A note on the zeros of Freud–Sobolev orthogonal polynomials

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Abstract

We prove that the zeros of a certain family of Sobolev orthogonal polynomials involving the Freud weight function $e^{-x^4}$ on $\mathbb{R}$ are real, simple, and interlace with the zeros of the Freud polynomials, i.e., those polynomials orthogonal with respect to the weight function $e^{-x^4}$. Some numerical examples are shown.

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1. Introduction

In [2] the Sobolev inner product was introduced:

$$(p, q)_S = \int_{-\infty}^{\infty} p(x)q(x)e^{-x^4} \, dx + \lambda \int_{-\infty}^{\infty} p'(x)q'(x)e^{-x^4} \, dx, \quad p, q \in \mathcal{P}, \quad \lambda \geq 0,$$

where $\mathcal{P}$ is the linear space of polynomials with real coefficients. Algebraic and asymptotic properties of the monic polynomials $\{Q_n^\lambda\}$ orthogonal with respect to (1) were obtained. Later on, in [3] the authors have considered a more general Sobolev inner product involving exponential weight functions, the inner product (1) remaining as a particular case and they have made a deep study of the asymptotic properties of the corresponding family of Sobolev orthogonal polynomials.

The aim of our contribution is to prove that the polynomials $\{Q_n^\lambda\}$ orthogonal with respect to (1) have all their zeros real and simple.

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2. Basic facts

We denote by \( \{P_n\} \) the sequence of monic polynomials orthogonal with respect to the inner product

\[
(p, q) = \int_{-\infty}^{\infty} p(x)q(x)e^{-x^4} \, dx, \quad p, q \in \mathbb{P}.
\]

It is well known (see [5]) that these polynomials satisfy a three-term recurrence relation

\[
x P_n(x) = P_{n+1}(x) + c_n P_{n-1}(x), \quad n \geq 1,
\]

with initial conditions \( P_0(x) = 1 \) and \( P_1(x) = x \), where the parameters \( c_n \) satisfy a non-linear recurrence relation

\[
n = 4c_n(c_{n+1} + c_n + c_{n-1}), \quad n \geq 1,
\]

with the initial conditions \( c_0 = 0, \ c_1 = \Gamma\left(\frac{3}{4}\right) / \Gamma\left(\frac{1}{4}\right) \), where \( \Gamma(z) \) denotes Euler’s Gamma function. This non-linear equation is a special case of the discrete Painlevé equation \( d-P_1 \) (see [4]). Furthermore, the polynomials \( \{P_n\} \) satisfy

\[
P'_n(x) = nP_{n-1}(x) + 4c_n c_{n-1} c_{n-2} P_{n-3}(x), \quad n \geq 3,
\]

(2)

where (see, for example, [6])

\[
\lim_{n \to \infty} \frac{c_n}{\sqrt{n}} = \frac{\sqrt{3}}{6}.
\]

Now we consider the Freud–Sobolev inner product

\[
(p, q)_S = \int_{-\infty}^{\infty} p(x)q(x)e^{-x^4} \, dx + \lambda \int_{-\infty}^{\infty} p'(x)q'(x)e^{-x^4} \, dx, \quad p, q \in \mathbb{P}, \quad \lambda \geq 0,
\]

\( \{Q^\lambda_n\} \) being the corresponding sequence of monic orthogonal polynomials. Taking into account (2) and the fact that \( Q^\lambda_n(-x) = (-1)^n Q^\lambda_n(x) \) we get (see [2, Proposition 2])

\[
P_n(x) = Q^\lambda_n(x) + a_{n-2}(\lambda) Q^\lambda_{n-2}(x),
\]

(3)

where

\[
a_{n-2}(\lambda) := a_{n-2} = 4\lambda(n - 2) \int_{-\infty}^{\infty} P^2_n(x)e^{-x^4} \, dx / (Q^\lambda_{n-2}, Q^\lambda_{n-2})_S > 0, \quad n \geq 3,
\]

(4)

and \( Q^\lambda_k(x) = P_k(x), \ k = 0, 1, 2 \). In [2] the asymptotic behaviour of the sequence \( \{a_n\} \) has been established using the asymptotic behaviour of the sequence \( \{c_n\} \), lower and upper bounds of \( \{c_n\} \) for all \( n \geq 1 \), as well as the fact that \( Q^\lambda_n \) is an extremal polynomial with respect to the Sobolev norm.

**Proposition 1** (Cachafeiro et al. [2, Proposition 3]). For the sequence \( \{a_n/\sqrt{n}\} \) one gets the upper bound

\[
a_n / \sqrt{n} < \frac{\sqrt{5}}{3}, \quad n \geq 1.
\]

Furthermore, the sequence \( \{a_n/\sqrt{n}\} \) is convergent and

\[
\lim_{n \to \infty} \frac{a_n}{\sqrt{n}} = \frac{\sqrt{3}}{18}.
\]

In [2] the authors also found a non-linear recurrence relation for \( a_n \) in terms of \( c_n \), that is,

\[
a_n = \frac{4nc_{n+2}c_{n+1}}{1/\lambda + n^2/c_n + 16c_n c_{n-1} c_{n-2} - 4(n - 2)a_{n-2}}, \quad n \geq 3,
\]
Table 1

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<th>(a_n)</th>
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with initial conditions
\[
a_1 = \frac{4c_3c_2c_1}{1 + c_1\lambda^{-1}}, \quad a_2 = \frac{8c_4c_3c_2}{4 + c_2\lambda^{-1}}.
\] (5)

From the numerical point of view, we highlight that it is possible to compute the sequences \(\{c_n\}\) and \(\{a_n\}\) in a suitable way using software such as Mathematica. We show some numerical experiments with \(\lambda = 5\) in Table 1.

3. Main results

**Theorem 1.** The polynomials \(\{Q_n^\lambda\}\) have all their zeros real and simple. For \(n \geq 3\) the positive zeros of \(Q_n^\lambda\) interlace with those of \(P_n\).

**Proof.** We distinguish two cases: the even and the odd one, respectively. The proofs are similar with slight differences.

**Even case:** Let \(x_{2m,k}, k = 1, \ldots, m\), be the positive zeros of \(P_{2m}(x)\) in increasing order, that is, \(x_{2m,1} < \cdots < x_{2m,m}\).

First, we need to study the sign of the integrals
\[
I_{2m,k} := \int_{-\infty}^{\infty} Q_{2m}^\lambda(x) \frac{P_{2m}(x)}{x^2 - x_{2m,k}^2} e^{-x^4} \, dx, \quad m \geq 2, \quad k = 1, \ldots, m.
\]

We have
\[
I_{2m,k} = \int_{-\infty}^{\infty} Q_{2m}^\lambda(x) \prod_{j=1, j \neq k}^{m} (x^2 - x_{2m,j}^2) e^{-x^4} \, dx = \sum_{r=0}^{m-1} b_r \int_{-\infty}^{\infty} Q_{2m}^\lambda(x) x^{2r} e^{-x^4} \, dx
\]
\[
=: \sum_{r=0}^{m-1} b_r J_{2m,r},
\] (6)

where \(\text{sgn}(b_r) = (-1)^{m-1-r}, r = 0, \ldots, m - 1\). On the other hand,
\[
J_{2m,r} = (Q_{2m}^\lambda, x^{2r})_S - 2\lambda r \int_{-\infty}^{\infty} \left(\frac{Q_{2m}^\lambda}{x_{2m}^2}\right)' x^{2r-1} e^{-x^4} \, dx.
\]
Thus, applying the Sobolev orthogonality and integration by parts we get
\[
J_{2m,r} = 2r(2r - 1)\lambda J_{2m,r-1} - 8r\lambda J_{2m,r+1},
\]
and therefore
\[
J_{2m,r+1} = -\frac{1}{8r\lambda} J_{2m,r} + \frac{2r - 1}{4} J_{2m,r-1}, \quad r \geq 1.
\] (7)
The initial conditions for the linear recurrence (7) are determined as follows:

\[ J_{2m,0} = \int_{-\infty}^{\infty} Q^2_m(x)e^{-x^4} \, dx = (Q^2_m, 1)_S = 0, \quad m \geq 1. \]  

(8)

To compute \( J_{2m,1} \) we use the expansion of the polynomial \( Q^2_m \) in terms of the Freud polynomials \( P_i \)

\[ Q^2_m(x) = \sum_{j=0}^{m-1} (-1)^j \left( \prod_{i=1}^{j} a_{2m-2i} \right) P_{2m-2j}(x), \quad m \geq 2, \]

which is obtained by using reiteratively (3). Then,

\[ J_{2m,1} = \int_{-\infty}^{\infty} Q^2_m(x)x^2e^{-x^4} \, dx \]

\[ = \sum_{j=0}^{m-1} (-1)^j \left( \prod_{i=1}^{j} a_{2m-2i} \right) \int_{-\infty}^{\infty} P_{2m-2j}(x)x^2e^{-x^4} \, dx \]

\[ = (-1)^{m-1} \left( \prod_{i=1}^{m-1} a_{2m-2i} \right) \int_{-\infty}^{\infty} P_2(x)x^2e^{-x^4} \, dx. \]

From (4) we deduce that \( \text{sgn}(J_{2m,1}) = (-1)^{m-1} \) and then using (7) we get \( \text{sgn}(J_{2m,2}) = (-1)^{m-2} \). Thus, applying (7) successively we obtain \( \text{sgn}(J_{2m,r}) = (-1)^{m-r} \), \( r \geq 1 \). Therefore,

\[ \text{sgn}(b_r J_{2m,r}) = (-1)^{m-1-r}(-1)^{m-r} = \text{sgn}(P'_{2m}(x_{2m,k})), \quad k = 1, \ldots, m. \]  

(9)

On the other hand, using Gaussian quadrature in all the zeros of \( P_{2m} \) and taking into account the symmetry of the polynomials \( Q^2_m(x) \), the Christoffel numbers (see, for example, [7, p. 140])

\[ \mu_{2m,i} = \frac{1}{\sum_{j=0}^{2m-1} P_j^2(x_{2m,i})} > 0, \quad i = 1, \ldots, m, \]

together with the fact

\[ \prod_{j=1, j \neq k}^{m} (x_{2m,k}^2 - x_{2m,j}^2) = \frac{P'_{2m}(x_{2m,k})}{2x_{2m,k}}, \quad k = 1, \ldots, m, \]

we get

\[ I_{2m,k} = \sum_{i=1}^{m} \mu_{2m,i} Q^2_m(x_{2m,i}) \prod_{j=1, j \neq k}^{m} (x_{2m,i}^2 - x_{2m,j}^2) \]

\[ = \mu_{2m,k} Q^2_m(x_{2m,k}) \frac{P'_{2m}(x_{2m,k})}{x_{2m,k}}, \]

and from (9) we deduce

\[ \text{sgn}(Q^2_m(x_{2m,k})) = -\text{sgn}(P'_{2m}(x_{2m,k})), \quad k = 1, \ldots, m. \]  

(10)

Since \( P'_{2m}(x) \) has opposite sign in two consecutive zeros of \( P_{2m}(x) \), from (10) we deduce that it also occurs for \( Q^2_m(x) \) and therefore \( Q^2_m(x) \) has one zero in each interval \( (x_{2m,k}, x_{2m,k+1}) \), \( k = 1, \ldots, m - 1 \) (and from the symmetry it has
one zero in each interval \((-x_{2m,k+1}, -x_{2m,k})\). Thus, \(Q_{2m}^j(x)\) has at least \(2m - 2\) real and simple zeros interlacing with those of \(P_{2m}(x)\). Finally, as \(P_{2m}'(x_{2m,m}) > 0\) then \(Q_{2m}^j(x_{2m,m}) < 0\) and since \(Q_{2m}^j(x)\) is monic we deduce the existence of one zero of \(Q_{2m}^j(x)\) in \((x_{2m,m}, \infty)\) and another zero in \((-\infty, -x_{2m,m})\), which proves the result for the even case.

**Odd case:** Let \(x_{2m+1,k}, k = 1, \ldots, m\), be the positive zeros of \(P_{2m+1}(x)\) in increasing order. Now, we need to know the sign of the integrals

\[
I_{2m+1,k} := \int_{-\infty}^{\infty} Q_{2m+1}^j(x) \frac{P_{2m+1}(x)}{x^2 - x_{2m+1,k}^2} e^{-x^2} \, dx, \quad m \geq 1, \quad k = 1, \ldots, m.
\]

We have

\[
I_{2m+1,k} = \int_{-\infty}^{\infty} Q_{2m+1}^j(x) x \prod_{j=1, j \neq k}^{m} (x^2 - x_{2m+1,j}^2) e^{-x^2} \, dx \\
= \sum_{r=0}^{m-1} \tilde{b}_r \int_{-\infty}^{\infty} Q_{2m+1}^j(x) x^{2r+1} e^{-x^2} \, dx := \sum_{r=0}^{m-1} \tilde{b}_r J_{2m+1,r}, \quad (11)
\]

where \(\text{sgn}(\tilde{b}_r) = (-1)^{m-1-r}, r = 0, \ldots, m - 1\). As in the even case we obtain a linear recurrence for the integrals \(J_{2m+1,r}\), that is,

\[
J_{2m+1,r+1} = J_{2m+1,r} - \frac{1}{4(2r + 1)} \lambda J_{2m+1,r} + \frac{r}{2} J_{2m+1,r-1}, \quad r \geq 1. \quad (12)
\]

In order to find the initial conditions we will use the Sobolev orthogonality, (2), and

\[
Q_{2m+1}^j(x) = \sum_{j=0}^{m} (-1)^j \left( \prod_{i=1}^{j} a_{2m+1-2i} \right) P_{2m+1-2j}(x), \quad m \geq 1.
\]

Then,

\[
J_{2m+1,0} = \int_{-\infty}^{\infty} x Q_{2m+1}^j(x) e^{-x^2} \, dx = (Q_{2m+1}^j, x)_S - \lambda \int_{-\infty}^{\infty} (Q_{2m+1}^j(x))' e^{-x^2} \, dx \\
= -\lambda \sum_{j=0}^{m} (-1)^j \left( \prod_{i=1}^{j} a_{2m+1-2i} \right) \int_{-\infty}^{\infty} P_{2m+1-2j}'(x) e^{-x^2} \, dx \\
= -\lambda \sum_{j=0}^{m-1} (-1)^j \left( \prod_{i=1}^{j} a_{2m+1-2i} \right) \int_{-\infty}^{\infty} \left(2m + 1 - 2j\right) P_{2m-2j}(x) e^{-x^2} \, dx \\
+ 4c_{2m+1-2j} e_{2m-2j} c_{2m-2j} P_{2m-2j-1}(x) e^{-x^2} \, dx \\
- \lambda \left(4c_3 e_{2} c_1 - a_1\right) \left( \prod_{i=1}^{m-1} a_{2m+1-2i} \right) \int_{-\infty}^{\infty} e^{-x^2} \, dx.
\]

Taking into account (5) we get \(a_1 < 4c_3 e_{2} c_1\), and therefore

\[
\text{sgn}(J_{2m+1,0}) = (-1)^m.
\]
Using integration by parts and the computations made to obtain the value of $J_{2m+1,0}$ we can find the other initial condition:

$$J_{2m+1,1} = \int_{-\infty}^{\infty} x^{3} Q_{2m+1}^{j}(x) e^{-x^{4}} \, dx = \frac{1}{4} \int_{-\infty}^{\infty} (Q_{2m+1}^{j}(x))^{'} e^{-x^{4}} \, dx$$

$$= (-1)^{m-1} \frac{4}{4} (4c_{2}c_{2}a_{1} - \prod_{i=1}^{m-1} a_{2m+1-2i}) \int_{-\infty}^{\infty} e^{-x^{4}} \, dx,$$

and thus

$$\text{sgn}(J_{2m+1,1}) = (-1)^{m-1}. \tag{14}$$

From (13) and (14), applying (12) successively, we obtain $\text{sgn}(J_{2m+1,r}) = (-1)^{m-r}$, $r \geq 0$. Therefore,

$$\tilde{b}_{r} J_{2m+1,r} < 0, \quad m \geq 1, \quad r \geq 0,$$

and from (11) we get

$$I_{2m+1,k} < 0, \quad k = 1, \ldots, m, \quad m \geq 1.$$

On the other hand, using Gaussian quadrature in all the zeros of $P_{2m+1}$ we obtain

$$I_{2m+1,k} = \mu_{2m+1,k} Q_{2m+1}^{j}(x_{2m+1,k}) P_{2m+1}^{j}(x_{2m+1,k}),$$

and now taking into account $Q_{2m+1}^{j}(0) = 0$ it only remains to proceed as in the even case to prove the result. \qed

**Proposition 2.** Let $n \geq 4$. Let $x_{n,k}, q_{n,k}^{j}, k = 1, \ldots, [n/2]$, be the positive zeros in increasing order of $P_{n}$ and $Q_{n}^{j}$, respectively, and $q_{n-2,k}^{j}, k = 1, \ldots, [n/2] - 1$, be the positive zeros in increasing order of $Q_{n-2}^{j}$. Then, we have

$$x_{n,1} < q_{n,1}^{j} < q_{n-2,1}^{j} < \cdots < x_{n,[n/2]-1} < q_{n,[n/2]-1}^{j} < q_{n-2,[n/2]-1}^{j} < x_{n,[n/2]} < q_{n,[n/2]}. \tag{15}$$

**Proof.** Formula (3) can be expressed as

$$Q_{n}^{j}(x) = P_{n}(x) - a_{n-2}(\lambda) Q_{n-2}^{j}(x).$$

Thus, evaluating the above expression in a positive zero $x_{n,k}$ of $P_{n}$ and taking into account the positivity of $a_{n-2}(\lambda)$ for $n \geq 3$ we get

$$\text{sgn}(Q_{n}^{j}(x_{n,k})) = -\text{sgn}(Q_{n-2}^{j}(x_{n,k})), \quad k = 1, \ldots, [n/2].$$

From the proof of Theorem 1 we know that $\text{sgn}(Q_{n}^{j}(x_{n,k})) = (-1)^{[n/2]-k+1}$ and therefore

$$\text{sgn}(Q_{n-2}^{j}(x_{n,k})) = (-1)^{[n/2]-k}, \quad k = 1, \ldots, [n/2].$$

This proves that

$$x_{n,1} < q_{n-2,1}^{j} < x_{n,2} < q_{n-2,2}^{j} < \cdots < x_{n,[n/2]-1} < q_{n-2,[n/2]-1}^{j} < x_{n,[n/2]}.$$

Finally, we can prove with the same technique that the positive zeros of $Q_{n-2}^{j}(x)$ separate the positive zeros of $Q_{n}^{j}(x)$, and using Theorem 1 we have the result. \qed

We illustrate the above proposition with some numerical examples. In Tables 2 and 3 we give an example of the interlacing property of the zeros of the polynomials $P_{n}$, $Q_{n}^{j}$, and $Q_{n-2}^{j}$.

**Remark.** At the Conference held in honour of N. M. Temme in Santander, D.K. Dimitrov and F. Marcellán asked about the monotony of the zeros $q_{n,i}^{j}$ of $Q_{n}^{j}$ as a function of the parameter $\lambda$ for $n$ fixed. There exist numerical examples illustrating that these zeros can be either an increasing function or a decreasing function of the parameter $\lambda$ when
the degree of the polynomials is low. However, our numerical experiments allow us to think that there exist certain properties of monotony when the degree of the polynomials is large. Perhaps, by adapting the results appearing in [1] it could be possible to obtain results about the monotony of these zeros as a function of $\lambda$, but till now this remains an open question.

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