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# ANOTHER CHARACTERIZATION OF HYPERCUBES

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Nous montrons que dans la classe des graphes connexes tels que deux arêtes incidentes quelconques appartiennent à un et un seul quadrilatère, les hypercubes finis sont les graphes de degré minimum n fini et possédant $2^n$ sommets.

The following theorem[1] is proved: Let $\mathscr{C}$ be the class of connected graphs such that each pair of distinct adjacent edges lies in exactly one 4-cycle. Then $G$ in $\mathscr{C}$ is a finite hypercube iff the minimum degree $\delta$ of $G$ is finite and[2] $|V(G)| = 2^\delta$.

By a 4-cycle we mean a set of 4 edges, each of them being adjacent with two others; a finite hypercube or a $n$-cube is defined to be the graph $C_n$ with $V(C_n) = \{0, 1\}^n$ and where $x$ and $y$ are joined iff the $n$-tuples $x$ and $y$ differ in exactly one position[3]; the distance $d(x, y)$ between 2 vertices in a connected graph $G$ is the minimum number of edges in a path joining $x$ to $y$.

**Remark 1.** The 6 smallest graphs in $\mathscr{C}$ are shown in Fig. 1.

**Remark 2.** Every $G$ in $\mathscr{C}$ is a simple graph (i.e. a graph without loops or multiple edges) unless $G$ itself is a loop.

**Proof.** Suppose $e$ an edge of a 4-cycle to be a loop. Let $f \neq e$ and $g \neq e$ be the two others edges adjacent with $e$. Each $e$, $f$ or $g$ is adjacent with two others edges, so it is not possible to add any fourth edge giving a 4-cycle with the first three. It follows that $G$ in $\mathscr{C}$ cannot contain a loop unless $G$ itself is a loop.

In the same way it is easy to prove that 2 edges of a 4-cycle cannot have the

[1] After the first version of this paper was written a book by H.M. Mulder [6] was published. Here a similar result but with different approach and proofs may be found.

[2] $V(G)$ denotes the vertex set of graph $G$. For any $X$, $|X|$ denotes the cardinal (number of elements) of $X$.

[3] $C_n$ can also be obtained from $C_0 = K_1$ by the following induction $C_{n+1} = C_n + K_2$ (for the definition of the cartesian sum of two graphs see for example [1]).

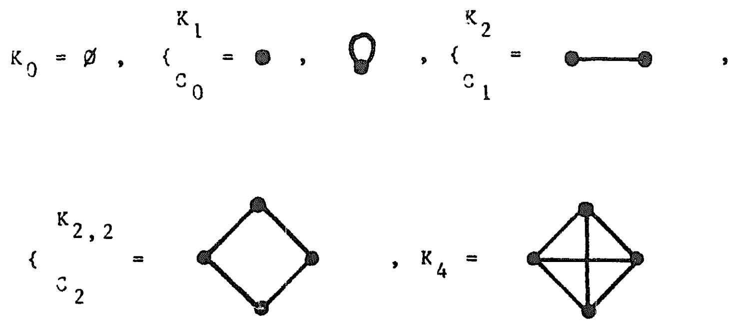


Fig. 1.

same ends. Then $G$ in $\mathscr{C}$ does not contain multiple edges and the remark is settled.

So we shall remove the unique non simple graph from $\mathscr{C}$ and from now *we consider* $\mathscr{C}$ *as a class of simple graphs* and denote an edge joining two vertices $x$ and $y$ by $xy$.

**Remark 3.** The class $\mathscr{C}$ verifies $\mathscr{C}+\mathscr{C}=\mathscr{C}$. In particular if G is in $\mathscr{C}$, then $G+K_2$ is also in $\mathscr{C}$.

**Theorem.** *Let $\mathscr{C}$ be the class of connected graphs such that each pair of distinct adjacent edges lies in exactly one 4-cycle. Then G in $\mathscr{C}$ is a finite hypercube iff the minimum degree $\delta$ of G is finite and $|V(G)|=2^{\delta}$.*

We decompose the proof of the theorem in three parts:

Every $G$ belonging to $\mathscr{C}$ is regular (Proposition 1).

If $\delta$ is finite, then

$$|V(G)|\leq 2^{\delta} \qquad \text{(Proposition 2).} \tag{1}$$

Study of equality in (1).

**Proposition 1.** *Every G in $\mathscr{C}$ is regular.*

**Proof.** If $G=\emptyset$ or $G=K_1$, then $G$ is trivially regular. If not let $x$ be a vertex and $xy$ be an edge incident with $x$.

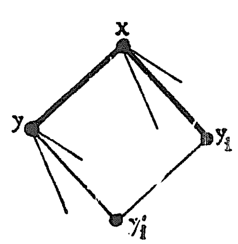


Fig. 2.

To each edge $xy_i$ different from $xy$ we can associate a unique vertex $y'_i$ such that $\{xy, yy'_i, y'_iy_i, y_ix\}$ constitutes a 4-cycle (see Fig. 2). This 4-cycle, or according to the context the fourth vertex $y'_i$ will be called the closure of $xy$ and $xy_i$.

Since all $y'_i$ are distinct we have $d(y) \geqslant d(x)$. Since $G$ is connected the result follows.

**Proposition 2.** *If $\delta$ is finite ($\delta = n$), then*

$$|V(G)| \leqslant 2^n. \tag{1}$$

**Proof.** Consider the following level decomposition of $G$: we first choose any $x \in V(G)$ and then define for $i \in \mathbb{N}$ the $i$th level $N(i) = \{y \in V(G) \mid d(x, y) = i\}$. It is sufficient to prove

$$|N(i)| \leqslant \binom{n}{i}, \tag{2}$$

since then

$$|V(G)| = \sum |N(i)| \leqslant 1 + \binom{n}{1} + \cdots + \binom{n}{n} + 0 + 0 + \cdots = 2^n.$$

*Proof of* (2). Trivially $|N(0)| = |\{x\}| = 1$. Now assume $n \geqslant 1$ and let us prove by induction on $i \geqslant 1$ the following property:

$$P(i) \qquad |N(i)| \leqslant \binom{n}{i} \quad \text{and} \quad \forall v \in N(i) \quad |N(i-1) \cap \Gamma v| \geqslant i,$$

where $\Gamma v$ denotes the set of all vertices adjacent to $v$.

For $i = 1$ this is immediate because $|N(1)| = n$ and each $v \in N(1)$ is joined to $x$. So assume $P(i)$ and establish $P(i+1)$. If $N(i+1) = \emptyset$, then $P(i+1)$. If not, let $v \in N(i+1)$ and $w \in N(i)$ be such that $vw$ is an edge of $G$. For each $w_k \in N(i-1) \cap \Gamma w$ let $w_k$ be the closure of $wv$ *and* $ww_k$ (see Fig. 3).

The $w_k$'s are by induction in number at least $i$, hence also the $w'_k$'s. The $w'_k$'s belong obviously to $N(i)$, then $|N(i) \cap \Gamma v| \geqslant i + 1$. Furthermore

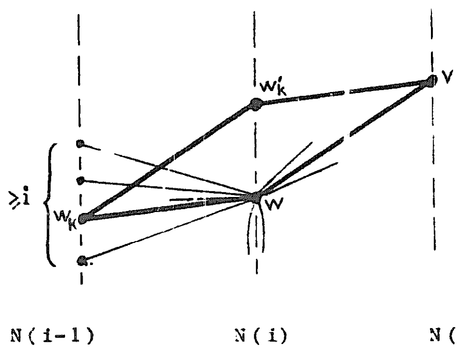


Fig. 3.

$|\Gamma w \cap N(i+1)| + |\Gamma w \cap N(i-1)| \leq n$, hence $|\Gamma w \cap N(i+1)| \leq n-i$ and finally by counting the edges of the bipartite graph induced by the edges with one end in $N(i)$ and the other in $N(i+1)$:

$$|N(i+1)| \leq |N(i)| \cdot \frac{n-i}{i+1} \leq \binom{n}{i+1}. \tag{3}$$

So the proofs of (2) and (1) are complete.

*Equality* $|V(G)| = 2^n$

This equality implies equality in (2) and (3). Then every edge intersects two (consecutive) levels, hence $G$ is bipartite. Furthermore all pairs of edges starting from any $v \in N(i+1)$ and ending in $N(i)$ have their closure in $N(i-1)$ (see Fig. 3). *So any 4-cycle intersects exactly* 3 *levels.* The closure of $i$ edges having a common vertex, is defined to be the smallest subgraph of $G$, containing these edges and belonging to $\mathscr{C}$. We will show by induction on $i \geq 1$ that all closures are hypercubes and therefore that $G$ contains in particular a $n$-cube and since $G$ has $n \cdot 2^{n-1}$ edges, that $G$ itself is a $n$-cube.

This affirmation is trivial for $i = 1$ or 2.

Assume the property for $i$ ($i \geq 2$) and prove it for $i+1$:

If $i+1 > n$ it is clearly true, otherwise consider a level decomposition of $G$ starting at the common vertex $x$ of $i+1$ given edges $xx_j$.

Let $H$ be the closure of $i$ from the $i+1$ given edges; let $x'$ be the other end of the remaining edge. Consider now the different closures of $xx'$ and $xx_j$ denoted by $x_j'$. Finally consider $H'$ the closure of the $x'x_j'$ (see Fig. 4).

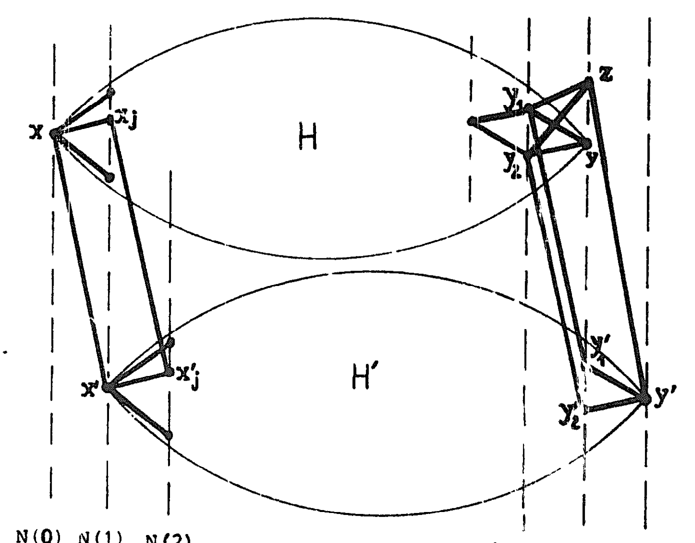


Fig. 4.

Let $y$ and $y'$ be the vertices of $H$ and $H'$ at distance $i$ from $x$ and $x'$ respectively. In $H'-\{y'\}$ consider a vertex $v$ at distance $j<i$ from $x'$. There are $j+1$ edges adjacent to $v$ and intersecting $N(j)$, from which $j$ have their closures contained in $H'$ and from which the remaining intersects $H$. It follows that *the graph induced by $H \cup H'-\{y'\}$ is isomorphic to an $i+1$-cube, from which one vertex is removed.* Left to prove that $yy'$ is an edge of $G$. For this purpose consider the unique edge starting at $y'$ whose other end $z$ belongs to $N(i)$ and not to $H'$ (it exists because of $|\Gamma y' \cap N(i)| = i+1$).

We claim $z = y$. Indeed the closure of $y'z$ and $y'y_1'$ ($y_1'$ in $H'$) cannot belong to $H'$, therefore this is necessarily a certain vertex in $H$, say $y_1$. In the same way the closure of $y'z$ and $y'y_2'$ is $y_2$ ($\neq y_1$). Hence $y = z$, for if not, $yy_1$ and $yy_2$ would lie in more than one 4-cycle.

**Remarks.** (a) The theorem answers a question [5] from one of the authors, namely to know whether the exclusion of a given configuration was essential to obtain a preceding characterization of finite hypercubes [4]; this characterization reads as follows: simple connected graphs with $2^n$ vertices and $n \cdot 2^{n-1}$ edges, with neither triangles nor the following configuration (see Fig. 5) and in which each pair of distinct edges lies in exactly one 4-cycle.

(b) The theorem implies the *characterization of Foldes* [3]: a simple connected finite bipartite graph is a $n$-cube iff between any two vertices at distance $d$, there are exactly $d!$ paths of $d$ edges.

Actually if $G$ is bipartite, then the two end vertices of a path of length 2 are joined by $2! = 2$ paths of length 2 and $G$ is in $\mathscr{C}$. Let $n$ be the maximum distance between two vertices $x$ and $y$ of $G$ and consider a level decomposition with $N(0) = \{x\}$; we have $d(y) = n$ and then $|N(1)| = n$, $|N(2)| = \binom{n}{2}, \ldots, |N(n)| = 1$, hence $|V(G)| = 2^n$ and, by our theorem, $G$ is a $n$-cube.

(c) If we make use of the characterization by Foldes the equality in (1) is rather obvious, since it is not difficult to see that between 2 vertices $x$ and $y$ at distance $d$ there exist precisely $d!$ distinct paths with length $d$ (for this consider a level decomposition with $N(0) = \{x\}$).

**Question.** The hypercubes are maximal graphs of $\mathscr{C}$. What about the mimimal graphs?

Trivial counting arguments show that for every graph in $\mathscr{C}$: if $n \geq 3$, then

$$|V(G)| \geq 1 + n + \frac{n(n-3)}{2} = 1 + \binom{n}{2},$$

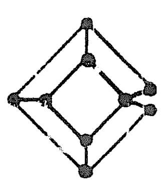

Fig. 5.

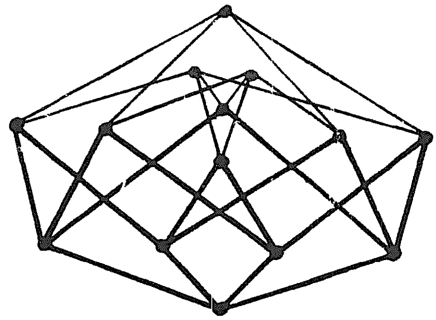

Fig. 6.

and if $G$ contains neither triangles nor pentagons, then for $n \geq 2$

$$|V(G)| \geq 1+n+\frac{n(n-1)}{2}+\frac{n(n-1)}{2}\cdot\frac{n-2}{n}=2\left(1+\binom{n}{2}\right).$$

Furthermore graphs in $\mathscr{C}$ with neither triangles nor pentagons and with $2(1+\binom{n}{2})$ vertices are bipartite and in 1–1 correspondance with the so called *biplanes* [2], i.e. symmetric blocks designs with parameters $(1+\binom{n}{2}, n, 2)$ (they can be interpreted as SBD with 'point set' $N(0) \cup N(2)$ and 'block set' $N(1) \cup N(3)$). The first corresponding graphs are $C_2$, $C_3$, and the following on 14 vertices [4] depicted in Fig. 6.

All biplanes are known for $n \leq 15$ [2] and the conjecture about the existence of finite biplanes for arbitrary large $n$ is then expressed in terms of minimal graphs of $\mathscr{C}$.

Note finally that the first minimal non bipartite graphs are $K_4$, $K_4+C_1$ and the icosahedron with $12\,(>1+\binom{5}{2}=11)$ vertices as shown by M. Mollard (private communication).

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