{\boldsymbol{\pi}})$ and we conclude that $M(\Lambda) \otimes V(\boldsymbol{\pi})$ is generated by $m_\Lambda \otimes v_{\boldsymbol{\pi}}$. To see that it is also generated by $m_\Lambda \otimes v^*_{\boldsymbol{\pi}}$ one proceeds as above using an observation that, by Proposition 2.7, $\widehat{\mathbf{U}}^0_q(\gg)v^*_{\boldsymbol{\pi}} = V(\boldsymbol{\pi})$ and that $\widehat{\mathbf{U}}^0_q(\gg)m_\Lambda = 0$. This proves (i) for the modules $M(\Lambda) \otimes V(\boldsymbol{\pi})$ and hence for the quotient module $X(\Lambda) \otimes V(\boldsymbol{\pi})$.

The proof of (ii) is similar. To see that induction starts, notice that by Corollary 2.7 and Lemma 1.6, we have

$$\sum_{n\in\mathbf{Z}}\widehat{\mathbf{U}}_q(m_\Lambda\otimes v_{\boldsymbol{\pi}}t^{mn+s})=\sum_{n\in\mathbf{Z}}\widehat{\mathbf{U}}_q^-(m_\Lambda\otimes\widehat{\mathbf{U}}_q^+(v_{\boldsymbol{\pi}}t^{mn+s}))=\sum_{n\in\mathbf{Z}}\widehat{\mathbf{U}}_q(m_\Lambda\otimes L^s(V(\boldsymbol{\pi}))).$$

The inductive step is now completed as before. □

**3.2.** Let $\Lambda\in\widehat{P}^+$. Recall from (2.1) that when we regard $X(\Lambda)$ as a $\widehat{\mathbf{U}}_q'$-module, any weight vector $u\in X(\Lambda)$ of weight $\Lambda-\widehat{\gamma}$ can be written uniquely as a sum $u=\sum_k u_k$ of linearly independent elements $u_k\in X(\Lambda)_{\Lambda-\gamma_k-n_k\alpha_0}$, where $\gamma_k\in Q^+$, $n_k\in\mathbf{N}$ and $\gamma_k+n_k\alpha_0=\widehat{\gamma}$. Denote by $\deg u$ the maximal value of the $n_k$.

Given a weight vector $v\in V(\boldsymbol{\pi})$, set $\mathrm{ht}_{\boldsymbol{\pi}}(v):=\mathrm{ht}(\lambda_{\boldsymbol{\pi}}-\mathrm{wt}\,v)$. Let $w\in X(\Lambda)\otimes V(\boldsymbol{\pi})$ be a weight vector and write

$$w=\sum_{k=1}^r u_k\otimes v_k,$$

where the $u_k$ are linearly independent weight vectors in $X(\Lambda)$ and the $v_k$ are weight vectors in $V(\boldsymbol{\pi})$. Fix an integer $j_0(w)=j_0$, $1\leqslant j_0\leqslant r$ such that the following two conditions hold:

$$\deg u_{j_0}\geqslant\deg u_j,\qquad\forall\,1\leqslant j\leqslant r,\tag{3.1}$$

$$\deg u_{j_0}=\deg u_j\Longrightarrow\mathrm{ht}_{\boldsymbol{\pi}}(v_{j_0})\geqslant\mathrm{ht}_{\boldsymbol{\pi}}(v_j).\tag{3.2}$$

**Proposition.** *Let $w=\sum_{k=1}^r u_k\otimes v_k\in X(\Lambda)\otimes V(\boldsymbol{\pi})$ be a weight vector and let $j_0=j_0(w)$ be as above.*

(i) *Assume that $v_{j_0}\notin\mathbf{C}(q)v_{\boldsymbol{\pi}}$. Then there exists $i\in I$, $s\geqslant 0$ such that*

$$0\neq x_{i,s}^+w=\sum_{\substack{j\,:\,\deg u_j=\deg u_{j_0},\\ \mathrm{ht}_{\boldsymbol{\pi}}(v_j)=\mathrm{ht}_{\boldsymbol{\pi}}(v_{j_0})}}q_i^{\mathrm{wt}\,u_j(\alpha_i^\vee)}(u_j\otimes x_{i,s}^+v_j)+S,\tag{3.3}$$

*where $S$ is a sum of terms of the form $u_j'\otimes v_j'$ with either $\deg u_j'<\deg u_{j_0}$ or $\deg u_j'=\deg u_{j_0}$ and $\mathrm{ht}_{\boldsymbol{\pi}}(v_j')<\mathrm{ht}_{\boldsymbol{\pi}}(x_{i,s}^+v_{j_0})$.*

(ii) *Suppose that $\deg u_{j_0}=0$. Then, for all $s\gg 0$ there exist $x\in\widehat{\mathbf{U}}_q^s(\gg)$ such that $xw=v_\Lambda\otimes v_{\boldsymbol{\pi}}$ and $y\in\widehat{\mathbf{U}}_q^s(\ll)$ such that $yw=v_\Lambda\otimes v_{\boldsymbol{\pi}}^*$.*

(iii) *Suppose that $\deg u_{j_0}=N$. Then, there exists $s>0$ and $x\in\widehat{\mathbf{U}}_q^s(\gg)$ such that $xw=v_\Lambda\otimes v_{\boldsymbol{\pi}}$ and an element $y\in\widehat{\mathbf{U}}_q^s(\ll)$ such that $yw=v_\Lambda\otimes v_{\boldsymbol{\pi}}^*$.*

**Proof.** By 2.2 and Proposition 2.7 there exist $i \in I$ and $s \in \mathbf{N}^+$ such that $x_{i,s}^+ u_j = 0$ for all $j$ and $x_{i,s}^+ v_{j_0} \neq 0$. Observe that by (1.2),

$$x_{i,s}^+(u_j \otimes v_j) = q_i^{\operatorname{wt} u_j(\alpha_j^\vee)}(u_j \otimes x_{i,s}^+ v_j) + \sum_k u'_k \otimes v'_k,$$

where $\operatorname{ht}_{\boldsymbol{\pi}}(v'_k) < \operatorname{ht}_{\boldsymbol{\pi}}(v_j)$ and $\deg u'_k < \deg u_j$. It follows that we can write $x_{i,s}^+ w$ as in (3.3). Notice that the term $u_{j_0} \otimes x_{i,s}^+ v_{j_0}$ occurs with a non-zero coefficient on the right-hand side of (3.3) and is clearly linearly independent from the other terms in this equation. Hence $x_{i,s}^+ w \neq 0$ and (i) is proved.

To prove (ii), notice that if $\deg u_{j_0} = 0$, then $\deg u_j = 0$ for all $j$ and hence $x u_j = 0$ for all $x \in \widehat{\mathbf{U}}_q^+$ which are homogenous of positive degree. It follows from (1.2) that for all $x \in \widehat{\mathbf{U}}_q^1(\gg)$ we have

$$xw = \sum_k u_k \otimes x v_k.$$

Choose $k$ such that $\operatorname{ht}_{\boldsymbol{\pi}}(v_k)$ is maximal and $x \in \widehat{\mathbf{U}}_q^1(\gg)$ such that $x v_k = v_{\boldsymbol{\pi}}$. Then for all $j$ we have $x v_j = a_j v_{\boldsymbol{\pi}}$ for some $a_j \in \mathbf{C}(q)$. It follows that $xw = u \otimes v_{\boldsymbol{\pi}}$ for some $u \in X(\Lambda)$ and $\deg u = 0$. Since $\deg u = 0$ and $X(\Lambda)$ is irreducible it follows that $u \in U_q(\mathfrak{g}) v_\Lambda$ and hence there exists $x' \in U_q(\mathfrak{g}) \cap \widehat{\mathbf{U}}_q^+$ such that $x'u = v_\Lambda$ and so we get $x'xw = v_\Lambda \otimes v_{\boldsymbol{\pi}}$. Furthermore, there exists $y \in \widehat{\mathbf{U}}_q^1(\ll)$ such that $y v_{\boldsymbol{\pi}} = v_{\boldsymbol{\pi}}^*$. It follows that $y(v_\Lambda \otimes v_{\boldsymbol{\pi}}) = v_\Lambda \otimes v_{\boldsymbol{\pi}^*}$ which completes the proof of (ii).

We prove (iii) by induction on $N$. Notice that (ii) proves that induction starts. Consider first the case when $v_{j_0} = a v_{\boldsymbol{\pi}}$ for some $a \in \mathbf{C}(q)^\times$. Then we can write

$$w = u \otimes v_{\boldsymbol{\pi}} + w',$$

where $w' = \sum_{j\,:\,\deg u_j < N} u_j \otimes v_j$. Choose $s > 0$ so that $x_{i,s}^+ u = 0$ for all $i \in I$. By (1.2), $x_{i,s}^+(u \otimes v_{\boldsymbol{\pi}}) = 0$. The induction hypothesis applies to $w'$ and so there exists $x \in \widehat{\mathbf{U}}_q^s(\gg)$ such that $xw' = v_\Lambda \otimes v_{\boldsymbol{\pi}}$. It follows that $xw = v_\Lambda \otimes v_{\boldsymbol{\pi}}$ and we are done.

Suppose then that $\operatorname{ht}_{\boldsymbol{\pi}}(v_{j_0}) = M$ and that (iii) is established for $\operatorname{ht}_{\boldsymbol{\pi}}(v_{j_0}) < M$. By part (i) there exist $i \in I$ and $s > 0$ such that $w' = x_{i,s}^+ w \neq 0$. Furthermore write $w' = \sum_j u'_j \otimes v'_j$ and set $j'_0 = j_0(w')$. Observe that $\deg u'_{j'_0} = \deg u_{j_0}$ and $\operatorname{ht}_{\boldsymbol{\pi}}(v'_{j'_0}) = M - 1$. Hence the induction hypothesis on $M$ applies and we conclude that there exists $x' \in \widehat{\mathbf{U}}_q^{s'}(\gg)$, $s' > s$ with $x(x_{i,s}^+ w) = v_\Lambda \otimes v_{\boldsymbol{\pi}}$. $\square$

**Corollary.** *Let $W$ be a non-zero submodule of $X(\Lambda) \otimes V(\boldsymbol{\pi})$ with $\Lambda \in \widehat{P}$ dominant. Then $W$ contains both $v_\Lambda \otimes v_{\boldsymbol{\pi}}$ and $v_\Lambda \otimes v_{\boldsymbol{\pi}}^*$.*

Theorem 1 follows immediately from the above Corollary and Proposition 3.1.

## 4. A filtration of $M(\Lambda)\otimes L(V(\pi))$ and $X(\Lambda)\otimes L(V(\pi))$

**4.1.** Let $M$ be a $\widehat{\mathbf{U}}_q$-module

**Definition.** We call a collection $\{\mathscr{F}_n\}_{n\in\mathbf{Z}}$ of $\widehat{\mathbf{U}}_q$-submodules of $M$ a decreasing $\mathbf{Z}$-filtration of $M$ if $M=\sum_{n\in\mathbf{Z}}\mathscr{F}_n$ and $\mathscr{F}_n\supseteq\mathscr{F}_{n+1}$ for all $n\in\mathbf{Z}$. We say that the filtration $\{\mathscr{F}_n\}_{n\in\mathbf{Z}}$ is *strictly decreasing* if $\mathscr{F}_n\neq\mathscr{F}_{n+1}$ for all $n\in\mathbf{Z}$ and is *trivial* if $\mathscr{F}_m=\mathscr{F}_n$ for all $m,n\in\mathbf{Z}$.

In this section we prove that for $s=0,\dots,m(\pi)-1$, the modules $M(\Lambda)\otimes L^s(V(\pi))$ and $X(\Lambda)\otimes L^s(V(\pi))$ admit a $\mathbf{Z}$-filtration $\mathscr{M}_n^{(s)}$ (respectively, $\mathscr{X}_n^{(s)}$), $n\in\mathbf{Z}$, whose successive quotients are in the category $\mathscr{O}$ and are isomorphic as $\widehat{\mathbf{U}}'_q$-modules. We prove that $\mathscr{M}_n\supsetneq\mathscr{M}_{n+1}$ for all $n\in\mathbf{Z}$ and that $\bigcap_{n\in\mathbf{Z}}\mathscr{M}_n^{(s)}=0$ We also show that the filtration $\mathscr{X}_n$, $n\in\mathbf{Z}$ is either trivial or strictly decreasing. In the first case we prove that this implies that $X(\Lambda)\otimes L^s(V(\pi))$ is irreducible and in the second case we prove that $\bigcap_{n\in\mathbf{Z}}\mathscr{X}_n^{(s)}=0$.

**4.2.** Set $m=m(\pi)$.

**Proposition.** *Let $\Lambda\in\widehat{P}$. Given $n\in\mathbf{Z}$, let $\mathscr{M}_n$ be the $\widehat{\mathbf{U}}_q$-submodule of $M(\Lambda)\otimes L(V(\pi))$ generated by the vectors $m_\Lambda\otimes v_\pi t^{mn+s}$, $s=0,\dots,m-1$. Then the modules $\mathscr{M}_n$ form a $\mathbf{Z}$-filtration of $M(\Lambda)\otimes L(V(\pi))$. Moreover, for all $n\in\mathbf{Z}$, the modules $\mathscr{M}_n/\mathscr{M}_{n+1}$ are in the category $\mathscr{O}$ and are isomorphic as $\widehat{\mathbf{U}}'_q$-modules.*

*Further, if $\mathscr{X}_n$ is the submodule of $X(\Lambda)\otimes L(V(\pi))$ generated by the vectors $v_\Lambda\otimes v_\pi t^{mn+s}$, $s=0,\dots,m-1$, then the modules $\mathscr{X}_n$ form a $\mathbf{Z}$-filtration of $X(\Lambda)\otimes L(V(\pi))$. Moreover, for all $n\in\mathbf{Z}$ the modules $\mathscr{X}_n/\mathscr{X}_{n+1}$ are in the category $\mathscr{O}$ and are isomorphic as $\widehat{\mathbf{U}}'_q$-modules.*

**Proof.** We prove only the statement for the Verma modules, the proof of the one for $X(\Lambda)$ being similar. Let $\pi=(\pi_i(u))_{i\in I}$, where $\pi_i(u)=\sum_k\pi_{i,k}u^k\in\mathbf{C}(q)[u]$. By the choice of $m$, there exists $i\in I$ such that $\pi_{i,r}=0, 0<r<m$ and $\pi_{i,m}\neq 0$. Then $P_{i,m}v_\pi=\pi_{i,m}v_\pi$ and so $P_{i,m}(m_\Lambda\otimes v_\pi t^{mn+s})=m_\Lambda\otimes(\pi_{i,m}v_\pi)t^{m(n+1)+s}$ by (1.4). Therefore, $\mathscr{M}_n\supseteq\mathscr{M}_{n+1}$. Since $M(\Lambda)\bigotimes L(V(\pi))=\sum_{n\in\mathbf{Z}}\mathscr{M}_n$ by Proposition 3.1(ii), it follows that $\{\mathscr{M}_n\}_{n\in\mathbf{Z}}$ is a $\mathbf{Z}$-filtration on $M(\Lambda)\bigotimes L(V(\pi))$.

To show that $\mathscr{M}_n/\mathscr{M}_{n+1}$ is in the category $\mathscr{O}$, it suffices to prove that the subspaces $\widehat{\mathscr{U}}_q^+(m_\Lambda\otimes v_\pi t^{mn+s})$, $s=0,\dots,m-1$ of $\mathscr{M}_n$ are finite-dimensional modulo $\mathscr{M}_{n+1}$. Equivalently, it is sufficient to prove that the subspace $\widehat{\mathbf{U}}_q^+(v_\pi t^{mn})$ is finite-dimensional modulo the subspace $\widehat{\mathbf{U}}_q^+(v_\pi t^{m(n+1)})$. Now, by Proposition 1.5,

$$\widehat{\mathbf{U}}_q^+(v_\pi t^{mn})\subset\widehat{\mathbf{U}}_q^0(\ll)\widehat{\mathbf{U}}_q^0(0)(v_\pi t^{mn})=\widehat{\mathbf{U}}_q^0(\ll)v_\pi t^{mn}\ (\mathrm{mod}\,\widehat{\mathbf{U}}_q^+v_\pi t^{m(n+1)})$$

since $P_{i,r}v_\pi=0$ unless $r$ is divisible by $m$. Since $V(\pi)$ is finite dimensional, by Lemma 2.7 there exist homogeneous $X_1,\dots,X_N\in\widehat{\mathbf{U}}_q^r(\ll)$ for some $r\geqslant 0$ such that $X_1v_\pi,\dots,X_Nv_\pi$ form a basis of $V(\pi)$. Let $x\in\widehat{\mathbf{U}}_q^0(\ll)$. Then there exist

$a_j \in \mathbf{C}(q),\ j = 1, \ldots, N$ such that

$$x v_{\boldsymbol{\pi}} = \sum_{j=1}^{N} a_j X_j v_{\boldsymbol{\pi}}.$$

We may assume that $x$ is homogeneous of degree $k$. Then $x v_{\boldsymbol{\pi}} \in V(\boldsymbol{\pi})^{(-k)}$ and so $a_j = 0$ unless $\deg X_j = k \,(\mathrm{mod}\, m)$ by Lemma 2.9. Then we can write

$$x(v_{\boldsymbol{\pi}} t^{mn}) = (x v_{\boldsymbol{\pi}}) t^{mn+k} = \left( \sum_{j=1}^{N} a_j X_j v_{\boldsymbol{\pi}} \right) t^{mn+k} = \sum_{j=1}^{N} a_j X_j (v_{\boldsymbol{\pi}} t^{mn+k-\deg X_j}),$$

the only non-zero terms being those with $\deg X_j = k \,(\mathrm{mod}\, m)$. It follows that $x(v_{\boldsymbol{\pi}} t^{mn}) = 0 \,(\mathrm{mod}\, \widehat{\mathbf{U}}_q^+(v_{\boldsymbol{\pi}} t^{m(n+1)}))$ if $k$ is sufficiently large. Therefore, the dimension of $\widehat{\mathbf{U}}_q^+(v_{\boldsymbol{\pi}} t^{mn}) \ \ (\mathrm{mod}\, \widehat{\mathbf{U}}_q^+(v_{\boldsymbol{\pi}} t^{m(n+1)}))$ is bounded above by the dimension of the subspace of $\widehat{\mathbf{U}}_q^0(\ll)$ spanned by homogeneous elements whose degree does not exceed $\max_j\{\deg X_j\}$. Evidently, such a subspace of $\widehat{\mathbf{U}}_q^0(\ll)$ is finite-dimensional.

To prove that $\mathscr{M}_n/\mathscr{M}_{n+1} \cong \mathscr{M}_{n-1}/\mathscr{M}_n$ as a $\widehat{\mathbf{U}}'_q$-module for all $n$, consider the map

$$\begin{aligned} M(\Lambda) \otimes L(V(\boldsymbol{\pi})) &\longrightarrow M(\Lambda) \otimes L(V(\boldsymbol{\pi})), \\ v \otimes w t^k &\mapsto v \otimes w t^{k+m} \end{aligned}$$

for all $v \in M(\Lambda)$, $w \in V(\boldsymbol{\pi})$ and $k \in \mathbf{Z}$. This is obviously a map of $\widehat{\mathbf{U}}'_q$-modules (but, of course, not $\widehat{\mathbf{U}}_q$-modules) which takes $\mathscr{M}_n$ isomorphically onto $\mathscr{M}_{n+1}$. Moreover, this operation corresponds to tensoring $\mathscr{M}_n$ with the 1-dimensional highest weight integrable module $X(m\delta)$. Thus we have $\mathscr{M}_n \cong \mathscr{M}_{n+1} \otimes X(-m\delta)$ and so in fact $\mathscr{M}_n/\mathscr{M}_{n+1}$ is isomorphic to $(\mathscr{M}_{n-1} \otimes X(m\delta))/(\mathscr{M}_n \otimes X(m\delta))$ as a $\widehat{\mathbf{U}}'_q$-module. □

**4.3.** For $s = 0, \ldots, m-1$, let $\mathscr{M}_n^{(s)}$ (respectively, $\mathscr{X}_n^{(s)}$) be the $\widehat{\mathbf{U}}_q$-submodule of $M(\Lambda) \otimes L(V(\boldsymbol{\pi}))$ (respectively, of $X(\Lambda) \otimes L(V(\boldsymbol{\pi}))$) generated by $m_\Lambda \otimes v_{\boldsymbol{\pi}} t^{mn+s}$ (respectively, by $v_\Lambda \otimes v_{\boldsymbol{\pi}} t^{mn+s}$).

**Lemma.** *For all $n \in \mathbf{Z}$, we have*

$$\mathscr{M}_n = \bigoplus_{s=0}^{m-1} \mathscr{M}_n^{(s)},$$

*and*

$$\mathscr{M}_n^{(s)} = \mathscr{M}_n \cap (M(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))).$$

*Further, the* $\mathscr{M}_n^{(s)}$, $n \in \mathbf{Z}$ *form a decreasing filtration of* $M(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))$, $s = 0, \ldots, m-1$. *Analogous statements hold for* $\mathscr{X}_n^{(s)}$.

**Proof.** This follows immediately from the trivial observation that for any $\widehat{\mathbf{U}}_q$-module $M$ we have $M \otimes L(V(\boldsymbol{\pi})) = \bigoplus_{s=0}^{m-1} M \otimes L^s(V(\boldsymbol{\pi}))$. □

**4.4.** Fix an ordered basis of $V(\boldsymbol{\pi})$ of weight vectors $v_0 = v_{\boldsymbol{\pi}}, v_1, \ldots, v_N = v_{\boldsymbol{\pi}}^*$ such that $\mathrm{ht}_{\boldsymbol{\pi}}(v_i) \leqslant \mathrm{ht}_{\boldsymbol{\pi}}(v_{i+1})$ for all $i = 0, \ldots, N-1$. Furthermore, we may assume, without loss of generality, that $v_j \in V(\boldsymbol{\pi})^{(k_j)}$ for some $0 \leqslant k_j \leqslant m-1$. It is clear that $F_i v_j$ is a linear combination of $v_{j'}$ with $j' > j$ if $i \in I$ and with $j' < j$ if $i = 0$. Let $\tilde{\mathscr{M}}_n^{(s)}$ be the $\widehat{\mathscr{U}}_q^-$-submodule of $M(\Lambda) \otimes L(V(\boldsymbol{\pi}))$ generated by the set $\{m_\Lambda \otimes v_j t^r : r \geqslant mn + s,\ j = 0, \ldots, N,\ r = s - k_j \pmod m\}$ and set $\sum_{s=0}^{m-1} \tilde{\mathscr{M}}_n^{(s)} = \tilde{\mathscr{M}}_n$. Similarly, let $\tilde{\mathscr{X}}_n^{(s)}$ be the $\widehat{\mathscr{U}}_q^-$-submodule of $X(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))$ generated by the set $\{v_\Lambda \otimes v_j t^r : r \geqslant mn + s,\ j = 0, \ldots, N,\ r = s - k_j \pmod m\}$.

**Lemma.** *For* $0 \leqslant s \leqslant m-1$, *and* $n \in \mathbf{Z}$, *we have*

$$\mathscr{M}_n^{(s)} \subset \tilde{\mathscr{M}}_n^{(s)}, \qquad \mathscr{X}_n^{(s)} \subset \tilde{\mathscr{X}}_n^{(s)}.$$

**Proof.** Immediate. □

**4.5.** The following proposition plays a crucial role in the reminder of the paper.

**Proposition.**

(i) *Let* $v \in V(\boldsymbol{\pi})$, $n \in \mathbf{N}$ *and suppose that there exist elements* $X_{j,r} \in (\widehat{\mathbf{U}}_q^-)_+$, $r \geqslant 0$, $j = 0, \ldots, N$ *such that in* $M(\Lambda) \otimes L(V(\boldsymbol{\pi}))$ *we have,*

$$m_\Lambda \otimes v t^n = \sum_{j=0}^{N} \sum_{r \geqslant n} X_{j,r}(m_\Lambda \otimes v_j t^r). \tag{4.1}$$

*Then* $v = 0$.

(ii) *Let* $w \in M(\Lambda) \otimes L(V(\boldsymbol{\pi}))$, $w \neq 0$. *Then there exists* $n \in \mathbf{Z}$ *such that* $w \notin \tilde{\mathscr{M}}_{n+1}$.

(iii) *Let* $v \in V(\boldsymbol{\pi})$ *and suppose that there exist elements* $X_{j,r} \in (\widehat{\mathbf{U}}_q^-)_+$, $r \geqslant 0$, $j = 0, \ldots, N$ *such that in* $X(\Lambda) \otimes L(V(\boldsymbol{\pi}))$ *we have,*

$$v_\Lambda \otimes v t^n = \sum_{j=0}^{N} \sum_{r \geqslant n} X_{j,r}(m_\Lambda \otimes v_j t^r). \tag{4.2}$$

*Let $R$ be the maximal $r \geqslant n$ such that there exists $0 \leqslant j \leqslant N$ with $X_{j,r}(v_\Lambda \otimes v_j t^r) \neq 0$ and let $j_0$ be the minimal $j$ such that $X_{j,R}(v_\Lambda \otimes v_j t^R) \neq 0$. Then $X_{j_0,R} \in \operatorname{Ann}_{\widehat{\mathbf{U}}_q^-} v_\Lambda$.*

**Proof.** To prove (i), suppose for a contradiction that $v \neq 0$. Let $R$ be the maximal $r \geqslant n$ such that there exists $j$ with $X_{j,r}(m_\Lambda \otimes v_j t^r) \neq 0$ and let $j_0$ be the minimal $0 \leqslant j \leqslant N$ such that $X_{j,R}(m_\Lambda \otimes v_j t^R) \neq 0$. Then $X_{j_0,R}(m_\Lambda \otimes v_{j_0} t^R)$ contains a term $c(X_{j_0,R} m_\Lambda) \otimes v_{j_0} t^R$ for some $c \in \mathbf{C}(q)^\times$. Since $\Delta(\widehat{\mathbf{U}}_q^-) \subset \widehat{\mathscr{U}}_q^- \otimes \widehat{\mathscr{U}}_q^-$, it follows that all other elements in (4.1) are terms of the form $m' \otimes v_{j'} t^r$ where either $r < R$ or $r = R$ and $j' > j_0$. If $R > n$, then this forces $X_{j_0,R} \in \operatorname{Ann}_{\widehat{\mathbf{U}}_q^-} m_\Lambda$ and hence $X_{j_0,R} = 0$ which is a contradiction. If $R = n$, then (4.1) reduces to

$$m_\Lambda \otimes v t^n = \sum_{j=0}^{N} X_{j,n}(m_\Lambda \otimes v_j t^n).$$

Let $k = \#\{j : X_{j,n}(m_\Lambda \otimes v_j t^n) \neq 0\}$. If $k = 0$ then we are done. Suppose that $k = 1$. Then $m_\Lambda \otimes v t^n = X_{j,n}(m_\Lambda \otimes v_j t^n)$ for some $0 \leqslant j \leqslant N$. If $X_{j,n} m_\Lambda \neq 0$ then $X_{j,n}(m_\Lambda \otimes v_j t^n)$ contains a non-zero term lying in $\bigoplus_{\gamma \in \widehat{Q}^+ \setminus \{0\}} M(\Lambda)_{\Lambda-\gamma} \otimes L(V(\boldsymbol{\pi}))$ which is clearly impossible. Hence $X_{j,n} m_\Lambda = 0$ and we get a contradiction. Suppose then that we have proved that either $k = 0$ or $k \geqslant s$ for some $s \in \mathbf{N}^+$. If $k = s$ then $X_{j_r,n}(m_\Lambda \otimes v_{j_r} t^{j_r}) \neq 0$ for some $0 \leqslant j_1 < \cdots < j_s \leqslant N$. If $X_{j_r,n} m_\Lambda \neq 0$ for any $j_r$, then again the right hand side of (4.1) contains a non-zero term in $\bigoplus_{\gamma \in \widehat{Q}^+ \setminus \{0\}} M(\Lambda)_{\Lambda-\gamma} \otimes L(V(\boldsymbol{\pi}))$ which is a contradiction. Thus $k = 0$ or $k > s$. Since $V(\boldsymbol{\pi})$ is finite-dimensional it follows that $k = 0$ and we are done.

To prove (ii), write $w = \sum_{j=1}^{s} m_j \otimes w_j t^{r_j}$, where $m_j \in M(\Lambda)$, $w_j \in V(\boldsymbol{\pi})$ and $r_j \in \mathbf{Z}$. Let $n_0 = \max\{r_j : 1 \leqslant j \leqslant s\}$ and suppose that $w \in \tilde{\mathscr{M}}_n$ for some $n > n_0$. This means that we can write

$$w = \sum_{j=0}^{N} \sum_{r \geqslant n} X_{j,r}(m_\Lambda \otimes v_j t^r)$$

for some choice of $X_{j,r} \in \widehat{\mathbf{U}}_q^-$. But now arguing exactly as in the $R > n$ case of (i), we see that $w = 0$ which is a contradiction.

The proof of (iii) is an obvious modification of the argument in (i). $\square$

**4.6.**

**Proposition.** *The $\mathbf{Z}$-filtration $\mathscr{M}_n^{(s)}$ of $M(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))$, $s = 0, \ldots, m-1$ is strictly decreasing and $\bigcap_{n \in \mathbf{Z}} \mathscr{M}_n^{(s)} = 0$.*

**Proof.** In view of Proposition 4.2 for the first statement it is sufficient to prove that $\mathscr{M}_0^{(s)} \neq \mathscr{M}_1^{(s)}$. Assume for a contradiction that $\mathscr{M}_0^{(s)} = \mathscr{M}_1^{(s)}$. Then $m_\Lambda \otimes v_{\boldsymbol{\pi}} t^s \in \mathscr{M}_1^{(s)}$ and hence it follows from Lemma 4.4 that there exist $X_{j,r} \in \widehat{\mathbf{U}}_q^-$ such that

$$m_\Lambda \otimes v_0 t^s = \sum_{j=0}^{N} \sum_{r \geqslant m+s} X_{j,r}(m_\Lambda \otimes v_j t^r). \tag{4.3}$$

Since $X_{j,r} \in (\widehat{\mathbf{U}}_q^-)_+$ we get a contradiction by Proposition 4.5(i).

Furthermore, let $w \in \bigcap_{n \in \mathbf{N}} \mathscr{M}_n^{(s)}$. If $w \neq 0$, then by Proposition 4.5(ii) we can choose $n_0 \in \mathbf{Z}$ such that $w \notin \tilde{\mathscr{M}}_{n_0}^{(s)}$ contradicting $\mathscr{M}_{n_0}^{(s)} \subset \tilde{\mathscr{M}}_{n_0}^{(s)}$. $\square$

**4.7.** To analyse the filtration on $X(\Lambda) \otimes L(V(\boldsymbol{\pi}))$, $\Lambda \in \widehat{P}^+$, we need the following analogue of Proposition 3.2.

**Proposition.** *Let $w = \sum_{k=1}^r u_k \otimes v_k t^{r_k} \in X(\Lambda) \otimes L(V(\boldsymbol{\pi}))$ be a weight vector and let $j_0 = j_0(w)$ be the integer associated with the element $\sum_{k=1}^r u_k \otimes v_k \in X(\Lambda) \otimes V(\boldsymbol{\pi})$ as in* 3.2.

(i) *Assume that $v_{j_0} \neq v_{\boldsymbol{\pi}}$. There exists $i \in I$, $s \geqslant 0$ such that*

$$0 \neq x_{i,s}^+ w = \sum_{\substack{j\,:\,\deg u_j = \deg u_{j_0},\\ \mathrm{ht}_{\boldsymbol{\pi}}(v_j) = \mathrm{ht}_{\boldsymbol{\pi}}(v_{j_0})}} q_i^{\mathrm{wt}\, u_j(\alpha_i^\vee)} (u_j \otimes x_{i,s}^+ v_j t^{r_j + s}) + S, \tag{4.4}$$

*where $S$ is a sum of terms of the form $u'_j \otimes v'_j t^{r'_j}$ where either $\deg u'_j < \deg u_{j_0}$ or $\deg u'_j = \deg u_{j_0}$ and $\mathrm{ht}_{\boldsymbol{\pi}}(v'_j) < \mathrm{ht}_{\boldsymbol{\pi}}(x_{i,s}^+ v_{j_0})$.*

(ii) *Suppose that $\deg u_{j_0} = 0$. Then $r_j = R$ for all $j$ for some $R \in \mathbf{N}$. Furthermore, for all $s \gg 0$ there exists $x \in \mathbf{U}_q^s(\gg)$ and an integer $L$ such that $xw = v_\Lambda \otimes v_{\boldsymbol{\pi}} t^L$ and an element $y \in \mathbf{U}_q^s(\ll)$ and an integer $L'$ such that $yw = v_\Lambda \otimes v_{\boldsymbol{\pi}}^* t^{L'}$.*

(iii) *Suppose that $\deg u_{j_0} = N$. Then there exists $s > 0$ and $x \in \mathbf{U}_q^s(\gg)$ and an integer $K$ such that $xw = v_\Lambda \otimes v_{\boldsymbol{\pi}} t^K$ and an element $y \in \mathbf{U}_q^K(\ll)$ and an integer $K'$ such that $yw = v_\Lambda \otimes v_{\boldsymbol{\pi}}^* t^{K'}$.*

**Proof.** The first statement in part (ii) is an obvious consequence of the fact that $w$ is a weight vector. The proposition is now proved in exactly the same way as Proposition 3.2 and we omit the details. $\square$

**Corollary.** *Let $W$ be a non-zero submodule of $X(\Lambda) \otimes L(V(\boldsymbol{\pi}))$ with $\Lambda \in \widehat{P}$ dominant. Then $W$ contains $v_\Lambda \otimes v_{\boldsymbol{\pi}} t^s$ for some $s \in \mathbf{Z}$ and $v_\Lambda \otimes v_{\boldsymbol{\pi}}^* t^r$ for some $r \in \mathbf{Z}$.*

**4.8.** We note the following consequence of Corollary 4.7

**Proposition.** *The* $\mathbf{Z}$*-filtration* $\mathscr{X}_n^{(s)}$ *of* $X(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))$, $s = 0, \ldots, m-1$ *is either*

(i) *trivial and* $X(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))$ *is irreducible, or*
(ii) *strictly decreasing and*

$$\bigcap_{n \in \mathbf{Z}} \mathscr{X}_n^{(s)} = 0.$$

**Proof.** Suppose that $\mathscr{X}_n^{(s)} = \mathscr{X}_m^{(s)}$ for some $m > n \in \mathbf{Z}$. Then $\mathscr{X}_n^{(s)} = \mathscr{X}_{n+1}^{(s)}$ and it follows from Proposition 4.2 that $\mathscr{X}_m^{(s)} = \mathscr{X}_n^{(s)}$ for all $m, n \in \mathbf{Z}$. This proves that the filtration is either trivial or strictly decreasing.

Let $W$ be a non-zero submodule of $X(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))$. By Corollary 4.7, $v_\Lambda \otimes v_{\boldsymbol{\pi}} t^{mr+s} \in W$ for some $r \in \mathbf{Z}$ and so $\mathscr{X}_r^{(s)} \subset W$. If the filtration is trivial, then this implies that $\mathscr{X}_n^{(s)} \subset W$ for all $n \in \mathbf{Z}$. It follows from Proposition 3.1(ii) that $W = X(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))$ and (i) is proved.

Suppose that the filtration is strictly decreasing and set $W = \bigcap_{n \in \mathbf{Z}} \mathscr{X}_n^{(s)}$. Suppose that $W \neq 0$. Then it follows from Corollary 4.7 that $v_\Lambda \otimes v_{\boldsymbol{\pi}} t^{mr+s} \in W$ for some $r \in \mathbf{Z}$ and hence $\mathscr{X}_r^{(s)} \subset W$. Then $\mathscr{X}_r^{(s)} \subset \mathscr{X}_{r+1}^{(s)}$ and so $\mathscr{X}_r^{(s)} = \mathscr{X}_{r+1}^{(s)}$ which is a contradiction whence (ii). $\square$

**4.9.** The results of this section allow us to complete the tensor products $X(\Lambda) \otimes L(V(\boldsymbol{\pi}))$ and $M(\Lambda) \otimes L(V(\boldsymbol{\pi}))$. We restrict ourselves to the first case, the second one being similar. Let $\Lambda \in \widehat{P}$ and suppose that the filtration $\mathscr{X}_r^{(s)}$, $r \in \mathbf{Z}$ is strictly decreasing. Let $X(\Lambda) \widehat{\otimes} L^s(V(\boldsymbol{\pi}))$ be the completion of $X(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))$ with respect to the topology induced by the filtration $\mathscr{X}_r^{(s)}$. It is well-known that then there exists a canonical map $\phi_{\mathscr{X}} : X(\Lambda) \otimes L^s(V(\boldsymbol{\pi})) \longrightarrow X(\Lambda) \widehat{\otimes} L^s(V(\boldsymbol{\pi}))$ and $\ker \phi_{\mathscr{X}} = \bigcap_{r \in \mathbf{Z}} \mathscr{X}_r^{(s)} = 0$ by Proposition 4.8(ii). Therefore, $X(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))$ embeds into the completion. On the other hand,

$$X(\Lambda) \widehat{\otimes} L^s(V(\boldsymbol{\pi})) \cong \varprojlim (X(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))) / \mathscr{X}_r^{(s)}.$$

Furthermore, let $\widehat{\mathscr{X}}_n^{(s)}$ be the completion of $\mathscr{X}_n^{(s)}$,

$$\widehat{\mathscr{X}}_n^{(s)} = \varprojlim \mathscr{X}_m^{(s)} / \mathscr{X}_n^{(s)}.$$

Then $\widehat{\mathscr{X}}_n^{(s)}$ is a $\mathbf{Z}$-filtration on $X(\Lambda) \widehat{\otimes} L^s(V(\boldsymbol{\pi}))$ and $\mathscr{X}_n^{(s)} / \mathscr{X}_{n+1}^{(s)} \cong \widehat{\mathscr{X}}_n^{(s)} / \widehat{\mathscr{X}}_{n+1}^{(s)}$ and so the associated graded space of $X(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))$ with respect to the filtration $\mathscr{X}_n^{(s)}$

is isomorphic to the associated graded space of $X(\Lambda)\widehat{\otimes}L^s(V(\boldsymbol{\pi}))$ with respect to the filtration $\widehat{\mathscr{X}}_n^{(s)}$. One also has $(X(\Lambda)\widehat{\otimes}L(V(\boldsymbol{\pi})))/\widehat{\mathscr{X}}_n \cong (X(\Lambda)\otimes L(V(\boldsymbol{\pi})))/\mathscr{X}_n$.

## 5. An irreducibility criterion for $X(\Lambda)\otimes L(V(\pi))$

In this section we establish a sufficient condition for the simplicity of the $\widehat{\mathbf{U}}_q$-modules $X(\Lambda)\otimes L^s(V(\boldsymbol{\pi}))$, $s=0,\ldots,m(\boldsymbol{\pi})-1$, $\Lambda\in\widehat{P}^+$.

**Theorem 2.** *Let $\Lambda\in\widehat{P}^+$ and let $\boldsymbol{\pi}=(\pi_i(u))_{i\in I}$ be an $\ell$-tuple of polynomials with constant term* 1. *Suppose that either*

$$(k(\boldsymbol{\pi})+m(\boldsymbol{\pi}))(\Lambda,\delta)<(\Lambda+\lambda_{\boldsymbol{\pi}},\alpha_i),$$

*for some $i\in I$ satisfying $k(\boldsymbol{\pi})=\dim V(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}-\alpha_i}$ or*

$$k(\boldsymbol{\pi})(\Lambda,\delta)<-(\Lambda+w_\circ\lambda_{\boldsymbol{\pi}},\alpha_i),$$

*for some $i\in I$ satisfying $k(\boldsymbol{\pi})=\dim V(\boldsymbol{\pi})_{w_\circ\lambda_{\boldsymbol{\pi}}+\alpha_i}$. Then for all $s=0,\ldots,m(\boldsymbol{\pi})-1$, the filtration $\mathscr{X}_n^{(s)}$, $n\in\mathbf{Z}$ of $X(\Lambda)\otimes L^s(V(\boldsymbol{\pi}))$ is trivial and hence $X(\Lambda)\otimes L^s(V(\boldsymbol{\pi}))$ is an irreducible $\widehat{\mathbf{U}}_q$-module.*

**Proof.** It suffices by Proposition 4.8 to show that the filtration is trivial. Let $\boldsymbol{\pi}^0$ be as defined in 2.9, and set $k=k(\boldsymbol{\pi}^0)$, clearly $k(\boldsymbol{\pi})=mk$ where $m=m(\boldsymbol{\pi})$. Fix $i\in I$ so that $k(\boldsymbol{\pi})=\dim V(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}-\alpha_i}$. By Lemma 2.4,

$$x^-_{i,m(k+1)}v_{\boldsymbol{\pi}}=\sum_{r=1}^{mk}a_r x^-_{i,r}v_{\boldsymbol{\pi}}$$

for some $a_r\in\mathbf{C}(q)$, $r=1,\ldots,mk$. Applying (1.3) we see that

$$\begin{aligned}x^-_{i,m(k+1)}(v_\Lambda\otimes v_{\boldsymbol{\pi}}t^{mn+s}) &= v_\Lambda\otimes(x^-_{i,m(k+1)}v_{\boldsymbol{\pi}})t^{m(n+k+1)+s}\\ &= v_\Lambda\otimes\left(\sum_{r=1}^{mk}a_r x^-_{i,r}v_{\boldsymbol{\pi}}\right)t^{m(n+k+1)+s}\\ &= \sum_{r=1}^{mk}a_r x^-_{i,r}(v_\Lambda\otimes v_{\boldsymbol{\pi}}t^{m(n+k+1)+s-r}),\end{aligned}$$

which proves that $x^-_{i,m(k+1)}(v_\Lambda\otimes v_{\boldsymbol{\pi}}t^{mn+s})$ is contained in $\mathscr{X}^{(s)}_{n+1}$. Consider the subalgebra of $\widehat{\mathbf{U}}_q$ generated by $E=x^-_{i,m(k+1)}$, $F=x^+_{i,-m(k+1)}$ and $K=C^{m(k+1)}K_i^{-1}$ which is isomorphic to $U_{q_i}(\mathfrak{sl}_2)$ with standard generators $E, F, K^{\pm 1}$. Note that in $\mathscr{X}_n^{(s)}/\mathscr{X}_{n+1}^{(s)}$ we have

$$E(v_\Lambda \otimes v_{\boldsymbol{\pi}} t^{mn+s}) = 0, \quad K(v_\Lambda \otimes v_{\boldsymbol{\pi}} t^{mn+s}) = q_i^r (v_\Lambda \otimes v_{\boldsymbol{\pi}} t^{mn+s}),$$

where $d_i r = m(k+1)(\Lambda, \delta) - (\Lambda + \lambda_{\boldsymbol{\pi}}, \alpha_i) < 0$. Since $\mathscr{X}_n^{(s)}/\mathscr{X}_{n+1}^{(s)}$ is an integrable module for $\widehat{\mathbf{U}}_q$ and hence for this copy of $\mathbf{U}_{q_i}(\mathfrak{sl}_2)$, this forces

$$v_\Lambda \otimes v_{\boldsymbol{\pi}} t^{mn+s} \in \mathscr{X}_{n+1} \cap X(\Lambda) \otimes L^s(V(\boldsymbol{\pi})) = \mathscr{X}_{n+1}^{(s)},$$

and so $\mathscr{X}_n^{(s)} = \mathscr{X}_{n+1}^{(s)}$.

The second assertion is proved similarly. We work with the elements $v_\Lambda \otimes v_{\boldsymbol{\pi}}^* t^{mn+s}$ and $x_{i,mk}^+$ and we omit the details. □

## 6. A reducibility criterion for $X(\Lambda) \otimes L(V(\boldsymbol{\pi}))$

In this section we analyze the structure of $X(\Lambda) \otimes L(V(\boldsymbol{\pi}))$ and give a sufficient condition for the tensor product to be reducible.

**Theorem 3.** *Let $\Lambda \in \widehat{P}$ be dominant and suppose that $(\Lambda, \delta) \geq (\Lambda + \lambda_{\boldsymbol{\pi}})(\theta^\vee) + m(\boldsymbol{\pi}) - 1$ or, equivalently, $\Lambda(\alpha_0^\vee) \geqslant \lambda_{\boldsymbol{\pi}}(\theta^\vee) + m(\boldsymbol{\pi}) - 1$. Then the modules $X(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))$, $s = 0, \ldots, m(\boldsymbol{\pi}) - 1$ are reducible.*

The theorem is proved in the rest of the section.

**6.1.**

**Lemma.** *Let $\Lambda$ be a dominant weight. Take $v \in V(\boldsymbol{\pi})$. Then*

$$F_i^{\Lambda_i+1}(v_\Lambda \otimes v t^r) = \sum_{k=1}^{\Lambda_i+1} c_k F_i^{\Lambda_i+1-k}(v_\Lambda \otimes (F_i^k v) t^{r-k\delta_{i,0}})$$

*where $c_k \in \mathbf{C}(q)$.*

**Proof.** Recall that $F_i^{\Lambda_i+1} v_\Lambda = 0$. One has

$$F_i^{\Lambda_i+1}(v_\Lambda \otimes v t^r) = F_i^{\Lambda_i}(c_1 F_i v_\Lambda \otimes v t^r + v_\Lambda \otimes (F_i v) t^{r-\delta_{i,0}}).$$

The second term has the desired form. If $\Lambda_i = 0$ then the first term equals zero and we are done. Otherwise, we can write

$$F_i^{\Lambda_i}(F_i v_\Lambda \otimes vt^r) = F_i^{\Lambda_i - 1}(c' F_i^2 v_\Lambda \otimes vt^r + F_i v_\Lambda \otimes (F_i v) t^{r-\delta_{i,0}}).$$

Clearly, the element $F_i v_\Lambda \otimes F_i vt^{r-\delta_{i,0}}$ is a linear combination of $F_i(v_\Lambda \otimes (F_i v)t^{r-\delta_{i,0}})$ and $v_\Lambda \otimes (F_i^2 v)t^{r-2\delta_{i,0}}$ which are both of the required form.

Now suppose that, for all $k = 1, \ldots, s-1$, we can write $F_i^{\Lambda_i+1-k}(F_i^k v_\Lambda \otimes vt^r)$ as a linear combination of terms which have the required form and $F_i^{\Lambda_i - k}(F_i^{k+1} v_\Lambda \otimes vt^r)$. Then

$$F_i^{\Lambda_i+1-s}(F_i^s v_\Lambda \otimes vt^r) = F_i^{\Lambda_i - s}(F_i^{s+1} v_\Lambda \otimes vt^r + F_i^s v_\Lambda \otimes F_i vt^{r-\delta_{i,0}}).$$

Now, the second term can be written as a linear combination of terms which have the required form by the induction hypothesis. Hence we can repeat the process until we get to $s = \Lambda_i$ in which case $F_i^{s+1}$ annihilates $v_\Lambda$ and we are done. $\square$

**6.2.** Set $m = m(\boldsymbol{\pi})$. By Proposition 4.2, the module $X(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))$ admits a **Z**-filtration $\mathscr{X}_n^{(s)} = \widehat{\mathbf{U}}_q(v_\Lambda \otimes v_{\boldsymbol{\pi}} t^{mn+s})$. Let $v_0, \ldots, v_N$ be the basis of $V(\boldsymbol{\pi})$ introduced in 4.4.

**Lemma.** *Let $v \in V(\boldsymbol{\pi})$, $s, n \in \mathbf{N}$. Suppose that there exists $R \in \mathbf{N}$ and $X_{j,r} \in \widehat{\mathbf{U}}_q^-$, $j = 0, \ldots, N$, $n + s \leqslant r \leqslant R$ such that*

$$v_\Lambda \otimes vt^n = \sum_{j=0}^{N} \sum_{r=n+s}^{R} X_{j,r}(v_\Lambda \otimes v_j t^r). \tag{6.1}$$

*Let $R_0$ be the minimal value of $R$ for which such an expression exists. Then $R_0 < \lambda_{\boldsymbol{\pi}}(\theta^\vee) + n + s$.*

**Proof.** Assume for a contradiction that $R_0 \geqslant \lambda_{\boldsymbol{\pi}}(\theta^\vee) + n + s$. Let $j_0$ be minimal such that $X_{j_0,R_0}(v_\Lambda \otimes v_{j_0} t^{R_0}) \neq 0$. By Proposition 4.5(ii) and Proposition 2.2(ii) $X_{j_0,R_0} = \sum_{i \in \hat{I}} y_i F_i^{\Lambda_i+1}$ for some $y_i \in \widehat{\mathbf{U}}_q^-$. If $i \in I$, then by Lemma 6.1 $y_i F_i^{\Lambda_i+1}(v_\Lambda \otimes v_{j_0} t^{R_0})$ is a linear combination of the elements $y_i F_i^{\Lambda_i-k+1}(v_\Lambda \otimes F_i^k v_{j_0} t^{R_0})$, $k = 1, \ldots, \Lambda_i + 1$. But these terms are all of the form $X'_{j',R_0}(v_\Lambda \otimes v_{j'} t^{R_0})$ with $j' > j_0$ and $X'_{j',R_0} \in \widehat{\mathbf{U}}_q^-$. By Lemma 6.1 and Lemma 2.6 we conclude that $y_0 F_0^{\Lambda_0+1}(v_\Lambda \otimes v_{j_0} t^{R_0})$ is a linear combination of terms of the form $y_0 F_0^{\Lambda_0+1-k}(v_\Lambda \otimes F_0^k v_{j_0} t^{R_0-k})$ where $1 \leqslant k \leqslant \min\{\Lambda_0 + 1, \lambda_{\boldsymbol{\pi}}(\theta^\vee)\}$. Observe that $R_0 - k \geqslant R_0 - \lambda_{\boldsymbol{\pi}}(\theta^\vee) \geqslant n + s$. Thus, we have obtained another expression of form (6.1) where either $R < R_0$ or $R = R_0$ and the minimal value of $j$ such that $X_{j,R_0}(v_\Lambda \otimes v_j t^{R_0}) \neq 0$ is strictly greater than $j_0$. The former situation cannot occur by the choice of $R_0$. On the other hand, since $V(\boldsymbol{\pi})$ is finite-dimensional, using the above argument repeatedly we must eventually reach a stage where $n+s \leqslant R < R_0$ which is a contradiction. $\square$

**6.3.** Theorem 3 is an immediate consequence of the following.

**Proposition.** *Suppose that $\Lambda(\alpha_0^\vee) \geqslant \lambda_{\boldsymbol{\pi}}(\theta^\vee)+m-1$. Then the $\mathbf{Z}$-filtration $\mathscr{X}_n^{(s)}$ on $X(\Lambda) \otimes L^s(V(\boldsymbol{\pi}))$ is strictly decreasing.*

**Proof.** Assume for a contradiction that $\mathscr{X}_0^{(s)} = \mathscr{X}_1^{(s)}$. Then, as in Proposition 4.6, we can write

$$v_\Lambda \otimes v_0 t^s = \sum_{r=m+s}^{R} \sum_{j=0}^{N} X_{j,r}(v_\Lambda \otimes v_j t^r) \tag{6.2}$$

for some $R \geqslant m+s$ and $X_{j,r} \in (\widehat{\mathbf{U}}_q^-)_+$ with $X_{j,R}(v_\Lambda \otimes v_j t^R) \neq 0$ for some $0 \leqslant j \leqslant N$.

Assume that $R$ is minimal such that the expression of form (6.2) exists. Then $m+s \leqslant R < \lambda_{\boldsymbol{\pi}}(\theta^\vee)+m+s$ by Lemma 6.2. Furthermore, let $j_0$ be the minimal value of $j$, such that $X_{j,R}(v_\Lambda \otimes v_j t^R) \neq 0$. Then by Proposition 4.5(ii) and Proposition 2.2(ii), $X_{j_0,R} = \sum_{i\in I} y_i F_i^{\Lambda_i+1}$ for some $y_i \in \widehat{\mathbf{U}}_q^-$. Furthermore, $X_{j_0,R}$ is of weight $-(R-s)\delta + \mathrm{wt}\, v_0 - \mathrm{wt}\, v_j \in -(R-s)\alpha_0 + Q^+$. On the other hand, the weight of $y_0 F_0^{\Lambda_0+1}$ is contained in the set $-(\Lambda_0+1)\alpha_0 - \widehat{Q}^+$. Since $R-s < \lambda_{\boldsymbol{\pi}}(\theta^\vee)+m \leqslant \Lambda_0+1$ we conclude that $y_0 F_0^{\Lambda_0+1}(v_\Lambda \otimes v_{j_0} t^R) = 0$. It follows that $X_{j_0,R}(v_\Lambda \otimes v_{j_0} t^R) = \sum_{i\in I} y_i F_i^{\Lambda_i+1}(v_\Lambda \otimes v_{j_0} t^R)$. Thus, by Lemma 6.1 $X_{j_0,R}(v_\Lambda \otimes v_{j_0} t^R)$ is a linear combination of terms of the form $X_{j',R}(v_\Lambda \otimes v_{j'} t^R)$ with $j' > j_0$. Since $V(\boldsymbol{\pi})$ is finite-dimensional, repeating this process we obtain an expression of form (6.2) with $X_{j,R}(v_\Lambda \otimes v_j t^R) = 0$ for all $0 \leqslant j \leqslant N$ which contradicts the minimality of $R$. $\square$

## 7. Structure of $\mathscr{X}_n/\mathscr{X}_{n+1}$ in some special cases

In this section we analyze the quotient modules $\mathscr{X}_n/\mathscr{X}_{n+1}$ in the special case when $\mathfrak{g}$ is of type $A_\ell, B_\ell, C_\ell, D_\ell$ and $V(\boldsymbol{\pi})$ is isomorphic as a $U_q(\mathfrak{g})$-module to the natural representation of $U_q(\mathfrak{g})$.

**7.1.** Assume that the nodes of the Dynkin diagram of $\mathfrak{g}$ are numbered as in [17, Section 4.8]. Then $V(\varpi_1)$ is the natural representation of $U_q(\mathfrak{g})$ and for the rest of the section we set $\varpi = \varpi_1$. Moreover, it is known (cf. say [4]) that if we define an $\ell$-tuple of polynomials $\boldsymbol{\varpi} = (\pi_i(u))_{i\in I}$ by $\pi_i(u) = \delta_{i,1}(1-u)$, then $V(\boldsymbol{\varpi}) \cong V(\varpi)$ as $U_q(\mathfrak{g})$-modules.

**Lemma.** *We have* $\dim V(\boldsymbol{\varpi})_\mu = 1$ *for all* $\mu \in \Omega(V(\boldsymbol{\varpi}))$ *and hence* $k(\boldsymbol{\varpi}) = 1$. *Moreover, for all* $v \in V(\boldsymbol{\varpi})$, $i \in \widehat{I}$,

$$E_i^2 v = 0 = F_i^2 v$$

*if* $\mathfrak{g}$ *is of type* $A_\ell, C_\ell$ *or* $D_\ell$, *and*

$$E_i^{2+\delta_{i,\ell}} v = 0 = F_i^{2+\delta_{i,\ell}} v$$

*if* $\mathfrak{g}$ *is of type* $B_\ell$.

**Proof.** To prove the first statement, it is enough to note that by [19] the corresponding statements hold for the $U_q(\mathfrak{g})$-module $V(\varpi)$. If $i \in I$, then the second statements also follow for the same reason. If $i = 0$, then the result follows from Lemma 2.6. □

Notice that the conditions of Theorem 2 are not satisfied for the module $L(V(\boldsymbol{\varpi}))$ and $\Lambda \in \widehat{P}^+$ which is not a multiple of $\delta$. Indeed, the first condition of Theorem 2 reads $2(\Lambda, \delta) < (\Lambda, \alpha_1) + 1$ or, equivalently, $(\Lambda, \delta - \alpha_1) + (\Lambda, \delta) < 1$ which is a contradiction unless $\Lambda \in \mathbf{Z}\delta$. On the other hand, $w_\circ \varpi_1 = -\varpi_{r'}$ for some $r' \in I$ and so the second condition of Theorem 2 yields $(\Lambda, \delta) + (\Lambda, \alpha_{r'}) < 1$ which is impossible if $\Lambda \notin \mathbf{Z}\delta$.

Recall the function $n : V(\boldsymbol{\pi}) \to \mathbf{N}$ defined in Section 2. Since the weight spaces of $V(\boldsymbol{\varpi})$ are one-dimensional, it is convenient to think of this as a function from $\Omega(V(\boldsymbol{\varpi})) \to \mathbf{N}$. We continue to denote this function by $n$.

The main result of this section is the following.

**Theorem 4.** *Let* $\Lambda \in \widehat{P}^+$ *and assume that* $\Lambda$ *is not a multiple of* $\delta$. *Then the filtration* $\mathscr{X}_n$ *on* $X(\Lambda) \otimes L(V(\boldsymbol{\varpi}))$ *is strictly decreasing. Further,*

(i) *suppose that* $\mathfrak{g}$ *is of type* $A_\ell$, $C_\ell$ *or* $D_\ell$. *Then,*

$$\mathscr{X}_n/\mathscr{X}_{n+1} \cong \bigoplus_{\mu \in \Omega_\Lambda(\varpi)} X(\Lambda + \mu + (n + n(\mu))\delta),$$

*where* $\Omega_\Lambda(\boldsymbol{\varpi}) = \{\mu \in \Omega(V(\boldsymbol{\varpi})) : \Lambda + \mu \in \widehat{P}^+\}$.

(ii) *If* $\mathfrak{g}$ *is of type* $B_\ell$, *set*

$$\Omega_\Lambda(\boldsymbol{\varpi}) = \begin{cases} \{\mu \in \Omega(V(\boldsymbol{\varpi})) : \Lambda + \mu \in \widehat{P}^+\}, & \Lambda(\alpha_\ell^\vee) > 0 \\ \{\mu \in \Omega(V(\boldsymbol{\varpi})) \setminus \{0\} : \Lambda + \mu \in \widehat{P}^+\}, & \Lambda(\alpha_\ell^\vee) = 0. \end{cases}$$

*Then*

$$\mathscr{X}_n/\mathscr{X}_{n+1} \cong \bigoplus_{\mu \in \Omega_\Lambda(\varpi)} X(\Lambda + \mu + (n + n(\mu))\delta).$$

We prove this theorem in the rest of the section. By Proposition 4.2 it is enough to consider the case $n = 0$.

**7.2.** We will need the following.

**Lemma.** *Suppose that there exist a sequence of integrable* $\widehat{\mathbf{U}}_q$*-modules* $\mathscr{V}_0 \supseteq \mathscr{V}_1 \supseteq \cdots \supseteq \mathscr{V}_K$ *such that* $\mathscr{V}_0/\mathscr{V}_K$ *is a module in the category* $\mathcal{O}$ *and either* $\mathscr{V}_i = \mathscr{V}_{i+1}$ *or* $\mathscr{V}_i/\mathscr{V}_{i+1} \cong X(\mu_i)$ *for some* $\mu_i \in \widehat{P}^+$. *Set* $J = \{0 \leqslant i < K : \mathscr{V}_i \supsetneq \mathscr{V}_{i+1}\}$ *and suppose that* $J$ *is not empty. Then* $\mathscr{V}_0/\mathscr{V}_K \cong \bigoplus_{i\in J} X(\mu_i)$.

**Proof.** We argue by induction on the cardinality of $J$. Suppose first that $J = \{i\}$ for some $0 \leqslant i \leqslant K$. Then

$$\mathscr{V}_K = \cdots = \mathscr{V}_{i+1} \subsetneq \mathscr{V}_i = \mathscr{V}_{i-1} = \cdots = \mathscr{V}_0$$

and so $\mathscr{V}_0/\mathscr{V}_K = \mathscr{V}_i/\mathscr{V}_{i+1} \cong X(\mu_i)$.

If the cardinality of $J$ is greater than 1, let $i_1$ be the minimal element of $J$, that is. $\mathscr{V}_{i_1} = \mathscr{V}_{i_1-1} = \cdots = \mathscr{V}_0$ and $\mathscr{V}_{i_1+1} \subsetneq \mathscr{V}_{i_1}$. Then we have the following short exact sequence:

$$0 \longrightarrow \mathscr{V}_{i_1+1}/\mathscr{V}_K \longrightarrow \mathscr{V}_0/\mathscr{V}_K \longrightarrow \mathscr{V}_0/\mathscr{V}_{i_1+1} \longrightarrow 0.$$

All modules involved are in the category $\mathcal{O}$ and integrable. Therefore, this short exact sequence splits and so $\mathscr{V}_0/\mathscr{V}_K = \mathscr{V}_0/\mathscr{V}_{i_1+1} \oplus \mathscr{V}_{i_1+1}/\mathscr{V}_K \cong X(\mu_{i_1}) \oplus \mathscr{V}_{i_1+1}/\mathscr{V}_K$. The lemma follows by applying the induction hypothesis to the sequence $\mathscr{V}_K \subseteq \cdots \subseteq \mathscr{V}_{i_1+1}$. $\square$

**7.3.** Let $\mathfrak{g}$ be of type $A_\ell$ and let $v_0$ be a highest weight vector in $V(\varpi)$. Set

$$w_0 = v_0, \quad w_1 = E_0 w_0, \quad w_j = E_{\ell-j+2} w_{j-1}, \qquad 2 \leqslant j \leqslant \ell + 1.$$

Then $w_{\ell+1} = w_0$ and the elements $w_j$, $0 \leqslant j \leqslant \ell$ form a basis of the $\widehat{\mathbf{U}}'_q$-module $V(\boldsymbol{\varpi})$. Set $F_{\ell+1} = F_0$. It is easy to see that

$$E_j w_i = \delta_{j,\ell-i+1} w_{i+1}, \qquad F_j w_{i+1} = \delta_{j,\ell-i+1} w_i, \qquad j \in \widehat{I}, \quad 0 \leqslant j \leqslant \ell,$$

and $n(w_j) = 1 - \delta_{j,0}$. The elements $w_j t^n$, $0 \leqslant j \leqslant \ell$, $n \in \mathbf{Z}$, form a basis of the $\widehat{\mathbf{U}}_q$-module $L(V(\boldsymbol{\pi}))$ and we have

$$E_j w_i t^n = \delta_{j,\ell-i+1} w_{i+1} t^{n+\delta_{j,0}}, \qquad F_j w_{i+1} t^n = \delta_{j,\ell-i+1} w_{i-1} t^{n-\delta_{j,0}}. \tag{7.1}$$

Define $\mathscr{X}_{n,j} = \widehat{\mathbf{U}}_q(v_\Lambda \otimes w_j t^{n+1-\delta_{j,0}})$, $0 \leqslant j \leqslant \ell + 1$. Then $\mathscr{X}_{n,\ell+1} = \mathscr{X}_{n+1,0}$.

**Lemma.**

(i) *For all* $n \in \mathbf{Z}$, *we have*

$$\mathscr{X}_{n,0} \supseteq \mathscr{X}_{n,1} \supseteq \cdots \supseteq \mathscr{X}_{n,\ell} \supseteq \mathscr{X}_{n+1,0}.$$

*Further,* $\mathscr{X}_{n,j} \supsetneq \mathscr{X}_{n,j+1}$ *implies*

$$\mathscr{X}_{n,j}/\mathscr{X}_{n,j+1} \cong X(\Lambda + \mathrm{wt}\, w_j + (n+1-\delta_{j,0})\delta).$$

(ii) *For all* $i \in \widehat{I}$, $0 \leqslant j \leqslant \ell$, *we have*

$$F_i^{\Lambda_i+1}(v_\Lambda \otimes w_{j+1}t^n) = a_i \delta_{i,\ell-j+1} F_i^{\Lambda_i}(v_\Lambda \otimes w_j t^{n-\delta_{i,0}}),$$

*for some* $a_i \in \mathbf{C}(q)^\times$.

**Proof.** Part (i) is immediate from (7.1). Part (ii) follows from (7.1) as well by applying Lemmas 6.1 and 7.1. □

Applying Lemma 7.2 we conclude that

$$\mathscr{X}_n/\mathscr{X}_{n+1} \cong \bigoplus_{0 \leqslant j \leqslant \ell : \mathscr{X}_{n,j} \neq \mathscr{X}_{n,j+1}} X(\Lambda + \mathrm{wt}\, w_j + (n + n(w_j))\delta).$$

Thus, in this case Theorem 4 is equivalent to the following:

**Proposition.** *For all* $0 \leqslant j \leqslant \ell$, $\mathscr{X}_{0,j} = \mathscr{X}_{0,j+1}$ *if and only if* $\Lambda_{\ell-j+1} = 0$, *where* $\Lambda_{\ell+1} = \Lambda_0$.

**Proof.** If $\Lambda_{\ell-j+1} = 0$, then $F_{\ell-j+1} \in \mathrm{Ann}_{\widehat{\mathbf{U}}_q^-} v_\Lambda$ by Proposition 2.2(ii). Therefore, by (ii) of the above Lemma, $v_\Lambda \otimes w_j t^{1-\delta_{j,0}} = c F_{\ell-j+1}(v_\Lambda \otimes w_{j+1}t)$ for some $c \in \mathbf{C}(q)^\times$ and we are done.

For the converse, suppose that $v_\Lambda \otimes w_j t^{1-\delta_{j,0}} \in \mathscr{X}_{0,j+1}$. It follows from (7.1) that we can write,

$$v_\Lambda \otimes w_j t^{1-\delta_{j,0}} = \sum_{i=j+1}^{\ell+1} Y_i(v_\Lambda \otimes w_i t) + \sum_{r=2}^{R} \sum_{j=0}^{\ell} X_{j,r}(v_\Lambda \otimes w_j t^r) \tag{7.2}$$

for some $Y_i, X_{j,r} \in \widehat{\mathbf{U}}_q^-$. We first prove that there exists an expression of the above form in which the second term is zero. Indeed, suppose that the second term in the right-hand side is always non-zero and let $R$ be minimal such that an expression of form (7.2) exists. Let $j_0$ be such that $X_{j_0,R}(v_\Lambda \otimes w_{j_0} t^R) \neq 0$ and assume that

$\varpi - \mathrm{wt}\, w_{j_0}$ is minimal with this property. Then by Proposition 4.5 $X_{j_0,R} \in \mathrm{Ann}_{\widehat{\mathbf{U}}_q^-} v_\Lambda$, say $X_{j_0,R} = \sum_{i \in \hat{I}} y_i F^{\Lambda_i+1}$ for some $y_i \in \widehat{\mathbf{U}}_q^-$. If $j_0 = 1$, then it follows from Lemma 7.3 that

$$X_{j_0,R}(v_\Lambda \otimes w_{j_0} t^R) = y_0 F_0^{\Lambda_0}(v_\Lambda \otimes w_0 t^{R-1}).$$

Since this is impossible by the choice of $R$ we get $j_0 > 1$. But then Lemma 7.3 implies that we get another expression of form (7.2) with the minimal value of $\varpi - \mathrm{wt}\, w_j$ strictly greater than $\varpi - \mathrm{wt}\, w_{j_0}$. Repeating, we eventually obtain an expression of form (7.2) where the minimal value of $\varpi - \mathrm{wt}\, w_j$ such that $X_{j,R}(v_\Lambda \otimes w_j t^R) \neq 0$ is attained for $j = 1$ which is a contradiction.

Thus, we can write

$$v_\Lambda \otimes w_j t^{1-\delta_{j,0}} = \sum_{i=j+1}^{\ell+1} Y_i(v_\Lambda \otimes w_i t). \tag{7.3}$$

Let $i_0 > j$ be maximal such that an expression of the above from exists and $Y_{i_0}(v_\Lambda \otimes w_{i_0} t) \neq 0$. Then $\varpi - \mathrm{wt}\, v_{i_0}$ is minimal with this property since $i_0 > 0$. Hence $Y_{i_0} \in \mathrm{Ann}_{\widehat{\mathbf{U}}_q^-} v_\Lambda$ by Proposition 4.5. If $i_0 = j+1$, then $Y_{j+1}$ is of weight $-\alpha_{\ell-j+1}$. It follows that $Y_{j+1} = aF_{\ell-j+1}$ for some $a \in \mathbf{C}(q)^\times$. Thus, $F_{\ell-j+1} \in \mathrm{Ann}_{\widehat{\mathbf{U}}_q^-} v_\Lambda$ whence $\Lambda_{\ell-j+1} = 0$. If $i_0 > j+1$, then $Y_{i_0}$ is of weight $-(\alpha_{\ell-i_0+1} + \cdots + \alpha_{\ell-j+1})$. Since $F_j w_{i_0} = 0$ unless $j = \ell - i_0 + 2$, we conclude that $Y_{i_0}(v_\Lambda \otimes w_{i_0} t) = yF_{\ell-i_0+2}(v_\Lambda \otimes w_{i_0} t)$ and $\Lambda_{\ell-i_0+2} = 0$. Thus, we get an expression of form (7.3) where the maximal $i > j$ such that $Y_i(v_\Lambda \otimes w_i t) \neq 0$ is at most $i_0 - 1$. Repeating the argument, we get to the case $i_0 = j+1$ which has already been considered. □

**7.4.** Suppose that $\mathfrak{g}$ is of type $C_\ell$. Then $n(\boldsymbol{\varpi}) = 1$. Let $v_0$ be a highest weight vector of $V(\varpi)$ and set

$$\begin{aligned} w_0 &= v_0, \\ w_j &= E_{j-1} w_{j-1}, && 1 \leqslant j \leqslant \ell+1, \\ w_{\ell+j+1} &= E_{\ell-j} w_{\ell+j},\ j \leqslant \ell-2, && 1 \leqslant j \leqslant \ell-1. \end{aligned}$$

Then $w_0, \ldots, w_{2\ell-1}$ form a basis of $V(\boldsymbol{\varpi})$, $w_{2\ell} = w_0$ and $n(w_j) = 1 - \delta_{j,0}$. Set $\mathscr{X}_{n,j} = \widehat{\mathbf{U}}_q(v_\Lambda \otimes w_j t^{n+n(w_j)})$, $0 \leqslant j \leqslant 2\ell$. In particular, $\mathscr{X}_{n,2\ell} = \mathscr{X}_{n+1,0}$. Then, as in the case considered in 7.3, Theorem 4 is equivalent to the following:

**Proposition.** *For* $0 \leqslant j \leqslant \ell$, $\mathscr{X}_{0,j} = \mathscr{X}_{0,j+1}$ *if and only if* $\Lambda_j = 0$. *Similarly, for* $1 \leqslant j \leqslant \ell-1$, $\mathscr{X}_{0,\ell+j} = \mathscr{X}_{0,\ell+j+1}$ *if and only if* $\Lambda_{\ell-j} = 0$.

The proof repeats that of Proposition 7.3 with the obvious changes of notations and we omit the details.

**7.5.** Let $\mathfrak{g}$ be of type $B_\ell$. In this case the situation is somewhat more complicated since $n(\boldsymbol{\varpi}) = 2$. Let $v_0$ be a highest weight vector of $V(\boldsymbol{\varpi})$ and set

$$\begin{aligned} &w_0 = v_0, \quad w_1 = E_0 w_0, \\ &w_j = E_j w_{j-1}, && 2 \leqslant j \leqslant \ell, \\ &w_{\ell+j+1} = E_{\ell-j} w_{\ell+j}, && 0 \leqslant j \leqslant \ell - 2, \\ &w_{2\ell} = E_0 w_{2\ell-1}. \end{aligned}$$

Then $w_0, \ldots, w_{2\ell}$ form a basis of $V(\boldsymbol{\varpi})$. Set $w_{2\ell+1} = w_0$. We also have

$$\begin{aligned} &F_i w_0 = \delta_{i,1} a_0 w_{2\ell-1}, \quad F_i w_1 = \delta_{i,1} a_1 w_{2\ell} + \delta_{i,0} w_0, \\ &F_i w_j = \delta_{i,j} w_{j-1}, && 2 \leqslant j \leqslant \ell, \\ &F_i w_{\ell+j+1} = \delta_{i,\ell-j} w_{\ell+j}, && 0 \leqslant j \leqslant \ell - 2, \\ &F_i w_{2\ell} = \delta_{i,0} w_{2\ell-1}. \end{aligned}$$

One can easily check that $n(w_j) = 1 - \delta_{j,0} + \delta_{j,2\ell}$. Define $\mathscr{X}_{n,j} = \widehat{\mathbf{U}}_q(v_\Lambda \otimes w_j t^{n+n(w_j)})$, $j = 0, \ldots, 2\ell$.

We have the following analog of Lemma 7.3.

**Lemma.**

(i) *For all $n \in \mathbf{Z}$, we have*

$$\mathscr{X}_{n,0} \supseteq \mathscr{X}_{n,1} \supseteq \cdots \supseteq \mathscr{X}_{n,2\ell-1} \supseteq \mathscr{X}_{n,2\ell} + \mathscr{X}_{n+1,0} \supseteq \mathscr{X}_{n+1,0}.$$

*Furthermore,* $\mathscr{X}_{n,j} \supsetneq \mathscr{X}_{n,j+1}$, $0 \leqslant j \leqslant 2\ell - 2$ *implies*

$$\mathscr{X}_{n,j} / \mathscr{X}_{n,j+1} \cong X(\Lambda + \operatorname{wt} w_j + (n + n(w_j))\delta).$$

*Similarly,* $\mathscr{X}_{n,2\ell-1} \supsetneq \mathscr{X}_{n,2\ell} + \mathscr{X}_{n+1,1}$ *implies*

$$\mathscr{X}_{n,2\ell-1} / (\mathscr{X}_{n,2\ell} + \mathscr{X}_{n+1,1}) \cong X(\Lambda + \operatorname{wt} w_{2\ell-1} + (n + n(w_{2\ell-1}))\delta),$$

*whilst* $\mathscr{X}_{n,2\ell} + \mathscr{X}_{n+1,0} \supsetneq \mathscr{X}_{n+1,0}$ *implies*

$$(\mathscr{X}_{n,2\ell} + \mathscr{X}_{n+1,0}) / \mathscr{X}_{n+1,1} \cong X(\Lambda + \operatorname{wt} w_{2\ell} + (n + n(w_{2\ell}))\delta).$$

(ii) *For all $i \in \widehat{I}$, we have*

$$
\begin{aligned}
&F_i^{\Lambda_i+1}(v_\Lambda \otimes w_0 t^n) = a_0\delta_{i,1}F_i^{\Lambda_i}(v_\Lambda \otimes w_{2\ell-1}t^n),\\
&F_i^{\Lambda_i+1}(v_\Lambda \otimes w_1 t^n) = a_1\delta_{i,1}F_i^{\Lambda_i}(v_\Lambda \otimes w_{2\ell-1}t^n) + \delta_{i,0}F_i^{\Lambda_i}(v_\Lambda \otimes w_0 t^{n-1}),\\
&F_i^{\Lambda_i+1}(v_\Lambda \otimes w_j t^n) = \delta_{i,j}F_i^{\Lambda_i}(v_\Lambda \otimes w_{j-1}t^n), \qquad 2 \leqslant j \leqslant \ell,\\
&F_i^{\Lambda_i+1}(v_\Lambda \otimes w_{\ell+j+1} t^n) = \delta_{i,\ell-j}F_i^{\Lambda_i}(v_\Lambda \otimes w_{\ell+j}t^n)\\
&\qquad\qquad + \delta_{j,0}\delta_{i,\ell}F_i^{\Lambda_i-1}(v_\Lambda \otimes w_{\ell-1}t^n), \quad 0 \leqslant j \leqslant \ell-2,\\
&F_i^{\Lambda_i+1}(v_\Lambda \otimes w_{2\ell} t^n) = \delta_{i,0}F_i^{\Lambda_i}(v_\Lambda \otimes w_{2\ell-1}t^{n-1}).
\end{aligned}
$$

Thus, Theorem 4 reduces to the following:

**Proposition.**

$$
\begin{aligned}
&\mathscr{X}_{0,0} = \mathscr{X}_{0,1} && \Longleftrightarrow \Lambda_0 = 0, &&\\
&\mathscr{X}_{0,j-1} = \mathscr{X}_{0,j} && \Longleftrightarrow \Lambda_j = 0, && 2 \leqslant j \leqslant \ell,\\
&\mathscr{X}_{0,\ell+j+1} = \mathscr{X}_{0,\ell+j} && \Longleftrightarrow \Lambda_{\ell-j} = 0, && 0 \leqslant j \leqslant \ell-2,\\
&\mathscr{X}_{0,2\ell-1} = \mathscr{X}_{0,2\ell} + \mathscr{X}_{1,0} && \Longleftrightarrow \Lambda_0 = 0 \text{ or } \Lambda_1 = 0, &&\\
&\mathscr{X}_{0,2\ell} + \mathscr{X}_{1,0} = \mathscr{X}_{1,0} && \Longleftrightarrow \Lambda_1 = 0. &&
\end{aligned}
$$

**Proof.** The only if direction follows in all cases from Proposition 2.2(ii) and the formulae in (ii) of the above Lemma.

For the converse, we consider three separate cases.

*Case* 1. $\mathscr{X}_{0,j} = \mathscr{X}_{0,j+1}$, $0 \leqslant j < 2\ell - 1$.

We can write

$$
\begin{aligned}
v_\Lambda \otimes w_j t^{1-\delta_{j,0}} = &\sum_{i=j+1}^{2\ell-1} Y_i(v_\Lambda \otimes w_i) + Y_{2\ell+1}(v_\Lambda \otimes w_{2\ell+1}t)\\
&+ \sum_{r=2}^{R}\sum_{i=0}^{2\ell} X_{i,r}(v_\Lambda \otimes w_i t^r).
\end{aligned}
$$

Arguing exactly as in Proposition 7.3, but using Lemma 7.5 instead, we conclude that there exists an expression of the form

$$
v_\lambda \otimes w_j t^{1-\delta_{j,0}} = \sum_{i=j+1}^{2\ell-1} Y_i(v_\Lambda \otimes w_i t) + Y_{2\ell+1}(v_\Lambda \otimes w_{2\ell+1}t). \tag{7.4}
$$

Let $i_0 > j$ be maximal such that an expression of the above from exists and $Y_{i_0}(v_\Lambda \otimes w_{i_0}t) \neq 0$. Then $\varpi - \mathrm{wt}\, v_{i_0}$ is minimal with this property since $i_0 > 0$. Hence $Y_{i_0} \in \mathrm{Ann}_{\widehat{\mathbf{U}}_q^-} v_\Lambda$ by Proposition 4.5. If $i_0 = j+1$, then $Y_{j+1}$ is of weight $-\alpha_{j+1}$ if $2 \leqslant j < \ell$ and of weight $-\alpha_{2\ell-j+1}$ if $j \geqslant 1$. It follows that $Y_{j+1} = aF_{j+1}$ (respectively, $aF_{2\ell-j+1}$) for some $a \in \mathbf{C}(q)^\times$ and hence $\Lambda_{j+1} = 0$ (respectively, $\Lambda_{2\ell-j+1} = 0$). Suppose that $i_0 > j+1$ and set

$$k = \begin{cases} i_0, & 2 \leqslant i_0 \leqslant \ell, \\ 2\ell - i_0 + 1, & \ell < i_0 \leqslant 2\ell - 1, \\ 1, & i_0 = 2\ell + 1. \end{cases}$$

Then $F_i w_{i_0} = 0$ unless $i = k$. Therefore, $Y_{i_0}(v_\Lambda \otimes w_{i_0}t) = yF_k^{\Lambda_k+1}(v_\Lambda \otimes w_{i_0}t)$ for some $y \in \widehat{\mathbf{U}}_q^-$. Using Lemma 6.1 and the formulae in Lemma 7.5(ii) we obtain an expression of form (7.4) where the maximal $i > j$ such that $Y_i(v_\Lambda \otimes w_i t) \neq 0$ is at most $i_0 - 1$ if $i_0 \leqslant 2\ell - 1$ and at most $2\ell - 1$ if $i_0 = 2\ell + 1$. Repeating this argument we reduce to the case $i_0 = j + 1$ which has already been considered.

*Case* 2. $\mathscr{X}_{0,2\ell-1} = \mathscr{X}_{0,2\ell} + \mathscr{X}_{1,0}$.

In this case we should prove that either $\Lambda_0 = 0$ or $\Lambda_1 = 0$. Suppose that there exists an expression,

$$v_\Lambda \otimes w_{2\ell-1}t = Y(v_\Lambda \otimes w_0 t) + \sum_{r=2}^{R} \sum_{i=0}^{2\ell} X_{i,r}(v_\Lambda \otimes w_i t^r).$$

Using Lemma 7.5 we see that as usual there must exist an expression of the form

$$v_\Lambda \otimes w_{2\ell-1}t = Y_1(v_\Lambda \otimes w_0 t) + Y_2(v_\Lambda \otimes w_{2\ell}t^2)$$

for some $Y_1, Y_2 \in \widehat{\mathbf{U}}_q^-$. If $Y_2(v_\Lambda \otimes w_{2\ell}t^2) \neq 0$, then by Proposition 4.5 we get that $Y_2 \in \mathrm{Ann}_{\widehat{\mathbf{U}}_q^-} v_\Lambda$. On the other hand since $\mathrm{wt}\, w_{2\ell-1}t - \mathrm{wt}\, w_{2\ell}t^2 = -\alpha_0$, we see that $Y_2 = aF_0 \in \mathrm{Ann}_{\widehat{\mathbf{U}}_q^-} v_\Lambda$ for some $a \in \mathbf{C}(q)^\times$. Hence $\Lambda_0 = 0$ and we are done. Otherwise $Y_2(v_\Lambda \otimes w_{2\ell}t^2) = 0$ and then $Y_1 \in \mathrm{Ann}_{\widehat{\mathbf{U}}_q^-} v_\Lambda$. Again since $Y_1$ has weight $-\alpha_1$, it follows that $\Lambda_1 = 0$ and the proof of case 2 is complete.

*Case* 3. $\mathscr{X}_{0,2\ell} + \mathscr{X}_{1,0} = \mathscr{X}_{1,0}$.

In this case we can write

$$v_\Lambda \otimes w_{2\ell}t^2 = \sum_{i=0}^{2\ell-1} Y_i(v_\Lambda \otimes w_i t^{2-\delta_{i,0}}) + Y_{2\ell+1}(v_\Lambda \otimes w_{2\ell+1}t^2) + \sum_{r=3}^{R} \sum_{i=0}^{2\ell} X_{i,r}(v_\Lambda \otimes w_i t^r)$$

for some $Y_i, X_{i,r} \in \widehat{\mathbf{U}}_q^-$. Observe first that $Y_0 \in \widehat{\mathbf{U}}_q^-$ must be of weight $\mathrm{wt}\, w_{2\ell}t^2 - \mathrm{wt}\, w_0 t \in \alpha_0 - Q^+ \notin -\widehat{Q}^+$. Thus, $Y_0 = 0$. Furthermore, using Lemma 7.5, we can reduce the above expression to

$$v_\Lambda \otimes w_{2\ell} t^2 = \sum_{i=1}^{2\ell-1} Y_i(v_\Lambda \otimes w_i t^2) + Y_{2\ell+1}(v_\Lambda \otimes w_{2\ell+1} t^2). \tag{7.5}$$

Let $i_0$ be maximal $i$ such that $Y_i(v_\Lambda \otimes w_i t^2) \neq 0$. Then $\varpi - \mathrm{wt}\, v_{i_0} < \varpi_i - \mathrm{wt}\, v_i$ for all $i < i_0$, $i \neq 2\ell$ and so $Y_{i_0} \in \mathrm{Ann}_{\widehat{\mathbf{U}}_q^-} v_\Lambda$ by Proposition 4.5. Suppose first that $i_0 = 1$. Then $Y_{i_0}$ is of weight $-\alpha_1$ and so $Y_{i_0} = aF_1$ for some $a \in \mathbf{C}(q)^\times$. Then $\Lambda_1 = 0$ by Proposition 2.2(ii). To complete the proof, it remains to observe that the case $i_0 > 1$ can be reduced to the case $i_0 = 1$ by an argument similar to the one in Case 1. $\square$

**7.6.** Finally, let $\mathfrak{g}$ be of type $D_\ell$, $\ell \geqslant 4$. In this case we also have $n(\boldsymbol{\varpi}) = 2$. Define

$$\begin{aligned}
&w_0 = v_0, \quad w_1 = E_0 w_0,\\
&w_j = E_j w_{j-1}, && 2 \leqslant j \leqslant \ell - 1,\\
&w_\ell = E_\ell w_{\ell-2},\\
&w_{\ell+1} = E_\ell w_{\ell-1} = E_{\ell-1} w_\ell,\\
&w_{\ell+j} = E_{\ell-j} w_{\ell+j-1}, && 2 \leqslant j \leqslant \ell - 2,\\
&w_{2\ell-1} = E_0 w_{2\ell-2}.
\end{aligned}$$

Then $w_0, \ldots, w_{2\ell-1}$ form a basis of $V(\boldsymbol{\varpi})$. One can easily check that $n(w_j) = 1 - \delta_{j,0} + \delta_{j,2\ell-1}$. Define $\mathscr{X}_{n,j} = \widehat{\mathbf{U}}_q(v_\Lambda \otimes w_j t^{n+n(w_j)})$, $j = 0, \ldots, 2\ell - 1$. Then

$$\begin{aligned}
\mathscr{X}_{n,0} &\supseteq \mathscr{X}_{n,1} \supseteq \cdots \supseteq \mathscr{X}_{n,\ell-2} \supseteq \mathscr{X}_{n,\ell-1} + \mathscr{X}_{n,\ell} \supseteq \mathscr{X}_{n,\ell+1}\\
&\supseteq \mathscr{X}_{n,\ell+2} \supseteq \cdots \supseteq \mathscr{X}_{n,2\ell-2} \supseteq \mathscr{X}_{n+1,0} + \mathscr{X}_{n,2\ell-1} \supseteq \mathscr{X}_{n+1,0}
\end{aligned}$$

Theorem 4 is thus equivalent to the following:

**Proposition.**

$$\begin{aligned}
&\mathscr{X}_{0,0} = \mathscr{X}_{0,1} && \iff \Lambda_0 = 0,\\
&\mathscr{X}_{0,j} = \mathscr{X}_{0,j+1} && \iff \Lambda_{j+1} = 0, \qquad 1 \leqslant j \leqslant \ell - 3,\\
&\mathscr{X}_{0,\ell-2} = \mathscr{X}_{0,\ell-1} + \mathscr{X}_{0,\ell} && \iff \Lambda_{\ell-1} = 0 \textit{ or } \Lambda_\ell = 0,\\
&\mathscr{X}_{0,\ell-1} = \mathscr{X}_{0,\ell+1} && \iff \Lambda_\ell = 0,\\
&\mathscr{X}_{0,\ell} = \mathscr{X}_{0,\ell+1} && \iff \Lambda_{\ell-1} = 0,\\
&\mathscr{X}_{0,\ell+j-1} = \mathscr{X}_{0,\ell+j} && \iff \Lambda_{\ell-j} = 0, \qquad 1 \leqslant j \leqslant \ell - 2,\\
&\mathscr{X}_{0,2\ell-2} = \mathscr{X}_{0,2\ell-1} + \mathscr{X}_{1,0} && \iff \Lambda_0 = 0 \textit{ or } \Lambda_1 = 0,\\
&\mathscr{X}_{0,2\ell-1} + \mathscr{X}_{1,0} = \mathscr{X}_{1,0} && \iff \Lambda_1 = 0.
\end{aligned}$$

The proof is similar to that of Proposition 7.5 with the obvious changes in notations.

**Remark.** It is known (cf. for example [21]) that the modules $L(V(\boldsymbol{\varpi}))$ considered in 7.3–7.6 admit crystal bases which in turn admit a simple realization in the framework of Littelmann's path model. Let $\widehat{\mathscr{B}}(\boldsymbol{\varpi})$ (respectively, $\mathscr{B}(\Lambda)$) be a subcrystal of Littelmann's path crystal isomorphic to a crystal basis of $L(V(\boldsymbol{\varpi}))$ (respectively, to a crystal basis of $X(\Lambda)$). Then the concatenation product $\mathscr{B}(\Lambda) \otimes \widehat{\mathscr{B}}(\boldsymbol{\varpi})$ contains a subcrystal which is a disjoint union of indecomposable crystals isomorphic to $\mathscr{B}(\Lambda + \mathrm{wt}\, b)$, where $b$ runs over the set of $\Lambda$-dominant elements in $\widehat{\mathscr{B}}(\boldsymbol{\varpi})$. For the special cases considered above the two are actually isomorphic (this is proven for the type $A_\ell$ in [14], but the argument given where remains valid for the modules considered in 7.4 and 7.6 and can be easily modified for the module considered in 7.5). Moreover, one can check that there is a bijection between the set of $\Lambda$-dominant elements of $\widehat{\mathscr{B}}(\boldsymbol{\varpi})$ and the set $\Omega_\Lambda(\boldsymbol{\varpi}) \times \mathbf{Z}$.

**List of notations**

| | |
|---|---|
| $I$ | 1.1 |
| $\varpi_i$ | 1.1 |
| $P,\ Q,\ Q^+,\ \mathrm{ht}$ | 1.1 |
| $\widehat{\mathfrak{g}}$ | 1.2 |
| $\widehat{I}$ | 1.2 |
| $\omega_i$ | 1.2 |
| $\widehat{P},\ \widehat{Q},\ \widehat{Q}^+,\ \mathrm{ht}$ | 1.2 |
| $\widehat{\mathbf{U}}_q$ | 1.3 |
| $x_{i,k}^{\pm},\ h_{i,k}$ | 1.3 |
| $K_i$ | 1.3 |
| $C,\ D$ | 1.3 |
| $\widehat{\mathbf{U}}_q^r(\ll),\ \widehat{\mathbf{U}}_q^r(\gg),\ \widehat{\mathbf{U}}_q^r(0)$ | 1.3 |
| $\widehat{\mathbf{U}}_q^\circ$ | 1.3 |
| $\phi_z$ | 1.4 |
| $\widehat{\mathbf{U}}_q^+,\ \widehat{\mathbf{U}}_q^-,\ \widehat{\mathbf{U}}_q'$ | 1.5 |
| $\Delta$ | 1.6 |
| $\widehat{\mathscr{U}}_q^+,\ \widehat{\mathscr{U}}_q^-$ | 1.6 |
| $P_{i,k}$ | 1.6 |
| $M_\mu$ | 2.1 |
| wt | 2.1 |
| $\Omega(M)$ | 2.1 |
| $M(\Lambda),\ m_\Lambda$ | 2.2 |
| $X(\Lambda),\ v_\Lambda$ | 2.2 |
| $V(\boldsymbol{\pi}),\ v_{\boldsymbol{\pi}},\ v_{\boldsymbol{\pi}}^*$ | 2.3 |
| $\lambda_{\boldsymbol{\pi}}$ | 2.3 |
| $w_\circ$ | 2.3 |
| $k(\boldsymbol{\pi})$ | 2.5 |
| $n(v),\ n(\boldsymbol{\pi})$ | 2.8 |
| $m(\boldsymbol{\pi})$ | 2.9 |
| $\eta_{\boldsymbol{\pi}}$ | 2.9 |
| $V(\boldsymbol{\pi})^{(k)}$ | 2.9 |
| $L(V(\boldsymbol{\pi})),\ L^s(V(\boldsymbol{\pi}))$ | 2.10 |
| $\widehat{\eta}_{\boldsymbol{\pi}}$ | 2.10 |
| $\mathrm{ht}_{\boldsymbol{\pi}}$ | 3.2 |

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